Hurwicz model of uncertain optimal control with jump

Liubao Deng [1]*, Jinzhong Shen[2] and Yuefen Chen[3]

[1]*School of Finance, Anhui University of Finance and Economics,*
No.962 Caoshan Road Bengbu 233030, Anhui, China, E-mail: dengliubao@163.com

[2]*School of Electrical and Information Engineering, Anhui University of Science and Technology,*
No.168 Taifeng Street Huainan 232001, Anhui, China, E-mail: jzshen009@163.com

[3]*School of Mathematics and Statistics, Xinyang Normal University,*
No.237 Nanhu Road Xinyang 464000, Henan, China, E-mail: yfchen@xynu.edu.cn

Abstract

How to choose the optimization criterion of the objective function is an important issue for uncertain optimal control. The Hurwicz criterion is a flexible optimization criterion attempting to find the intermediate area between the extremes posed by the optimistic and pessimistic criteria. Based on uncertainty theory, in this paper, we establish a new uncertain optimal control model with jump by making use of Hurwicz criterion to optimize an uncertain objective function. By applying Bellman's principle of optimality, the principle of optimality for the proposed model is presented and then the equation of optimality is derived. Finally, an example is given to show the the effectiveness of the results obtained.

Keywords: optimal control, Hurwicz criterion, uncertainty, jump

1 Introduction

In the past decades, optimal control of systems with jumps have received a lot of interest from many engineers and economists. This kind of optimal control problem has a practical background in engineering, economics and management, especially in financial market. Some rare events or catastrophes or machine failures, have a great influence on how the

*Corresponding author. Email: dengliubao@163.com

biological, physical or other system evolves. The stochastic differential equations driven by both Brownian motions and Poisson processes have become increasing popular for modelling the stochastic jump diffusion systems in various areas. Since Merton [1] first investigated the optimal control problem of consumption and investment with random poisson jumps in 1971, the study on optimal control of stochastic jump diffusion system has been made considerable advances both in theory and application. Some studied results can be seen in [2, 3, 4, 5, 6] and the references therein.

The state evolution of dynamic control systems with jumps usually are disturbed by many indeterministic factors. In general, the probability theory is an effective tool for dealing with these indeterminacy. However, a fundamental premise of realistic applications of probability theory is that there must be sufficient available sample data to estimate probability distribution of indeterminate event. For many systems in real life, the sample size usually is too small or even no sample such as bridge strength, oil filed reserves and the price of a new stock. In this case, we can't estimate a probability distribution by means of statistics and have to consult some domain experts to evaluate their belief degree that each indeterministic event will occur. Liu [7] found probability theory is no longer valid for describing belief degree. In order to rationally deal with the indeterminacy, an uncertainty theory was founded by Liu [8] in 2007 and refined by Liu [9] in 2010.

Based on the uncertainty theory, Zhu [10] first proposed and dealt with an uncertain optimal control problem without jump in which the expected value operator was used to optimize the uncertain objective function. An equation of optimality as a counterpart of HJB equation was obtained by employing dynamic programming. Then, by means of Zhu's equation of optimality, Xu and Zhu [11], Yan and Zhu [12] studied uncertain bang-bang optimal control models. Chen and Zhu [13] discussed the uncertain optimal control problem with time-delay. Yao and Qin [14], Li and Zhu [15] investigated the uncertain linear quadratic control problem. Yang and Gao [16] studied an uncertain differential game. Taking the influence of some external extreme events or noises on uncertain dynamic systems into account, Deng and Zhu [17, 18, 19, 20] further studied some expected value models of optimal control problem for uncertain dynamic systems with jump and the equations of optimality were derived.

How to choose the optimization criterion of the objective function is an important issue for uncertain optimal control. In fact, there are many optimization criteria such as expected value, optimistic value, pessimistic value, and Hurwicz criterion. We can't say which of these criteria is better or worse because they are suitable for different specific situations and specific problems. The expected value criterion is a traditional one in the sense of the weighted average. Both the optimistic criterion and the pessimistic criterion are extreme criteria. The Hurwicz criterion can also be called optimism coefficient method, designed by economics professor Leonid Hurwicz [21] in 1951. It is a more complex and more flexible optimization criterion attempting to find the intermediate area between the extremes posed by the optimistic and pessimistic criteria. Hurwicz criterion incorporates a measure of both by assigning a certain percentage weight to optimism and the balance to pessimism. By changing a coefficient denoting the optimism degree, the Hurwicz criterion actually becomes various criteria. The optimistic value criterion and the pessimistic value criterion are its two special cases.

In 2013, Sheng and Zhu [22] established an optimistic value model of uncertain optimal control without jump by making use of the optimistic value criterion to optimize the uncertain objective function. Afterward, Yan and Zhu [23] studied bang-bang control model with optimistic value criterion for uncertain switched systems. Li, Zhu and Chen [24] discussed approximating uncertain optimal control problems under optimistic value criterion by applying the piecewise optimisation method. Further, Deng and Chen [25, 26, 27] investigated optimistic value models of uncertain optimal control with jump. Besides, Sheng, Zhu and Hamalainen [28] proposed an uncertain optimal control model without jump under Hurwicz criterion.

The purpose of this paper is to further study the uncertain optimal control problem with jumps under Hurwicz criterion, where dynamic systems are modelled by a type of uncertain differential equation driven by canonical process and V jump uncertain process. To the best of our knowledge, this problem has not been investigated in the literature and remains open. The remainder of this paper proceeds as follows. Some concepts and theorems in uncertainty theory are reviewed in Section 2. In Section 3, we establish a new uncertain optimal control model with jump by making use of Hurwicz criterion to optimize an uncertain objective function and give the principle of optimality. In Section 4, we derive the equation of optimality of proposed model. Section 5 gives a conclusion. Final Section is Appendix, in which an estimation for the α-pessimistic value of $a\eta + b\eta^2$

if η is a V-jump uncertain variable ($\alpha \in (0,\, 1)$) is given.

2 Preliminary

In this section, we will review some basic concepts and theorems in uncertainty theory 8. Let Γ be a nonempty set, and $\mathcal{L}$ a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}$ defined on the σ-algebra $\mathcal{L}$ over Γ is called an uncertain measure if it satisfies $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$, and $\mathcal{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$ for every countable sequence of events $\{\Lambda_i\}$. The triplet $(\Gamma,\, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. An uncertain variable is a function ξ from an uncertainty space $(\Gamma,\, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that for any Borel set of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma |\ \xi(\gamma) \in B\}$ is an event. An uncertain variable ξ may be described by its uncertainty distribution Φ: $\Re \to [0,1]$ which is defined by $\Phi(x) = \mathcal{M}\{\xi \leq x\}$. The expected value of ξ is defined by $E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr$ provided that at least one of the two integrals is finite. The variance of ξ is $V[\xi] = E[(\xi - E[\xi])^2]$. The uncertain variables $\xi_1, \xi_2, ..., \xi_m$ are said to be independent if $\mathcal{M}\{\bigcap_{i=1}^{m} \{\xi_i \in B_i\}\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}$, for any Borel set $B_1, B_2, \cdots, B_m$ of real numbers.

Definition 2.1 . *(Liu [8]) Let ξ be an uncertain variable, and $\alpha \in (0,1]$. Then $\xi_{\sup}(\alpha) = \sup\{r | \mathcal{M}\{\xi \geq r\} \geq \alpha\}$ is called the α-optimistic value to ξ; and $\xi_{\inf}(\alpha) = \inf\{r | \mathcal{M}\{\xi \leq r\} \geq \alpha\}$ is called the α-pessimistic value to ξ.*

Theorem 2.1 . *(Liu [8, 9]) Let ξ and η be are independent uncertain variables and $\alpha \in (0,1]$. Then we have*
(i) if $c \geq 0$, then $(c\xi)_{\sup}(\alpha) = c\xi_{\sup}(\alpha)$ and $(c\xi)_{\inf}(\alpha) = c\xi_{\inf}(\alpha)$;
(ii) if $c < 0$, then $(c\xi)_{\sup}(\alpha) = c\xi_{\inf}(\alpha)$ and $(c\xi)_{\inf}(\alpha) = c\xi_{\sup}(\alpha)$;
(iii) $(\xi + \eta)_{\sup}(\alpha) = \xi_{\sup}(\alpha) + \eta_{\sup}(\alpha)$, $(\xi + \eta)_{\inf}(\alpha) = \xi_{\inf}(\alpha) + \eta_{\inf}(\alpha)$.

Theorem 2.2 . *(Sheng and Zhu [22]) Let ξ be a normal uncertain variable $N(0, \sigma)$, for any real number a, and any small enough $\varepsilon > 0$ and $\alpha \in (0,1)$,*

(1) if $b > 0$, then

$$(a\xi + b\xi^2)_{\sup}(\alpha) \geq \frac{\sqrt{3}|a|\sigma}{\pi} \ln \frac{1-\alpha}{\alpha} + b \left(\frac{\sqrt{3}\sigma}{\pi} \ln \frac{1-\alpha}{\alpha} \right)^2, \tag{2.1}$$

$$(a\xi+b\xi^2)_{\sup}(\alpha) \le \frac{\sqrt{3}|a|\sigma}{\pi}\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}+b\left(\frac{\sqrt{3}\sigma}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2. \tag{2.2}$$

(2) if $b<0$, then

$$(a\xi+b\xi^2)_{\sup}(\alpha) \ge \frac{\sqrt{3}|a|\sigma}{\pi}\ln\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon}+b\left(\frac{\sqrt{3}\sigma}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2, \tag{2.3}$$

$$(a\xi+b\xi^2)_{\sup}(\alpha) \le \frac{\sqrt{3}|a|\sigma}{\pi}\ln\frac{1-\alpha}{\alpha}+b\left(\frac{\sqrt{3}\sigma}{\pi}\ln\frac{1-\alpha}{\alpha}\right)^2. \tag{2.4}$$

(3) if $b=0$, then

$$(a\xi+b\xi^2)_{\sup}(\alpha) = \frac{\sqrt{3}|a|\sigma}{\pi}\ln\frac{1-\alpha}{\alpha}. \tag{2.5}$$

Definition 2.2 . *(Deng and Zhu [17]) An uncertain process V_t is said to be a V jump process with parameters r_1 and r_2 $(0<r_1<r_2<1)$ for $t\ge 0$ if*
(i) $V_0=0$,
(ii) V_t has stationary and independent increments,
(iii) every increment $V_{s+t}-V_s$ is a Z jump uncertain variable $Z(r_1,r_2,t)$, whose uncertainty distribution is

$$\Psi(x)=\begin{cases} 0, & \text{if } x<0\\ \dfrac{2r_1}{t}x, & \text{if } 0\le x<\dfrac{t}{2}\\ r_2+\dfrac{2(1-r_2)}{t}\left(x-\dfrac{t}{2}\right), & \text{if } \dfrac{t}{2}\le x<t\\ 1, & \text{if } x\ge t. \end{cases} \tag{2.6}$$

Let V_t be a V jump uncertain process, and $\eta=\Delta V_t=V_{t+\Delta t}-V_t$. Then for any $\alpha\in(0,1)$, it follows from the definition of α-optimistic value and α-pessimistic value that

$$\eta_{\sup}(\alpha)=\begin{cases} \left(1-\dfrac{\alpha}{2(1-r_2)}\right)\Delta t, & \text{if } 0<\alpha<1-r_2\\ \dfrac{\Delta t}{2}, & \text{if } 1-r_2\le\alpha<1-r_1\\ \dfrac{1-\alpha}{2r_1}\Delta t, & \text{if } 1-r_1\le\alpha<1 \end{cases} \tag{2.7}$$

and

$$\eta_{\inf}(\alpha)=\begin{cases}\dfrac{\alpha}{2r_1}\Delta t, & \text{if } 0<\alpha\le r_1\\ \dfrac{\Delta t}{2}, & \text{if } r_1<\alpha\le r_2\\ \left(1-\dfrac{1-\alpha}{2(1-r_2)}\right)\Delta t, & \text{if } r_2<\alpha<1,\end{cases} \tag{2.8}$$

respectively.

Theorem 2.3 *(Deng, You and Chen [27]) Let V_t a V jump uncertain process. Denote $\eta=\Delta V_t$, then for any real number a, b, any $\alpha\in(0,1)$, and any $\varepsilon>0$ small enough,*

(1) if $a\ge 0, b\ge 0$, then

$$[a\eta+b\eta^2]_{\sup}(\alpha)\ge\begin{cases}\left(1-\dfrac{\alpha}{2(1-r_2)}\right)a\Delta t, & \textit{if } 0<\alpha<1-r_2\\ \dfrac{a}{2}\Delta t, & \textit{if } 1-r_2\le\alpha<1-r_1\\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t, & \textit{if } 1-r_1\le\alpha<1,\end{cases} \tag{2.9}$$

$$\left[a\eta+b\eta^2\right]_{\sup}(\alpha)\le\begin{cases}\left(1-\dfrac{\alpha}{2(1-r_2)}\right)a\Delta t+b\Delta t^2, & \textit{if } 0<\alpha<1-r_2\\ \dfrac{a}{2}\Delta t+b\Delta t^2, & \textit{if } 1-r_2\le\alpha<1-r_1\\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t+b\Delta t^2, & \textit{if } 1-r_1\le\alpha<1,\end{cases} \tag{2.10}$$

(2) if $a<0, b\ge 0$, then

$$[a\eta+b\eta^2]_{\sup}(\alpha)\ge\begin{cases}\dfrac{\alpha a}{2r_1}\Delta t, & \textit{if } 0<\alpha\le r_1\\ \dfrac{a}{2}\Delta t, & \textit{if } r_1<\alpha\le r_2\\ \left(1-\dfrac{1-\alpha}{2(1-r_2)}\right)a\Delta t, & \textit{if } r_2<\alpha<1,\end{cases} \tag{2.11}$$

$$\left[a\eta+b\eta^2\right]_{\sup}(\alpha)\le\begin{cases}\dfrac{a\alpha}{2r_1}\Delta t+b\Delta t^2, & \textit{if } 0<\alpha\le r_1\\ \dfrac{a}{2}\Delta t+b\Delta t^2, & \textit{if } r_1<\alpha\le r_2\\ \left(1-\dfrac{1-\alpha}{2(1-r_2)}\right)a\Delta t+b\Delta t^2, & \textit{if } r_2<\alpha<1,\end{cases} \tag{2.12}$$

(3) if $a \geq 0, b < 0$, then

$$\left[a\eta + b\eta^2\right]_{\sup}(\alpha) \geq \begin{cases} \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) a\Delta t + b\Delta t^2, & \textit{if } 0 < \alpha < 1 - r_2 \\ \dfrac{a}{2}\Delta t + b\Delta t^2, & \textit{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t + b\Delta t^2, & \textit{if } 1 - r_1 \leq \alpha < 1, \end{cases} \tag{2.13}$$

$$[a\eta + b\eta^2]_{\sup}(\alpha) \leq \begin{cases} \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) a\Delta t, & \textit{if } 0 < \alpha < 1 - r_2 \\ \dfrac{a}{2}\Delta t, & \textit{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t, & \textit{if } 1 - r_1 \leq \alpha < 1, \end{cases} \tag{2.14}$$

(4) if $a < 0, b < 0$, then

$$\left[a\eta + b\eta^2\right]_{\sup}(\alpha) \geq \begin{cases} \dfrac{\alpha a}{2r_1}\Delta t + b\Delta t^2, & \textit{if } 0 < \alpha \leq r_1 \\ \dfrac{a}{2}\Delta t + b\Delta t^2, & \textit{if } r_1 < \alpha \leq r_2 \\ \left(1 - \dfrac{1-\alpha}{2(1-r_2)}\right) a\Delta t + b\Delta t^2, & \textit{if } r_2 < \alpha < 1, \end{cases} \tag{2.15}$$

$$[a\eta + b\eta^2]_{\sup}(\alpha) \leq \begin{cases} \dfrac{\alpha a}{2r_1}\Delta t, & \textit{if } 0 < \alpha \leq r_1 \\ \dfrac{a}{2}\Delta t, & \textit{if } r_1 < \alpha \leq r_2 \\ \left(1 - \dfrac{1-\alpha}{2(1-r_2)}\right) a\Delta t, & \textit{if } r_2 < \alpha < 1. \end{cases} \tag{2.16}$$

3 Uncertain optimal control model with jump under Hurwicz criterion

In this section, we present the following uncertain optimal control model with jump under Hurwicz criterion:

$$\begin{cases} J(t,\mathbf{x}) \equiv \sup\limits_{\mathbf{u}_s\in\mathbf{U}} \left[\lambda S_{\sup}(\alpha) + (1-\lambda) S_{\inf}(\alpha)\right], \\ \text{subject to} \\ \quad d\mathbf{X_s} = \mathbf{P}\left(s,\mathbf{X}_s,\mathbf{u}_s\right) ds + \mathbf{Q}\left(s,\mathbf{X}_s,\mathbf{u}_s\right) d\mathbf{C}_s + \mathbf{R}\left(s,\mathbf{X}_s,\mathbf{u}_s\right) d\mathbf{V}_s, \\ \quad \mathbf{X}_t = \mathbf{x}, \end{cases} \tag{3.1}$$

where the vector $\mathbf{X}_s$ denotes a state vector of dimension n, $\mathbf{u}_s$ denotes a control vector of dimension r subject to a constraint set $\mathbf{U}$. $S = \int_t^T L\left(s,\mathbf{X}_s,\mathbf{u}_s\right) ds + F\left(T,\mathbf{X}_T\right)$, and the function $L : [0,T) \times \Re^n \times \Re^r \to \Re$ is an objective function, and $F : [0,T) \times \Re^n \to \Re$ is the terminal reward function. $S_{\sup}(\alpha) = \sup\{\tilde{S} | \mathcal{M}\{S \geq \tilde{S}\} \geq \alpha\}$ and $S_{\inf}(\alpha) = \inf\{\tilde{S} | \mathcal{M}\{S \leq \tilde{S}\} \geq \alpha\}$ denote the α-optimistic value and the α–pessimistic value to S, respectively. In addition, $\mathbf{P} : [0,T) \times \Re^n \times \Re^r \to \Re^n$ is a vector function, $\mathbf{Q} : [0,T) \times \Re^n \times \Re^r \to \Re^n \times \Re^k$ and $\mathbf{R} : [0,T) \times \Re^n \times \Re^r \to \Re^n \times \Re^k$ are two matrix functions. All functions mentioned are continuous. Assume that $\mathbf{C}_t = (C_{t1}, C_{t2} \cdots, C_{tk})^\tau$ and $\mathbf{V}_t = (V_{t1}, V_{t2}, \cdots, V_{tk})^\tau$, where $C_{t1}, C_{t2}, \cdots, C_{tk}$ are independent canonical processes, $V_{t1}, V_{t2}, \cdots, V_{tk}$ are independent uncertain V-jump process, and C_{ti} and V_{tj} are independent for any $i, j = 1, 2, \cdots, k$ $(i \neq j)$. $\alpha \in (0,1)$ is given confidence level and $\lambda \in (0,1)$ is a selected coefficient denoting the optimism degree. The symbol υ^τ denotes the transpose of a vector or a matrix υ.

Theorem 3.1 *[Principle of optimality] For any $(t,\mathbf{x}) \in [0,T) \times \Re^n$, and $\Delta t > 0$ with $t + \Delta t < T$, we have*

$$\begin{aligned} J(t,\mathbf{x}) = \sup_{\mathbf{u}_t\in\mathbf{U}} \Big[& L\left(t,\mathbf{x},\mathbf{u}_t\right) \Delta t + o(\Delta t) + \lambda J\left(t+\Delta t, \mathbf{x} + \Delta\mathbf{X}_t\right)_{\sup}(\alpha) \\ & + (1-\lambda) J\left(t+\Delta t, \mathbf{x}+\Delta\mathbf{X}_t\right)_{\inf}(\alpha) \Big], \end{aligned} \tag{3.2}$$

where $\mathbf{x} + \Delta\mathbf{X}_t = \mathbf{X}_{t+\Delta t}$.

Proof. We denote the right side of (3.2) by $\tilde{J}(t,\mathbf{x})$. For arbitrary $\mathbf{u}\in\mathbf{U}$, it follows from the definition of $J(t,\mathbf{x})$ that

$$J(t,\mathbf{x}) \geq \lambda\left\{\int_t^{t+\Delta t} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t,t+\Delta t)})\mathrm{d}s + \int_{t+\Delta t}^{T} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t+\Delta t,T]})\mathrm{d}s + F(T,\mathbf{X}_T)\right\}_{\sup}(\alpha)$$
$$+(1-\lambda)\left\{\int_t^{t+\Delta t} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t,t+\Delta t)})\mathrm{d}s + \int_{t+\Delta t}^{T} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t+\Delta t,T]})\mathrm{d}s + F(T,\mathbf{X}_T)\right\}_{\inf}(\alpha),$$

for any $\mathbf{u}_s$, where $\mathbf{u}_s|_{[t,t+\Delta t)}$ and $\mathbf{u}_s|_{[t+\Delta t,T]}$ are the values of control variable $\mathbf{u}_s$ restricted on $[t,t+\Delta t)$ and $[t+\Delta t,T]$, respectively.

For any $\Delta t>0$, by using Taylor series expansion, we get

$$\int_t^{t+\Delta t} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t,t+\Delta t)})\mathrm{d}s = L(t,\mathbf{x},\mathbf{u}(t,\mathbf{x}))\Delta t + o(\Delta t).$$

Thus

$$J(t,\mathbf{x}) \geq L(t,\mathbf{x},\mathbf{u})\Delta t + o(\Delta t) + \lambda\left\{\int_{t+\Delta t}^{T} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t+\Delta t,T]})\mathrm{d}s + F(T,\mathbf{X}_T)\right\}_{\sup}(\alpha)$$
$$+(1-\lambda)\left\{\int_{t+\Delta t}^{T} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t+\Delta t,T]})\mathrm{d}s + F(T,\mathbf{X}_T)\right\}_{\inf}(\alpha). \tag{3.3}$$

Taking the supremum with respect to $|\mathbf{u}_{[t+\Delta t,T]}$ in (3.3), then we get $J(t,\mathbf{x})\geq\tilde{J}(t,\mathbf{x})$. On the other hand, for all $\mathbf{u}$, we have

$$\begin{aligned}\lambda S_{\sup}(\alpha)+(1-\lambda)S_{\inf}(\alpha) &= f(t,\mathbf{x},\mathbf{u})\Delta t + o(\Delta t)\\ &\quad+\lambda\left\{\int_{t+\Delta t}^{T} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t+\Delta t,T]})\mathrm{d}s + F(T,\mathbf{X}_T)\right\}_{\sup}(\alpha)\\ &\quad+(1-\lambda)\left\{\int_{t+\Delta t}^{T} L(s,\mathbf{X}_s,\mathbf{u}_s|_{[t+\Delta t,T]})\mathrm{d}s + F(T,\mathbf{X}_T)\right\}_{\inf}(\alpha)\\ &\leq L(t,\mathbf{x},\mathbf{u})\Delta t + o(\Delta t) + J\left(t+\Delta t,\mathbf{x}+\Delta\mathbf{X}_t\right)\\ &\leq \tilde{J}(t,\mathbf{x}).\end{aligned}$$

Hence, $J(t,\mathbf{x})\leq\tilde{J}(t,\mathbf{x})$, and then $J(t,\mathbf{x})=\tilde{J}(t,\mathbf{x})$. Theorem 3.1 is proved.

Theorem 3.2 (Equation of optimality) *Let $J(t,\mathbf{x})$ be twice differentiable on $[0,T]\times\Re^n$. Then we have*

$$-J_t = \sup_{u_t\in U}\left\{L + \nabla_{\mathbf{x}}J^{\tau}\mathbf{P} + (2\lambda-1)\left(\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\parallel\nabla_{\mathbf{x}}J^{\tau}\mathbf{Q}\parallel_1 + k\parallel\nabla_{\mathbf{x}}J^{\tau}\mathbf{R}\parallel\right\}, \tag{3.4}$$

where J_t is the partial derivative of the function $J(t,\mathbf{x})$ in t, $\nabla_{\mathbf{x}}J(t,\mathbf{x})$ is the gradient of $J(t,\mathbf{x})$ in $\mathbf{x}$. $\|\cdot\|_1$ and $\|\cdot\|$ are the 1-norm for vectors, that is, $\|\cdot\|_1=\sum_{i=1}^{k}|p_i|$ and $\|\cdot\|=\sum_{i=1}^{k}p_i$ for $p=(p_1,p_2,\ldots,p_k)$, respectively. $L,\mathbf{P},\mathbf{Q},\mathbf{R},J_t,\nabla_{\mathbf{x}}J$ denote $\mathbf{L}(t,\mathbf{x},\mathbf{u}),\mathbf{P}(t,\mathbf{x},\mathbf{u}),\mathbf{Q}(t,\mathbf{x},\mathbf{u}),\mathbf{R}(t,\mathbf{x},\mathbf{u}),J_t(t,\mathbf{x}),\nabla_{\mathbf{x}}J(t,\mathbf{x})$, respectively, and

(1) when $\nabla_{\mathbf{x}}J^{\tau}\mathbf{R}\geq\mathbf{0}$, (i) if $0<\alpha<1-r_2$, then $k=1-\dfrac{\alpha}{2(1-r_2)}$, (ii) if $1-r_2\leq\alpha<1-r_1$, then $k=\dfrac{1}{2}$, (iii) if $1-r_1\leq\alpha<1$, then $k=\dfrac{1-\alpha}{2r_1}$;

(2) when $\nabla_{\mathbf{x}}J^{\tau}\mathbf{R}<\mathbf{0}$, (i) if $0<\alpha\leq r_1$, then $k=\dfrac{\alpha}{2r_1}$, (ii) if $r_1<\alpha\leq r_2$, then $k=\dfrac{1}{2}$, (iii) if $r_2<\alpha<1$, then $k=1-\dfrac{1-\alpha}{2(1-r_2)}$.

Where $\nabla_{\mathbf{x}}J^{\tau}\mathbf{R}\geq\mathbf{0}$ and $\nabla_{\mathbf{x}}J^{\tau}\mathbf{R}<\mathbf{0}$ mean that their all terms are non-negative and negative, respectively.

Proof: Let $H_{\alpha}^{\lambda}[S]=\lambda S_{\sup}(\alpha)+(1-\lambda)S_{\inf}(\alpha)$. By Taylor expansion, we get

$$\begin{aligned}J(t+\Delta t,\mathbf{x}+\Delta\mathbf{X}_t)=&J(t,\mathbf{x})+J_t(t,\mathbf{x})\Delta t+\nabla_{\mathbf{x}}J(t,\mathbf{x})^{\tau}\Delta\mathbf{X}_t+\frac{1}{2}J_{tt}(t,\mathbf{x})\Delta t^2\\&+\frac{1}{2}\Delta\mathbf{X}_t{}^{\tau}\nabla_{\mathbf{xx}}J(t,\mathbf{x})\Delta\mathbf{X_t}+\nabla_{\mathbf{x}}J_t(t,\mathbf{x})^{\tau}\Delta\mathbf{X}_t\Delta t+o(\Delta t),\end{aligned}\quad(3.5)$$

where $\nabla_{\mathbf{xx}}J(t,\mathbf{x})$ is the Hessian matrix of $J(t,\mathbf{x})$ in $\mathbf{x}$. Note that $\Delta\mathbf{X_t}=\mathbf{P}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta t+\mathbf{Q}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta\mathbf{C}_t+\mathbf{R}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta\mathbf{V}_t$. Substituting equation (3.5) into equation (3.2) and simplifying the resulting expression yields that

$$\begin{aligned}0=\sup_{\mathbf{u}_t\in\mathbf{U}}\big\{&L(\mathbf{x},\mathbf{u}_t,t)\Delta t+J_t(t,\mathbf{x})\Delta t+\nabla_{\mathbf{x}}J(t,\mathbf{x})^{\tau}\mathbf{P}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta t+H_{\alpha}^{\lambda}[\mathbf{m}\Delta\mathbf{C}_t\\&+\mathbf{n}\Delta\mathbf{V}_t+\Delta\mathbf{C}_t{}^{\tau}\mathbf{A}\Delta\mathbf{C}_t+\Delta\mathbf{C}_t{}^{\tau}\mathbf{B}\Delta\mathbf{V}_t+\Delta\mathbf{V}_t{}^{\tau}\mathbf{H}\Delta\mathbf{V}_t]+o(\Delta t)\big\},\end{aligned}\quad(3.6)$$

where $\mathbf{m}=\nabla_{\mathbf{x}}J(t,\mathbf{x})^{\tau}\mathbf{Q}(t,\mathbf{X}_t,\mathbf{u}_t)+\nabla_{\mathbf{x}}J_t(t,\mathbf{x})^{\tau}\mathbf{Q}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta t+\mathbf{P}(t,\mathbf{X}_t,\mathbf{u}_t)^{\tau}\nabla_{\mathbf{xx}}J(t,\mathbf{x})\mathbf{Q}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta t$, $\mathbf{n}=\nabla_{\mathbf{x}}J(t,\mathbf{x})^{\tau}\mathbf{R}(t,\mathbf{X}_t,\mathbf{u}_t)+\nabla_{\mathbf{x}}J_t(t,\mathbf{x})^{\tau}\mathbf{R}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta t+\mathbf{P}(t,\mathbf{X}_t,\mathbf{u}_t)^{\tau}\nabla_{\mathbf{xx}}J(t,\mathbf{x})\mathbf{R}(t,\mathbf{X}_t,\mathbf{u}_t)\Delta t$, $\mathbf{A}=\frac{1}{2}\mathbf{Q}(t,\mathbf{X}_t,\mathbf{u}_t)^{\tau}\nabla_{\mathbf{xx}}J(t,\mathbf{x})\mathbf{Q}(t,\mathbf{X}_t,\mathbf{u}_t)$, $\mathbf{B}=\mathbf{Q}(t,\mathbf{X}_t,\mathbf{u}_t)^{\tau}\nabla_{\mathbf{xx}}J(t,\mathbf{x})\mathbf{R}(t,\mathbf{X}_t,\mathbf{u}_t)$, $\mathbf{H}=\frac{1}{2}\mathbf{R}(t,\mathbf{X}_t,\mathbf{u}_t)^{\tau}\nabla_{\mathbf{xx}}J(t,\mathbf{x})\mathbf{R}(t,\mathbf{X}_t,\mathbf{u}_t)$.

Let $\mathbf{m}=(m_i)_{1\times k},\mathbf{n}=(n_i)_{1\times k},\mathbf{A}=(a_{ij})_{k\times k},\mathbf{B}=(b_{ij})_{k\times k}$ and $\mathbf{H}=(h_{ij})_{k\times k}$. Then we have

$$\begin{aligned}&\mathbf{m}\Delta\mathbf{C}_t+\mathbf{n}\Delta\mathbf{V}_t+\Delta\mathbf{C}_t{}^{\tau}\mathbf{A}\Delta\mathbf{C}_t+\Delta\mathbf{C}_t{}^{\tau}\mathbf{B}\Delta\mathbf{V}_t+\Delta\mathbf{V}_t{}^{\tau}\mathbf{H}\Delta\mathbf{V}_t\\&=\sum_{i=1}^{k}(m_i\Delta C_{ti}+n_i\Delta V_{ti})+\sum_{i=1}^{k}\sum_{j=1}^{k}(a_{ij}\Delta C_{ti}\Delta C_{tj}+b_{ij}\Delta C_{ti}\Delta V_{tj}+h_{ij}\Delta V_{ti}\Delta V_{tj}).\end{aligned}$$

Since $|a_{ij}\Delta C_{ti}\Delta C_{tj}| \leq |a_{ij}|({\Delta C_{ti}}^2 + {\Delta C_{tj}}^2)/2$, $|b_{ij}\Delta C_{ti}\Delta V_{tj}| \leq |b_{ij}|({\Delta C_{ti}}^2 + {\Delta V_{tj}}^2)/2$, $|h_{ij}\Delta V_{ti}\Delta V_{tj}| \leq |h_{ij}|({\Delta V_{ti}}^2 + {\Delta V_{tj}}^2)/2$, we have

$$\begin{aligned}
&\sum_{i=1}^{k}\left(m_i\Delta C_{ti} - e_i{\Delta C_{ti}}^2 + n_i\Delta V_{ti} - e_i{\Delta V_{ti}}^2\right)\\
&\leq \mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + {\Delta\mathbf{C}_t}^{\tau}\mathbf{A}\Delta\mathbf{C}_t + {\Delta\mathbf{C}_t}^{\tau}\mathbf{B}\Delta\mathbf{V}_t + {\Delta\mathbf{V}_t}^{\tau}\mathbf{H}\Delta\mathbf{V}_t\\
&\leq \sum_{i=1}^{k}\left(m_i\Delta C_{ti} + e_i{\Delta C_{ti}}^2 + n_i\Delta V_{ti} + e_i{\Delta V_{ti}}^2\right),
\end{aligned}$$

where $e_i = \sum_{j=1}^{k}(|a_{ij}| + |b_{ij}|/2)\,(> 0)$. It follows from the independence of C_{ti} and $V_{tj}(i, j = 1, 2, \cdots, k)$ that

$$\begin{aligned}
&\sum_{i=1}^{k}H_\alpha^\lambda\left(m_i\Delta C_{ti} - e_i{\Delta C_{ti}}^2 + n_i\Delta V_{ti} - e_i{\Delta V_{ti}}^2\right)\\
&\leq H_\alpha^\lambda\left(\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + {\Delta\mathbf{C}_t}^{\tau}\mathbf{A}\Delta\mathbf{C}_t + {\Delta\mathbf{C}_t}^{\tau}\mathbf{B}\Delta\mathbf{V}_t + {\Delta\mathbf{V}_t}^{\tau}\mathbf{H}\Delta\mathbf{V}_t\right)\\
&\leq \sum_{i=1}^{k}H_\alpha^\lambda\left(m_i\Delta C_{ti} + e_i{\Delta C_{ti}}^2 + n_i\Delta V_{ti} + e_i{\Delta V_{ti}}^2\right).
\end{aligned}$$

According to Theorem 2.2 and Theorem 5.1 of Appendix A in 28, for any $\varepsilon > 0$ small enough, we have

$$\begin{aligned}
H_\alpha^\lambda\left(m_i\Delta C_{ti} - e_i{\Delta C_{ti}}^2\right) \geq\ & \frac{\lambda\sqrt{3}|m_i|\Delta t}{\pi}\ln\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon} + \frac{(\lambda-1)\sqrt{3}|m_i|\Delta t}{\pi}\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}\\
&-e_i\left(\frac{\sqrt{3}\Delta t}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
H_\alpha^\lambda\left(m_i\Delta C_{ti} + e_i{\Delta C_{ti}}^2\right) \leq\ & \frac{\lambda\sqrt{3}|m_i|\Delta t}{\pi}\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon} + \frac{(\lambda-1)\sqrt{3}|m_i|\Delta t}{\pi}\ln\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon}\\
&+e_i\left(\frac{\sqrt{3}\Delta t}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2.
\end{aligned}$$

It follows from Theorem 2.3 and Theorem 5.1 of Appendix that

(1) if $n_i \geq 0$ $(i = 1, 2, \cdots, k)$ and $r_1 + r_2 < 1$, then

$$H_\alpha^\lambda\left(n_i \Delta V_{ti} - e_i \Delta V_{ti}{}^2\right) \geq \begin{cases} \lambda n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t + (1-\lambda)\, n_i \dfrac{\alpha}{2r_1} \Delta t - e_i \left(\Delta t\right)^2, & \text{if } 0 < \alpha < r_1 \\ \dfrac{n_i}{2} \Delta t - e_i \left(\Delta t\right)^2, & \text{if } r_1 \leq \alpha < r_2 \\ \lambda \dfrac{n_i\left(1-\alpha\right)}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{1-\alpha}{2\left(1-r_2\right)}\right) \Delta t - e_i \left(\Delta t\right)^2, & \text{if } r_2 \leq \alpha < 1 \end{cases}$$

and

$$H_\alpha^\lambda\left(n_i \Delta V_{ti} + e_i \Delta V_{ti}{}^2\right) \leq \begin{cases} \lambda n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t + (1-\lambda)\, n_i \dfrac{\alpha}{2r_1} \Delta t + e_i \left(\Delta t\right)^2, & \text{if } 0 < \alpha < r_1 \\ \dfrac{n_i}{2} \Delta t + e_i \left(\Delta t\right)^2, & \text{if } r_1 \leq \alpha < r_2 \\ \lambda \dfrac{n_i\left(1-\alpha\right)}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{1-\alpha}{2\left(1-r_2\right)}\right) \Delta t + e_i \left(\Delta t\right)^2, & \text{if } r_2 \leq \alpha < 1. \end{cases}$$

(2) if $n_i \geq 0$ $(i = 1, 2, \cdots, k)$ and $r_1 + r_2 \geq 1$, then

$$H_\alpha^\lambda\left(n_i \Delta V_{ti} - e_i \Delta V_{ti}{}^2\right) \geq \begin{cases} \lambda n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t + (1-\lambda)\, n_i \dfrac{\alpha}{2r_1} \Delta t - e_i \left(\Delta t\right)^2, & \text{if } 0 < \alpha < 1 - r_2 \\ \dfrac{n_i}{2} \Delta t - e_i \left(\Delta t\right)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \lambda \dfrac{n_i\left(1-\alpha\right)}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{1-\alpha}{2\left(1-r_2\right)}\right) \Delta t - e_i \left(\Delta t\right)^2, & \text{if } 1 - r_1 \leq \alpha < 1. \end{cases}$$

and

$$H_\alpha^\lambda\left(n_i \Delta V_{ti} + e_i \Delta V_{ti}{}^2\right) \leq \begin{cases} \lambda n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t + (1-\lambda)\, n_i \dfrac{\alpha}{2r_1} \Delta t + e_i \left(\Delta t\right)^2, & \text{if } 0 < \alpha < 1 - r_2 \\ \dfrac{n_i}{2} \Delta t + e_i \left(\Delta t\right)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \lambda \dfrac{n_i\left(1-\alpha\right)}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{1-\alpha}{2\left(1-r_2\right)}\right) \Delta t + e_i \left(\Delta t\right)^2, & \text{if } 1 - r_1 \leq \alpha < 1. \end{cases}$$

(3) if $n_i < 0$ $(i = 1, 2, \cdots, k)$ and $r_1 + r_2 < 1$, then

$$
\begin{aligned}
&H_\alpha^\lambda\left(n_i \Delta V_{ti} - e_i \Delta {V_{ti}}^2\right) \\
&\geq \begin{cases}
\lambda n_i \dfrac{\alpha}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t - e_i\,(\Delta t)^2, & \text{if } 0 < \alpha < r_1 \\
\dfrac{n_i}{2} \Delta t - e_i\,(\Delta t)^2, & \text{if } r_1 \leq \alpha < r_2 \\
\lambda n_i \left(1 - \dfrac{1-\alpha}{2\,(1-r_2)}\right) \Delta t + (1-\lambda)\, \dfrac{n_i\,(1-\alpha)}{2r_1} \Delta t - e_i\,(\Delta t)^2, & \text{if } r_2 \leq \alpha < 1.
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&H_\alpha^\lambda\left(n_i \Delta V_{ti} + e_i \Delta {V_{ti}}^2\right) \\
&\leq \begin{cases}
\lambda n_i \dfrac{\alpha}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t + e_i\,(\Delta t)^2, & \text{if } 0 < \alpha < r_1 \\
\dfrac{n_i}{2} \Delta t + e_i\,(\Delta t)^2, & \text{if } r_1 \leq \alpha < r_2 \\
\lambda n_i \left(1 - \dfrac{1-\alpha}{2\,(1-r_2)}\right) \Delta t + (1-\lambda)\, \dfrac{n_i\,(1-\alpha)}{2r_1} \Delta t + e_i\,(\Delta t)^2, & \text{if } r_2 \leq \alpha < 1.
\end{cases}
\end{aligned}
$$

(4) if $n_i < 0$ $(i = 1, 2, \cdots, k)$ and $r_1 + r_2 \geq 1$, then

$$
\begin{aligned}
&H_\alpha^\lambda\left(n_i \Delta V_{ti} - e_i \Delta {V_{ti}}^2\right) \\
&\geq \begin{cases}
\lambda n_i \dfrac{\alpha}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t - e_i\,(\Delta t)^2, & \text{if } 0 < \alpha < 1 - r_2 \\
\dfrac{n_i}{2} \Delta t - e_i\,(\Delta t)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\
\lambda n_i \left(1 - \dfrac{1-\alpha}{2\,(1-r_2)}\right) \Delta t + (1-\lambda)\, \dfrac{n_i\,(1-\alpha)}{2r_1} \Delta t - e_i\,(\Delta t)^2, & \text{if } 1 - r_1 \leq \alpha < 1.
\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&H_\alpha^\lambda\left(n_i \Delta V_{ti} + e_i \Delta {V_{ti}}^2\right) \\
&\leq \begin{cases}
\lambda n_i \dfrac{\alpha}{2r_1} \Delta t + (1-\lambda)\, n_i \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) \Delta t + e_i\,(\Delta t)^2, & \text{if } 0 < \alpha < 1 - r_2 \\
\dfrac{n_i}{2} \Delta t + e_i\,(\Delta t)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\
\lambda n_i \left(1 - \dfrac{1-\alpha}{2\,(1-r_2)}\right) \Delta t + (1-\lambda)\, \dfrac{n_i\,(1-\alpha)}{2r_1} \Delta t + e_i\,(\Delta t)^2, & \text{if } 1 - r_1 \leq \alpha < 1.
\end{cases}
\end{aligned}
$$

Therefore, (1) if $\mathbf{n} \geq \mathbf{0}$, where $n_i \geq 0\,(i = 1, 2, \cdots, k)$ and $r_1 + r_2 < 1$, then we get

$$H_\alpha^\lambda \left(\mathbf{m}\Delta\mathbf{C_t} + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)$$
$$\geq \begin{cases} \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \dfrac{\alpha}{2(1-r_2)}\right) + (1-\lambda)\dfrac{\alpha}{2r_1}\right]\displaystyle\sum_{i=1}^k n_i\Delta t \\ -g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < r_1 \\ \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \dfrac{1}{2}\displaystyle\sum_{i=1}^k n_i\Delta t - g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_1 \leq \alpha < r_2 \\ \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\dfrac{\lambda(1-\alpha)}{2r_1} + (1-\lambda)\left(1 - \dfrac{1-\alpha}{2(1-r_2)}\right)\right]\displaystyle\sum_{i=1}^k n_i\Delta t \\ -g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_2 \leq \alpha < 1, \end{cases} \quad (3.7)$$

and

$$H_\alpha^\lambda \left(\mathbf{m}\Delta\mathbf{C_t} + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)$$
$$\leq \begin{cases} \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \dfrac{\alpha}{2(1-r_2)}\right) + (1-\lambda)\dfrac{\alpha}{2r_1}\right]\displaystyle\sum_{i=1}^k n_i\Delta t \\ +g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < r_1 \\ \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \dfrac{1}{2}\displaystyle\sum_{i=1}^k n_i\Delta t + g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_1 \leq \alpha < r_2 \\ \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\dfrac{\lambda(1-\alpha)}{2r_1} + (1-\lambda)\left(1 - \dfrac{1-\alpha}{2(1-r_2)}\right)\right]\displaystyle\sum_{i=1}^k n_i\Delta t \\ +g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_2 \leq \alpha < 1, \end{cases} \quad (3.8)$$

where $h_1(\lambda,\alpha,\varepsilon) = \lambda\ln\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon} + (\lambda-1)\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon}$, $h_2(\lambda,\alpha,\varepsilon) = \lambda\ln\frac{1-\alpha+\varepsilon}{\alpha-\varepsilon} + (\lambda-1)\ln\frac{1-\alpha-\varepsilon}{\alpha+\varepsilon}$ and $g(\varepsilon) = \left(\frac{\sqrt{3}}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2 + 1$.

(2) if $\mathbf{n} \geq \mathbf{0}$, where $n_i \geq 0\,(i = 1, 2, \cdots, k)$ and $r_1 + r_2 \geq 1$, then we get

$$
\begin{aligned}
&H_\alpha^\lambda\left(\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)\\
\geq &\begin{cases}
\dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \dfrac{\alpha}{2(1-r_2)}\right) + (1-\lambda)\dfrac{\alpha}{2r_1}\right]\sum_{i=1}^k n_i\Delta t & \\
-g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < 1 - r_2\\
\dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \dfrac{1}{2}\sum_{i=1}^k n_i\Delta t & \\
-g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1\\
\dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\dfrac{\lambda(1-\alpha)}{2r_1} + (1-\lambda)\left(1 - \dfrac{1-\alpha}{2(1-r_2)}\right)\right]\sum_{i=1}^k n_i\Delta t & \\
-g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_1 \leq \alpha < 1,
\end{cases}
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
&H_\alpha^\lambda\left(\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)\\
\leq &\begin{cases}
\dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \dfrac{\alpha}{2(1-r_2)}\right) + (1-\lambda)\dfrac{\alpha}{2r_1}\right]\sum_{i=1}^k n_i\Delta t & \\
+g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < 1 - r_2\\
\dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \dfrac{1}{2}\sum_{i=1}^k n_i\Delta t & \\
+g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1\\
\dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\dfrac{\lambda(1-\alpha)}{2r_1} + (1-\lambda)\left(1 - \dfrac{1-\alpha}{2(1-r_2)}\right)\right]\sum_{i=1}^k n_i\Delta t & \\
+g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_1 \leq \alpha < 1,
\end{cases}
\end{aligned}
\tag{3.10}
$$

(3) if $\mathbf{n} < \mathbf{0}$, where $n_i < 0\,(i = 1, 2, \cdots, k)$ and $r_1 + r_2 < 1$, then we have

$$H_\alpha^\lambda\left(\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)$$
$$\geq \begin{cases} \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\frac{\alpha}{2r_1} + (1-\lambda)\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right]\sum_{i=1}^k n_i\Delta t \\ -g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < r_1 \\ \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \frac{1}{2}\sum_{i=1}^k n_i\Delta t - g(\varepsilon)\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_1 \leq \alpha < r_2 \\ \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \frac{1-\alpha}{2(1-r_2)}\right) + \frac{(1-\lambda)(1-\alpha)}{2r_1}\right]\sum_{i=1}^k n_i\Delta t \\ -g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_2 \leq \alpha < 1, \end{cases} \quad (3.11)$$

and

$$H_\alpha^\lambda\left(\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)$$
$$\leq \begin{cases} \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\frac{\alpha}{2r_1} + (1-\lambda)\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right]\sum_{i=1}^k n_i\Delta t \\ +g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < r_1 \\ \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \frac{1}{2}\sum_{i=1}^k n_i\Delta t + g(\varepsilon)\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_1 \leq \alpha < r_2 \\ \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \frac{1-\alpha}{2(1-r_2)}\right) + \frac{(1-\lambda)(1-\alpha)}{2r_1}\right]\sum_{i=1}^k n_i\Delta t \\ +g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } r_2 \leq \alpha < 1, \end{cases} \quad (3.12)$$

(4) if $\mathbf{n} < \mathbf{0}$, where $n_i < 0\,(i = 1, 2, \cdots, k)$ and $r_1 + r_2 \geq 1$, then we have

$$H_\alpha^\lambda \left(\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)$$
$$\geq \begin{cases} \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\frac{\alpha}{2r_1} + (1-\lambda)\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right]\sum_{i=1}^k n_i\Delta t \\ -g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < 1 - r_2 \\ \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \frac{1}{2}\sum_{i=1}^k n_i\Delta t \\ -g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \dfrac{\sqrt{3}h_1(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \frac{1-\alpha}{2(1-r_2)}\right) + \frac{(1-\lambda)(1-\alpha)}{2r_1}\right]\sum_{i=1}^k n_i\Delta t \\ -g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_1 \leq \alpha < 1, \end{cases} \tag{3.13}$$

and

$$H_\alpha^\lambda \left(\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^\tau\mathbf{B}\Delta\mathbf{V}_t + \Delta\mathbf{V}_t{}^\tau\mathbf{H}\Delta\mathbf{V}_t\right)$$
$$\leq \begin{cases} \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\frac{\alpha}{2r_1} + (1-\lambda)\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right]\sum_{i=1}^k n_i\Delta t \\ +g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 0 < \alpha < 1 - r_2 \\ \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \frac{1}{2}\sum_{i=1}^k n_i\Delta t \\ +g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \dfrac{\sqrt{3}h_2(\lambda,\alpha,\varepsilon)}{\pi}\displaystyle\sum_{i=1}^k |m_i|\Delta t + \left[\lambda\left(1 - \frac{1-\alpha}{2(1-r_2)}\right) + \frac{(1-\lambda)(1-\alpha)}{2r_1}\right]\sum_{i=1}^k n_i\Delta t \\ +g(\varepsilon)\displaystyle\sum_{i=1}^k e_i(\Delta t)^2, & \text{if } 1 - r_1 \leq \alpha < 1. \end{cases} \tag{3.14}$$

Then by Equation (3.6), we know that

(1) If $\mathbf{n} \geq \mathbf{0}$ and $r_1 + r_2 < 1$, applying Inequality (3.8), for $\Delta t > 0$, if $0 < \alpha < r_1$, there exists a control $u \equiv u_{\varepsilon,\Delta t}$ such that

$$\begin{aligned}
-\varepsilon\Delta t \leq\, & L\Delta t + J_t\Delta t + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P}\Delta t + H_{\alpha}^{\lambda}\left[\mathbf{m}\Delta\mathbf{C}_t + \mathbf{n}\Delta\mathbf{V}_t + \Delta\mathbf{C}_t{}^{\tau}\mathbf{A}\Delta\mathbf{C}_t + \Delta\mathbf{C}_t{}^{\tau}\mathbf{B}\Delta\mathbf{V}_t\right. \\
& \left. + \Delta\mathbf{V}_t{}^{\tau}\mathbf{H}\Delta\mathbf{V}_t\right] + o(\Delta t) \\
\leq\, & L\Delta t + J_t\Delta t + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P}\Delta t + \frac{\sqrt{3}h_2\left(\lambda,\alpha,\varepsilon\right)}{\pi}\sum_{i=1}^{k}|m_i|\Delta t + \left[\lambda\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right. \\
& \left. + (1-\lambda)\frac{\alpha}{2r_1}\right]\sum_{i=1}^{k} n_i\Delta t + g\left(\varepsilon\right)\sum_{i=1}^{k} e_i\left(\Delta t\right)^2 + o(\Delta t).
\end{aligned} \tag{3.15}$$

Dividing both sides of the inequality (3.15) by Δt, and taking the supremum with respect to $\mathbf{u}$, we derive

$$\begin{aligned}
-\varepsilon \leq\, & J_t + \sup_{u_t\in U}\left\{L + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P} + \frac{\sqrt{3}h_2\left(\lambda,\alpha,\varepsilon\right)}{\pi}\parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{Q}\parallel_1 + \left[\lambda\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right.\right. \\
& \left.\left. + (1-\lambda)\frac{\alpha}{2r_1}\right]\parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{R}\parallel\right\} + l_1\left(\varepsilon,\Delta t\right) + l_2(\Delta t)
\end{aligned} \tag{3.16}$$

since $\sum_{i=1}^{k} n_i \to \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{R}\parallel$, $\sum_{i=1}^{k}|m_i| \to \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{Q}\parallel_1$ as $\Delta t \to 0$, where $l_1(\varepsilon,\Delta t) = g\left(\varepsilon\right)\sum_{i=1}^{k} e_i = \left[\left(\frac{\sqrt{3}}{\pi}\ln\frac{2-\varepsilon}{\varepsilon}\right)^2 + 1\right]\sum_{i=1}^{k}\sum_{j=1}^{k}\left(|a_{ij}| + \frac{|b_{ij}|}{2}\right)\Delta t \to 0, l_2(\Delta t) = \frac{o(\Delta t)}{\Delta t} \to 0$ as $\Delta t \to 0$. Letting $\Delta t \to 0$ and then $\varepsilon \to 0$ results in

$$\begin{aligned}
0 \leq\, & J_t + \sup_{u_t\in U}\left\{L + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P} + (2\lambda-1)\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{Q}\parallel_1 + \left[\lambda\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right.\right. \\
& \left.\left. + (1-\lambda)\frac{\alpha}{2r_1}\right]\parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{R}\parallel\right\}
\end{aligned} \tag{3.17}$$

if $\nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \geq \mathbf{0}$ and $0 < \alpha < r_1$.

On the other hand, by Equation (3.6) and Inequality (3.7), applying the similar process, we can get

$$\begin{aligned}
0 \geq\, & J_t + \sup_{u_t\in U}\left\{L + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P} + (2\lambda-1)\frac{\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{Q}\parallel_1 + \left[\lambda\left(1 - \frac{\alpha}{2(1-r_2)}\right)\right.\right. \\
& \left.\left. + (1-\lambda)\frac{\alpha}{2r_1}\right]\parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{R}\parallel\right\}
\end{aligned} \tag{3.18}$$

if $\nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \geq \mathbf{0}$ and $0 < \alpha < r_1$.

Combining Inequalities (3.17) and (3.18), we can obtain

$$-J_t = \sup_{u_t \in U} \left\{ L + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P} + (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{Q} \parallel_1 + \left[\lambda \left(1 - \frac{\alpha}{2(1-r_2)} \right) \right.\right.$$
$$\left.\left. + (1-\lambda)\frac{\alpha}{2r_1} \right] \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \parallel \right\} \tag{3.19}$$

if $\nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \geq \mathbf{0}$ and $0 < \alpha < r_1$.

By applying the similar process, we can derive

$$-J_t = \sup_{u_t \in U} \left\{ L + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P} + (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{Q} \parallel_1 + \frac{1}{2} \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \parallel \right\} \tag{3.20}$$

if $\nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \geq \mathbf{0}$ and $r_1 \leq \alpha < r_2$, and

$$-J_t = \sup_{u_t \in U} \left\{ L + \nabla_{\mathbf{x}} J^{\tau}\mathbf{P} + (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{Q} \parallel_1 + \left[\frac{\lambda(1-\alpha)}{2r_1} \right.\right.$$
$$\left.\left. + (1-\lambda)\left(1 - \frac{1-\alpha}{2(1-r_2)} \right) \right] \parallel \nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \parallel \right\} \tag{3.21}$$

if $\nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \geq \mathbf{0}$ and $r_2 \leq \alpha < 1$. Thus we derive the equation of optimality for $\nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \geq \mathbf{0}$ and $r_1 + r_2 < 1$.

(2) Similarly, results can be obtained for $\nabla_{\mathbf{x}} J^{\tau}\mathbf{R} \geq \mathbf{0}$ and $r_1 + r_2 \geq 1$ or $\mathbf{n} < \mathbf{0}$.

(3) If condition $\mathbf{n} \geq \mathbf{0}$ or $\mathbf{n} < \mathbf{0}$ are not been satisfied, namely $n_i > 0, n_j < 0 (i \neq j, i, j = 1, 2, \cdots, k)$, by using the similar process, the equation of optimality can also be derived. Thus the theorem 3.2 is proved.

4 Example

In this section, an example is presented to illustrate the effectiveness of results obtained. Consider the following optimal control problem for an uncertain system with jump under Hurwicz criterion:

$$\begin{cases} J(0, \mathbf{x_0}) \equiv \sup\limits_{\mathbf{u}_s \in \mathbf{U}} \left[\lambda S_{\sup}(\alpha) + (1-\lambda) S_{\inf}(\alpha) \right], \\ \text{subject to} \\ \quad dx_1(t) = u_1(t)x_1(t)dt + \sigma_1 u_1(t)x_1(t)dC_1 + \delta_1 u_1(t)x_1(t)dV_1, \\ \quad dx_2(t) = u_2(t)x_2(t)dt + \sigma_2 u_2(t)x_1(t)dC_2 + \delta_2 u_2(t)x_2(t)dV_2, \end{cases}$$

where $S = \int_0^\infty \frac{1}{2} e^{-\gamma t} \left[u_1^2(t)x_1^2(t) + u_2^2(t)x_2^2(t)\right] dt$ and γ denotes the constant discount rate. It follows from the equation of optimality (3.2) that

$$-J_t = \sup_u \left\{ \frac{1}{2} e^{-\gamma t}(u_1^2 x_1^2 + u_2^2 x_2^2) + u_1 x_1 J_{x_1} + u_2 x_2 J_{x_2} + (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \left(|\sigma_1 u_1 x_1 J_{x_1}| + |\sigma_2 u_2 x_2 J_{x_2}|\right) + k\left(\delta_1 u_1 x_1 J_{x_1} + \delta_2 u_2 x_2 J_{x_2}\right) \right\} = \max_u L\left(u_1, u_2\right), \tag{4.1}$$

where $L(u_1, u_2)$ represents the term in the braces. Next we solve the Eq.(4.1). Without loss of generality, assuming that $\sigma_i, u_i, x_i > 0$ $(i = 1, 2)$, then

(1) if $J_{x1} \geq 0, J_{x2} \geq 0$, the optimal control u_1, u_2 satisfies

$$\frac{\partial L(u_1, u_2)}{\partial u_i} = e^{-\gamma t} u_i x_i^2 + x_i J_{x_i} + (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \sigma_i x_i J_{x_i} + k\delta_i x_i J_{x_i} = 0 \quad (i = 1, 2). \tag{4.2}$$

or

$$u_i = \frac{\tilde{k}_i e^{\gamma t}}{x_i} J_{x_i}, \tag{4.3}$$

where $\tilde{k}_i = -1 - (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} \sigma_i - k\delta_i$.

Substituting (4.3) into (4.1), we have

$$\begin{aligned} -2e^{\gamma t} J_t = &\left(\tilde{k}_1^2 + 2\tilde{k}_1 + 2\sigma_1 \tilde{k}_1 (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} + 2k\tilde{k}_1 \delta_1\right)\left(e^{\gamma t} J_{x_1}\right)^2 \\ &+ \left(\tilde{k}_2^2 + 2\tilde{k}_2 + 2\sigma_2 \tilde{k}_2 (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} + 2k\tilde{k}_2 \delta_2\right)\left(e^{\gamma t} J_{x_2}\right)^2. \end{aligned} \tag{4.4}$$

We conjecture that $J(t, x) = \frac{1}{2} e^{-\gamma t}(a x_1^2 + b x_2^2)$. Then $J_{x_1} = a e^{-\gamma t} x_1$, $J_{x_2} = b e^{-\gamma t} x_2$ and $J_t = -\frac{1}{2}\gamma e^{-\gamma t}(a x_1^2 + b x_2^2)$. Substituting them into (4.4) yields

$$\begin{aligned} \gamma(a x_1^2 + b x_2^2) = &\left(\tilde{k}_1^2 + 2\tilde{k}_1 + 2\sigma_1 \tilde{k}_1 (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} + 2k\tilde{k}_1 \delta_1\right)\left(a x_1\right)^2 \\ &+ \left(\tilde{k}_2^2 + 2\tilde{k}_2 + 2\sigma_2 \tilde{k}_2 (2\lambda - 1)\frac{\sqrt{3}}{\pi} \ln \frac{1-\alpha}{\alpha} + 2k\tilde{k}_2 \delta_2\right)\left(b x_2\right)^2. \end{aligned} \tag{4.5}$$

We find then that:

$$\begin{aligned} a &= \frac{\gamma}{\tilde{k}_1^2 + 2\tilde{k}_1 + 2\sigma_1\tilde{k}_1(2\lambda - 1)\dfrac{\sqrt{3}}{\pi}\ln\dfrac{1-\alpha}{\alpha} + 2k\tilde{k}_1\delta_1}, \\ b &= \frac{\gamma}{\tilde{k}_2^2 + 2\tilde{k}_2 + 2\sigma_2\tilde{k}_2(2\lambda - 1)\dfrac{\sqrt{3}}{\pi}\ln\dfrac{1-\alpha}{\alpha} + 2k\tilde{k}_2\delta_2}. \end{aligned} \tag{4.6}$$

Therefore, the optimal control are determined, respectively, by

$$\begin{aligned} u_1 &= \frac{\gamma\tilde{k}_1}{\tilde{k}_1^2 + 2\tilde{k}_1 + 2\sigma_1\tilde{k}_1(2\lambda - 1)\dfrac{\sqrt{3}}{\pi}\ln\dfrac{1-\alpha}{\alpha} + 2k\tilde{k}_1\delta_1}, \\ u_2 &= \frac{\gamma\tilde{k}_2}{\tilde{k}_2^2 + 2\tilde{k}_2 + 2\sigma_2\tilde{k}_2(2\lambda - 1)\dfrac{\sqrt{3}}{\pi}\ln\dfrac{1-\alpha}{\alpha} + 2k\tilde{k}_2\delta_2}, \end{aligned} \tag{4.7}$$

where $\tilde{k}_i = -1 - (2\lambda - 1)\dfrac{\sqrt{3}}{\pi}\ln\dfrac{1-\alpha}{\alpha}\sigma_i - k\delta_i$ $(i = 1, 2)$. And it follows from theorem (3.2) that (i) if $0 < \alpha < 1 - r_2$, then $k = 1 - \dfrac{\alpha}{2\,(1 - r_2)}$; (ii) if $1 - r_2 \leq \alpha < 1 - r_1$, then $k = \dfrac{1}{2}$; (iii) if $1 - r_1 \leq \alpha < 1$, then $k = \dfrac{1-\alpha}{2r_1}$.

(2) If $J_{x_1} < 0, J_{x_2} < 0$, then applying the similar method to the above process, we can get result (4.7), where $\tilde{k}_i = -1 + (2\lambda - 1)\dfrac{\sqrt{3}}{\pi}\ln\dfrac{1-\alpha}{\alpha}\sigma_i - k\delta_i$ $(i = 1, 2)$. And similarly, (i) if $0 < \alpha \leq r_1$, then $k = \dfrac{\alpha}{2r_1}$; (ii) if $r_1 < \alpha \leq r_2$, then $k = \dfrac{1}{2}$; (iii) if $r_2 < \alpha < 1$, then $k = 1 - \dfrac{1-\alpha}{2(1 - r_2)}$.

5 Conclusion

This paper explored an uncertain optimal control problem subject to an uncertain dynamic system with jump under Hurwicz criterion. In order to dealt with this type of optimal control problem, the principle of optimality for proposed model was presented based on Bellman's principle of optimality in dynamic programming and some results in uncertainty theory. And then a fundamental result of equation of optimality as a counterpart of HJB equation was obtained. The essential difference between uncertain optimal control and stochastic optimal control is that the former is concerned with the study of dynamic uncertain phenomena while the latter is about the study of dynamic stochastic phenomena. Our model is suitable for situations where there is no sample data or the sample data is too small or cannot afford numerous experiments to obtain statistical data due to economic reasons or technical difficulties. In future research, the authors are intending to continue investigating uncertain linear quadratic optimal control for uncertain systems with jump under Hurwicz criterion and derive the necessary and sufficient condition for the existence of optimal control.

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Appendix

In this section, we give an estimation for the α-pessimistic value of $a\eta + b\eta^2$ if η is a V-jump uncertain variable ($\alpha \in (0, 1)$).

Theorem 5.1 *Let V_t a V jump uncertain process. Denote $\eta = \Delta V_t$, then for any real number a, b, any $\alpha \in (0,1)$, and any $\varepsilon > 0$ small enough,*

(1) if $a \geq 0, b \geq 0$, then

$$[a\eta + b\eta^2]_{\inf}(\alpha) \geq \begin{cases} \dfrac{\alpha a}{2r_1}\Delta t, & \text{if } 0 < \alpha \leq r_1 \\ \dfrac{a}{2}\Delta t, & \text{if } r_1 < \alpha \leq r_2 \\ \left(1 - \dfrac{1-\alpha}{2(1-r_2)}\right) a\Delta t, & \text{if } r_2 < \alpha < 1. \end{cases} \tag{5.1}$$

$$\left[a\eta+b\eta^2\right]_{\inf}(\alpha)\leq\begin{cases}\dfrac{\alpha a}{2r_1}\Delta t+b\Delta t^2, & \textit{if } 0<\alpha\leq r_1\\ \dfrac{a}{2}\Delta t+b\Delta t^2, & \textit{if } r_1<\alpha\leq r_2\\ \left(1-\dfrac{1-\alpha}{2(1-r_2)}\right)a\Delta t+b\Delta t^2, & \textit{if } r_2<\alpha<1.\end{cases} \tag{5.2}$$

(2) if $a\geq 0, b<0,$ *then*

$$\left[a\eta+b\eta^2\right]_{\inf}(\alpha)\geq\begin{cases}\dfrac{a\alpha}{2r_1}\Delta t+b\Delta t^2, & \textit{if } 0<\alpha\leq r_1\\ \dfrac{a}{2}\Delta t+b\Delta t^2, & \textit{if } r_1<\alpha\leq r_2\\ \left(1-\dfrac{1-\alpha}{2(1-r_2)}\right)a\Delta t+b\Delta t^2, & \textit{if } r_2<\alpha<1.\end{cases} \tag{5.3}$$

$$[a\eta+b\eta^2]_{\inf}(\alpha)\leq\begin{cases}\dfrac{\alpha a}{2r_1}\Delta t, & \textit{if } 0<\alpha\leq r_1\\ \dfrac{a}{2}\Delta t, & \textit{if } r_1<\alpha\leq r_2\\ \left(1-\dfrac{1-\alpha}{2(1-r_2)}\right)a\Delta t, & \textit{if } r_2<\alpha<1.\end{cases} \tag{5.4}$$

(3) if $a<0, b\geq 0,$ *then*

$$[a\eta+b\eta^2]_{\inf}(\alpha)\geq\begin{cases}\left(1-\dfrac{\alpha}{2(1-r_2)}\right)a\Delta t, & \textit{if } 0<\alpha<1-r_2\\ \dfrac{a}{2}\Delta t, & \textit{if } 1-r_2\leq\alpha<1-r_1\\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t, & \textit{if } 1-r_1\leq\alpha<1.\end{cases} \tag{5.5}$$

$$\left[a\eta+b\eta^2\right]_{\inf}(\alpha)\leq\begin{cases}\left(1-\dfrac{\alpha}{2(1-r_2)}\right)a\Delta t+b\Delta t^2, & \textit{if } 0<\alpha<1-r_2\\ \dfrac{a}{2}\Delta t+b\Delta t^2, & \textit{if } 1-r_2\leq\alpha<1-r_1\\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t+b\Delta t^2, & \textit{if } 1-r_1\leq\alpha<1.\end{cases} \tag{5.6}$$

(4) if $a < 0, b < 0$, then

$$\left[a\eta + b\eta^2\right]_{\inf}(\alpha) \geq \begin{cases} \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) a\Delta t + b\Delta t^2, & \text{if } 0 < \alpha < 1 - r_2 \\ \dfrac{a}{2}\Delta t + b\Delta t^2, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t + b\Delta t^2, & \text{if } 1 - r_1 \leq \alpha < 1. \end{cases} \tag{5.7}$$

$$[a\eta + b\eta^2]_{\inf}(\alpha) \leq \begin{cases} \left(1 - \dfrac{\alpha}{2(1-r_2)}\right) a\Delta t, & \text{if } 0 < \alpha < 1 - r_2 \\ \dfrac{a}{2}\Delta t, & \text{if } 1 - r_2 \leq \alpha < 1 - r_1 \\ \dfrac{(1-\alpha)\,a}{2r_1}\Delta t, & \text{if } 1 - r_1 \leq \alpha < 1. \end{cases} \tag{5.8}$$

Proof: According to the second proposition from Theorem 2.1, if $c < 0$, then $(c\xi)_{\sup}(\alpha) = c\xi_{\inf}(\alpha)$ and $(c\xi)_{\inf}(\alpha) = c\xi_{\sup}(\alpha)$. We have

$$[a\eta + b\eta^2]_{\inf}(\alpha) = -[-a\eta - b\eta^2]_{\sup}(\alpha).$$

And then, via applying Theorem 2.3, the conclusions are easily proved.

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