Nonlocal fractional elliptic equations and applications

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ABSTRACT

The maximal L_p-regularity properties of nonlocal fractional elliptic equations are studied. Particularly, it is proven that the fractional elliptic operator generated by these equation is sectorial and also is a generator of an analytic semigroup. Moreover, maximal regularity properties of nonlocal fractional parabolic equation are established.

AMS: 47GXX, 35JXX, 47FXX, 47DXX, 43AXX

Key Word: fractional-differential equations, Sobolev spaces, elliptic equations, maximal L_p regularity, parabolic equations, L_p- multipliers

1. Introduction, notations and background

In the last years, fractional elliptic and parabolic equations have found many applications in physics (see [2, 5], [7-9], [11, 18] and the references therein). Fractional derivative calculus has received a considerable attention in the last years. This is mainly because the fact that they have developed venues of feasibility in many areas of science and engineering in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, bode's analysis of feedback amplifiers, capacitor theory. In addition to they have found venues of applicability in many areas of science and engineering such as electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data viscoelasticity, acoustics, electromagnetic waves, control theory, polarization, heat conduction and mathematical biology. The regularity properties of fractional differential equations (FDEs) have been studied e.g. in [1, 3, 8, 9, 12-16]. They [20] have studied the Mittag-Leffler functions as their typical representatives, including many interesting special cases that have already proven their usefulness in fractional calculus and its applications. In this study[21], the homotopy analysis method is applied to obtain the solution of nonlinear fractional partial differential equations. A new solution technique for analytical solutions of fractional partial differential equations (FPDEs) is presented [22]. The main objective of the present paper is to dicuss the $L_p(\mathbb{R}^n)$-maximal regularity of the fractional abstract differential equation (ADE) with parameter

$$\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u + \lambda u = f(x),\ x \in \mathbb{R}^n, \tag{1.1}$$

where a_α are complex valued functions, λ is a complex parameter. Now, we give some information of the fractional calculus which helps us to solve the given problem. We have well known definitions of a fractional derivative of order $\alpha > 0$ such as Riemann-–Liouville, Grunwald-–Letnikow, Caputo and Generalized Functions Approach. The most commonly used definitions are the Riemann—Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used through in this work.

Definition 1.1. A real function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it said to be in the space C_μ^m iff $f^m \in C_\mu, m \in N$.

Definition 1.2. The left and right Riemann-–Liouville fractional integral of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$${}_aI_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \tag{1.2}$$

$${}_xI_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \tag{1.3}$$

It has the following properties:
For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > 1$:

$$1.\ \mathrm{I}^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x), \tag{1.4}$$

$$2.\ \mathrm{I}^\alpha I^\beta f(x) = I^\beta I^\alpha f(x), \tag{1.5}$$

$$3.\ \mathrm{I}^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \tag{1.6}$$

Definition 1.3. The left and right Riemann-–Liouville fractional derivative of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as

$$
{}_aD_x^\alpha f(x) = \frac{d^n}{dx^n}{}_aI_x^{n-\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_a^x (x-t)^{n-\alpha-1} f(t)dt, \tag{1.7}
$$

$$
\begin{aligned}
{}_xD_b^\alpha f(x) &= (-1)^n \frac{d^n}{dx^n}{}_xI_b^{n-\alpha} f(x) \\
&= \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int\limits_x^b (t-x)^{n-\alpha-1} f(t)dt,
\end{aligned} \tag{1.8}
$$

$$
D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n},
$$

here D^{α_k} is Riemann-Liouville type fractional partial derivatives of order $\alpha_k \in (1, 2]$, with respect to x_k i.e.

$$
D_k^{\alpha_k} u = \frac{1}{\Gamma(2-\gamma)} \frac{\partial^2}{\partial x_k^2} \int\limits_0^{x_k} \frac{u(y)\, dy}{(x_k - y)^{\alpha_k - 1}}, \tag{1.9}
$$

here $\Gamma(\alpha_k)$ is Gamma function for $\alpha_k > 0$ (see e.g. [5, 7, 10, 11]), $a_\alpha * D^\alpha u$ is a convolution formula.

For $\alpha_i \in [0, \infty)$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$. Here, $L_p(\Omega)$ denotes the space of strongly measurable complex-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm given by

$$
\|f\|_{L_p(\Omega)} = \left(\int\limits_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty .
$$

Let $S(\mathbb{R}^n)$ denote the comlex-valued Schwartz class, i.e., the space of all rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms.

A function $\Psi \in C(\mathbb{R}^n)$ is called a Fourier multiplier from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ if the map

$$
u \to \Lambda u = F^{-1} \Psi(\xi) Fu, \ u \in S(\mathbb{R}^n)
$$

is well defined and extends to a bounded linear operator

$$
\Lambda : \ L_p(\mathbb{R}^n) \to \ L_p(\mathbb{R}^n).
$$

We prove that problem (1.1) has a unique solution $u \in W_p^l(\mathbb{R}^n; H(A), H)$ for $f \in L_p(\mathbb{R}^n; H)$ and the following uniform coercive estimate holds

$$
{}_{|\alpha| \leq l} |\lambda|^{1 - \frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_{L_p(\mathbb{R}^n; H)} \leq C \|f\|_{L_p(\mathbb{R}^n; H)}. \tag{1.10}
$$

The estimate (1.3) implies that the operator O generated by problem (1.1) has a bounded inverse from $L_p(\mathbb{R}^n)$ into the Sobolev space $W_p^l(\mathbb{R}^n)$, which will be defined subsequently. Particularly, from the estimate (1.3) we obtain that the operator O is uniformly sectorial in $L_p(\mathbb{R}^n)$. By using these coercive properties of elliptic operator, we prove the well posedness of the Cauchy problem for the nonlocal fractional parabolic ADE:

$$\partial_t u + \sum_{|\alpha| \leq l} a_\alpha * D^\alpha u = f(t,x), \; u(0,x) = 0, \tag{1.11}$$

in mixed spaces $L_{\mathbf{p}}$ for $\mathbf{p} = (p, p_1)$. In other words, we show that problem (1.4) has a unique solution $u \in W_{\mathbf{p}}^{1,\gamma}\left(\mathbb{R}_+^2; H(A), E\right)$ for $f \in L_{\mathbf{p}}\left(\mathbb{R}_+^2; H\right)$ satisfying the following coercive estimate

$$\|\partial_t u\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)} + \sum_{|\alpha| \leq l} \|a_\alpha * D_x^\alpha u\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)} + \|A * u\|_{L_{\mathbf{p}};\left(\mathbb{R}_+^{n+1}\right)} \leq$$

$$M \|f\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)}, \tag{1.12}$$

here $L_{\mathbf{p}} = L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)$ denote the space of strongly measurable functions f defined on $\mathbb{R}_+^{n+1}$ equipped with the mixed norm

$$\|f\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)} = \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}_+} |f(t,x)|^{p_1} \, dt \right)^{\frac{p}{p_1}} dx \right)^{\frac{1}{p}} < \infty, \; p_1, p \in (1, \infty).$$

Let $\mathbb{C}$ denote the set of complex numbers and

$$S_\varphi = \{\lambda; \;\; \lambda \in \mathbb{C}, \; |\arg \lambda| \leq \varphi\} \cup \{0\}, \; 0 \leq \varphi < \pi.$$

$B(E_1, E_2)$ denotes the space of bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it denotes by $B(E)$. Let $D(A)$, $R(A)$ denote the domain and range of the linear operator in E, respectively. Let Ker A denote a null space of A. A closed linear operator A is said to be $\varphi-$ sectorial (or sectorial for $\varphi = 0$) in a Banach space E with bound $M > 0$ if Ker $A = \{0\}$, $D(A)$ and $R(A)$ are dense on E, and $\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \leq M |\lambda|^{-1}$ for all $\lambda \in S_\varphi$, $\varphi \in [0, \; \pi)$, where I is an identity operator in E. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by A_λ. It is known [17, §1.15.1] that the powers A^θ, $\theta \in (-\infty, \infty)$ for a positive operator A exist.

A sectorial operator $A(x)$, $x \in \mathbb{R}^n$ is said to be uniformly sectorial in a Banach space E if there exists a $\varphi \in [0\,, \pi)$ such that the uniformly estimate holds

$$\left\|(A(x) + \lambda I)^{-1}\right\|_{B(E)} \leq M |\lambda|^{-1}$$

for all $\lambda \in S_\varphi$.

Here, $S' = S'(\mathbb{R}^n)$ denotes the space of linear continuous mappings from $S(\mathbb{R}^n)$ into $\mathbb{C}$ and it is called the Schwartz distributions. For any $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in [0, \infty)$ the function $(i\xi)^\alpha$ will be defined as:

$$(i\xi)^{\alpha}=\left\{\begin{array}{c}(i\xi_1)^{\alpha_1},.,(i\xi_n)^{\alpha_n},\ \xi_1\xi_2,.,\xi_n\neq 0\\ 0,\ \xi_1,\xi_2,.,\xi_n=0,\end{array}\right.$$

where

$$(i\xi_k)^{\alpha_k}=\exp\left[\alpha_k\left(\ln|\xi_k|+i\frac{\pi}{2}\text{ sgn }\xi_k\right)\right],\ k=1,2,...,n.$$

Let $s\in\mathbb{R}$ and $\xi=(\xi_1,\xi_2,...,\xi_n)\in\mathbb{R}^n$. Consider the following fractional Sobolev space

$$W_p^s(\mathbb{R}^n)=\{u\ \ u\in S'(\mathbb{R}^n),\ F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}Fu\in L_p(\mathbb{R}^n),$$

$$\|u\|_{W_p^s(\mathbb{R}^n)}=\|u\|_{L_p(\mathbb{R}^n)}+\ \left\|F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}Fu\right\|_{L_p(\mathbb{R}^n)}<\infty\Bigg\}.$$

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α.

The embedding theorems in vector valued spaces play a key role in the theory of DOEs. From [15] we obtain the estimating lower order derivatives

Theorem $\mathbf{A}_1$. Suppose $1<p\leq q<\infty$ and $s\in(0,\infty)$ with $\varkappa=\frac{1}{s}\left[|\alpha|+n\left(\frac{1}{p}-\frac{1}{q}\right)\right]\leq 1$, $0\leq\mu\leq 1-\varkappa$, then the embedding

$$D^\alpha W_p^s(\mathbb{R}^n)\subset L_q(\mathbb{R}^n)$$

is continuous and there exists a constant $C_\mu>0$, depending only on μ such that

$$\|D^\alpha u\|_{L_q(\mathbb{R}^n)}\leq C_\mu\left[h^\mu\|u\|_{W_p^s(\mathbb{R}^n)}+h^{-(1-\mu)}\|u\|_{L_p(\mathbb{R}^n)}\right]$$

for all $u\in W_p^s(\mathbb{R}^n)$ and $0<h\leq h_0<\infty$.

2. Nonlocal fractional elliptic equation

Consider the problem (1.1).

Condition 2.1. Assume $a_\alpha\in L_\infty(\mathbb{R}^n)$ such that

$$L(\xi)=\sum_{|\alpha|\leq l}\hat{a}_\alpha(\xi)(i\xi)^\alpha\in S_{\varphi_1},\ \ |L(\xi)|\geq C\sum_{k=1}^{n}\left|\hat{a}_{\alpha(l,k)}\right||\xi_k|^l,\tag{2.1}$$

for

$$\alpha(l,k)=(0,0,...,l,0,0,...,0),\text{ i.e }\alpha_i=0,\ i\neq k,$$

Consider operator functions

$$\sigma_1(\xi,\lambda)=\lambda\sigma_0(\xi,\lambda),\ \sigma_2(\xi,\lambda)=\sum_{|\alpha|\le l}|\lambda|^{1-\frac{|\alpha|}{l}}\hat{a}_\alpha(\xi)(i\xi)^\alpha\sigma_0(\xi,\lambda), \tag{2.2.}$$

where

$$\sigma_0(\xi,\lambda)=[L(\xi)+\lambda]^{-1}\ .$$

Let

$$X=L_p(\mathbb{R}^n),\ Y=W_p^l(\mathbb{R}^n).$$

In this section we prove the following:

Theorem 2.1. Assume that the Condition 2.1 is satisfied. Suppose that $\gamma\in(1,2]$, and $\lambda\in S_{\varphi_2}$. Then for $f\in X$, $0\le\varphi_1<\pi-\varphi_2$ and $\varphi_1+\varphi_2\le\varphi$ there is a unique solution u of the equation (1.1) belonging to Y and the following coercive uniform estimate holds

$$\sum_{|\alpha|\le l}|\lambda|^{1-\frac{|\alpha|}{l}}\|a*D^\alpha u\|_X+\|u\|_X\le C\|f\|_X. \tag{2.3}$$

For the proving of Theorem 2.1 we need the followin lemmas:

Lemma 2.1. Assume Condition 2.1 holds and $\lambda\in S_{\varphi_2}$with $\varphi_2\in[0,\ \pi)$, where $\varphi_1+\varphi_2<\pi$, then the operator functions $\sigma_i(\xi,\lambda)$ are uniformly bounded, i.e.,

$$\|\sigma_i(\xi,\lambda)\|_{B(E)}\le C,\ i=0,1,2.$$

Proof. By virtue of [4, Lemma 2.3], for $L(\xi)\in S_{\varphi_1}$, $\lambda\in S_{\varphi_2}$ and $\varphi_1+\varphi_2<\pi$ there exists a positive constant C such that

$$|\lambda+L(\xi)|\ge C(|\lambda|+|L(\xi)|). \tag{2.4}$$

Since $L(\xi)\in S_{\varphi_1}$ in view of Condition 2.1 and (2.4) the function $\sigma_0(\xi,\lambda)$ is uniformly bounded for all $\xi\in\mathbb{R}^n$, $\lambda\in S_{\varphi_2}$, i.e.

$$\sigma_0(\xi,\lambda)\le(|\lambda|+|L(\xi)|)^{-1}\le M_0.$$

Moreover, we have

$$|\sigma_1(\xi,\lambda)|\le M|\lambda|(|\lambda|+|L(\xi)|)^{-1}\le M_1.$$

Next, let us consider σ_2. It is clear to see that

$$|\sigma_2(\xi,\lambda)|_{B(E)}\le C\sum_{|\alpha|\le l}|\lambda|\prod_{k=1}^n\left[|\xi|\,|\lambda|^{-\frac{1}{l}}\right]^{\alpha_k}|\sigma_0(\xi,\lambda)|. \tag{2.5}$$

By setting $y_k = \left(|\lambda|^{-\frac{1}{l}} |\xi_k| \right)^{\alpha_k}$ in the following well known inequality

$$y_1^{\alpha_1} y_2^{\alpha_2} ... y_n^{\alpha_n} \leq C \left(1 + \sum_{k=1}^{n} y_k^l \right), \; y_k \geq 0, \; |\alpha| \leq l \tag{2.6}$$

we get

$$\|\sigma_2(\xi, \lambda)\|_{B(E)} \leq C \sum_{|\alpha| \leq l} |\lambda| \left[1 + \sum_{k=1}^{n} |\xi_k|^l |\lambda|^{-1} \right] |\lambda + L(\xi)|^{-1}.$$

Taking into account the Condition 2.1 and $(2.5) - (2.6)$ we obtain

$$|\sigma_2(\xi, \lambda)| \leq C \left(|\lambda| + \sum_{k=1}^{n} |\xi_k|^l \right) (|\lambda| + |L(\xi)|)^{-1} \leq C.$$

Lemma 2.2. Assume Condition 2.1 holds. Suppose $\hat{a}_\alpha \in C^{(n)}(\mathbb{R}^n)$ and

$$|\xi|^{|\beta|} \left| D^\beta \hat{a}_\alpha(\xi) \right| \leq C_1, \; \beta_k \in \{0, 1\}, \; \xi \in \mathbb{R}^n \setminus \{0\}, \; 0 \leq |\beta| \leq n, \tag{2.7}$$

Then, operators $|\xi|^{|\beta|} D_\xi^\beta \sigma_i(\xi, \lambda)$, $i = 0, 1, 2$ are uniformly bounded.

Proof. Consider the term $|\xi|^{|\beta|} D_\xi^\beta \sigma_0(\xi, \lambda)$. By using the Conditin 2.1and the above estimates $(2.4) - (2.6)$

$$|\xi_k| \left| \frac{\partial}{\partial \xi_k} \sigma_0(\xi, \lambda) \right| \leq$$

$$\left[|\xi_k| \left| \frac{\partial}{\partial \xi_k} \hat{a}_\alpha(\xi) \right| + \alpha_k |\hat{a}_\alpha(\xi)| \right] \left| \prod_{k=1}^{n} (i\xi_k)^{\alpha_k} \right| \left| [L(\xi) + \lambda]^{-2} \right| < \infty.$$

It easy to see that operators $|\xi|^\beta D^{|\beta|} \sigma_0(\xi, \lambda)$ contain the similar terms as in $|\xi_k| |D_{\xi_k} \sigma_0(\xi, \lambda)|$ for all $\beta_k \in \{0, 1\}$. Hence we get

$$|\xi|^{|\beta|} \left| D_\xi^\beta \sigma_0(\xi, \lambda) \right| < \infty.$$

In a similar way, by using the Conditin 2.1and the above estimates $(2.4) - (2.7)$ we obtain

$$|\xi|^{|\beta|} \left| D_\xi^\beta \sigma_i(\xi, \lambda) \right| < \infty, \; i = 1, 2. \tag{2.8}$$

Proof. **of Theorem 2.1**. By applying the Fourier transform to equation (1.1) we get

$$\hat{u}(\xi) = \sigma_0(\xi, \lambda) \hat{f}(\xi), \; \sigma_0(\xi, \lambda) = [L_\varepsilon(\xi) + \lambda]^{-1}. \tag{2.9}$$

Hence, the solution of (1.1) can be represented as $u(x) = F^{-1}\sigma_0(\xi,\lambda)\hat{f}$ and by Lemma 2.1 there are positive constants C_1 and C_2 such that

$$C_1 |\lambda| \|u\|_X \leq \left\|F^{-1}\left[\lambda\sigma_0(\xi,\lambda)\hat{f}\right]\right\|_X \leq C_2 |\lambda| \|u\|_X ,$$

$$C_1 \sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X \leq \left\|F^{-1}\left[\sigma_2(\xi,\lambda)\hat{f}\right]\right\|_X \leq \tag{2.10}$$

$$C_2 \sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X ,$$

Therefore, it is sufficient to show that the operators $\sigma_i(\xi,\lambda)$ are multipliers in X. But, by Lemma 2.2 and by virtue of Mikhlin multiplier theorem (see e.g [17, § 2.2]) we get that $\sigma_i(\xi,\lambda)$ are multipliers in X. So, we obtai the assertion.

Result 2.1. Theorem 2.1 implies that the operator O is separable in X, i.e. for all $f \in X$ there is a unique solution $u \in Y$ of the problem (1.1), all terms of equation (1.1) are also from X and there are positive constants C_1 and C_2 so that

$$C_1 \|Ou\|_X \leq \sum_{|\alpha|\leq l} \|a_\alpha * D^\alpha u\|_X + \|u\|_X \leq C_2 \|Ou\|_X .$$

Indeed, if we put $\lambda = 1$ in (2.3), by Theorem 2.1 we get the second inequality. So it is remain to prove the first estimate. The first inequality is equivalent to the following estimate

$$\sum_{|\alpha|\leq l} \left\|F^{-1}\hat{a}_\alpha (i\xi)^\alpha \hat{u}\right\|_X \leq \sum_{|\alpha|\leq l} \left\|F^{-1}\hat{a}_\alpha (i\xi)^\alpha \sigma_0(\xi,\lambda)\hat{f}(\xi)\right\|_X .$$

So, it suffices to show that the operator functions

$$\sigma_0(\xi,\lambda), \sum_{|\alpha|\leq l} \hat{a}_\alpha (i\xi)^\alpha \sigma_0(\xi,\lambda)$$

are uniform Fourier multipliers in X. This fact is proved in a similar way as in the proof of Theorem 2.1.

From Theorem 2.1, we have:

Result 2.2. Assume all conditions of Theorem 2.1 hold. Then, for all $\lambda \in S_\varphi$ the resolvent of operator O exists and the following sharp coercive uniform estimate holds

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\|a * D^\alpha (O+\lambda)^{-1}\right\|_{B(X)} + \left\|(O+\lambda)^{-1}\right\|_{B(X)} \leq C. \tag{2.11}$$

Indeed, we infer from Theorem 2.1 that the operator $O+\lambda$ has a bounded inverse from X to Y. So, the solution u of the equation (1.1) can be expressed as $u(x) = (O+\lambda)^{-1} f$ for all $f \in X$. Then estimate (2.4) implies the estimate (2.11).

Theorem 2.2. Assume that the Condition 2.1 is satisfied. Suppose that $\gamma \in (1,2]$, and $\lambda \in S_{\varphi_2}$. Then for $f \in X$, $0 \leq \varphi_1 < \pi - \varphi_2$ and $\varphi_1 + \varphi_2 \leq \varphi$ there is a unique solution u of the equation (1.1) belonging to Y and the following coercive uniform estimate holds

$$\sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|D^\alpha u\|_X + \|u\|_X \leq C \|f\|_X . \tag{2.12}$$

Proof. The estimate (2.12) is derived by reasoning as in Theorem 2.2.
From Theorem 2.2, we have the following results:
Result 2.3. There are positive constants C_1 and C_2 so that

$$C_1 \|Ou\|_X \leq \sum_{|\alpha| \leq l} \|D^\alpha u\|_X + \|Au\|_X \leq C_2 \|Ou\|_X . \tag{2.13}$$

From theorem 2.2. we obtain
Result 2.4. Assume all conditions of Theorem 2.2 hold. Then, for all $\lambda \in S_\varphi$ the resolvent of operator O exists and the following sharp uniform estimate holds

$$\sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| D^\alpha (O+\lambda)^{-1} \right\|_{B(X)} + \left\| (O+\lambda)^{-1} \right\|_{B(X)} \leq C. \tag{2.15}$$

Result 2.5. Theorem 2.2 particularly implies that the operator O is sectorial in X. Then the operators O^s are generators of analytic semigroups in X for $s \leq \frac{1}{2}$ (see e.g. [17, §1.14.5]).

3. The Cauchy problem for fractional parabolic equation

In this section, we shall consider the following Cauchy problem for the parabolic FDOE

$$\frac{\partial u}{\partial t} + \sum_{|\alpha| \leq l} a_\alpha * D^\alpha u = f(t,x), \; u(0,x) = 0, \; t \in \mathbb{R}_+, \; x \in \mathbb{R}^n, \tag{3.1}$$

where a is a complex number, D_x^α is the fractional derivative in x for $\alpha_k \in (1,2]$, defined by (1.2).

By applying Theorem 2.1 we establish the maximal regularity of the problem (3.1) in mixed $L_{\mathbf{p}}$ spaces, where $\mathbf{p} = (p_1, p)$. Let O denote the operator generated by problem (1.1) for $\lambda = 0$. For $\mathbf{p} = (p, p_1)$, $Z = L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1}\right)$ will denote the space of all $\mathbf{p}$-summable complexvalued functions on $\mathbb{R}_+^{n+1}$ with mixed norm, i.e., the space of all measurable complex-valued functions f defined on $\mathbb{R}_+^{n+1}$ for which

$$\|f\|_{L_{\mathbf{p}}\left(\mathbb{R}_+^{n+1};H\right)} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}_+} |f(t,x)|^p \, dx \right)^{\frac{p_1}{p}} dt \right)^{\frac{1}{p_1}} < \infty.$$

Let $Z^{1,\gamma} = W_{\mathbf{p}}^{1,l}\left(\mathbb{R}_{+}^{n+1}\right)$ denotes the space of all functions $u \in L_{\mathbf{p}}\left(\mathbb{R}_{+}^{n+1}\right)$ possessing the generalized derivative $D_t u = \frac{\partial u}{\partial t} \in Z$ with respect to y and fractional derivatives $D_x^{\alpha} u \in Z$ with respect to x for $|\alpha| \leq l$ with the norm

$$\|u\|_{Z^{1,2}(A)} = \|u\|_Z + \|\partial_t u\|_Z + \|D_x^{\gamma} u\|_Z ,$$

where $u = u\left(t, x\right)$.

Now, we are ready to state the main result of this section.

Theorem 3.1. Assume the conditions of Theorem 2.1 hold for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$. Then for $f \in Z$ problem (3.1) has a unique solution $u \in Z^{1,\gamma}\left(A\right)$ satisfying the following unform coercive estimate

$$\|\partial_t u\|_Z + \sum_{|\alpha| \leq l} \|a_\alpha * D^\alpha u\|_Z + \|u\|_Z \leq C \|f\|_Z .$$

Proof. By definition of $X = L_p\left(\mathbb{R}^n\right)$ and mixed space $L_{\mathbf{p}}\left(\mathbb{R}_{+}^{n+1}\right)$, $\mathbf{p} = (p,\, p_1)$, we have

$$\|u\|_{L_{p_1}(0,\infty;)} = \left(\int_0^\infty \|u\left(t\right)\|_X^{p_1} dt\right)^{\frac{1}{p_1}} = \left(\int_0^\infty \|u\left(t\right)\|_{L_p(\mathbb{R}^n)}^{p_1} dt\right)^{\frac{1}{p_1}} = \|u\|_Z .$$

Therefore, the problem (3.1) can be expressed as the following Cauchy problem for the abstract parabolic equation

$$\frac{du}{dt} + Ou\left(t\right) = f\left(t\right),\ u\left(0\right) = 0,\ t \in \left(0, \infty\right). \tag{3.2}$$

Then, by virtue of [19, Theorem 4.2], we obtain that for $f \in L_{p_1}\left(0, \infty; X\right)$ the problem (3.2) has a unique solution $u \in W_{p_1}^1\left(0, \infty; D\left(O\right), X\right)$ satisfying the following estimate

$$\left\|\frac{du}{dt}\right\|_{L_{p_1}(0,\infty;X)} + \|Ou\|_{L_{p_1}(0,\infty;X)} \leq C \|f\|_{L_{p_1}(0,\infty;X)} .$$

From the Theorem 2.2, relation (3.2) and from the above estimate we get the assertion.

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