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Mixed boundary-transmission problems for composite layered elastic structures

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Summary

We investigate mixed type boundary-transmission problems of the generalized thermo-electro-magneto elasticity (GTEME) theory for complex elastic anisotropic layered structures containing interfacial cracks. This type of problems are described mathematically by systems of partial differential equations with appropriate transmission and boundary conditions for six dimensional unknown physical field (three components of the displacement vector, electric potential function, magnetic potential function, and temperature distribution function). We apply the potential method and the theory of pseudodifferential equations and prove uniqueness and existence theorems of solutions to different type mixed boundary-transmission problems in appropriate Sobolev spaces. We analyze smoothness properties of solutions near the edges of interfacial cracks and near the curves where different type boundary conditions collide.
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1 | INTRODUCTION

Mathematical modeling and analysis of complex multi-component structures in the presence of coupled electro-magnetic and thermo-mechanical fields became very important from the theoretical and practical points of view, due to the rapidly increasing use of composite materials in modern technological processes, as well as in biology and medicine. Recently, many applications have arisen that involve the dynamics of response of new material systems in the presence of strongly coupled electro-magnetic and thermo-mechanical fields (see, e.g., Brown[1], Eringen etc[2], Aouadi[3], Aouadi[4], Avellaneda etc[5], Benveniste[6], Cady[7], Bracke etc[8], Chandrasekharaiah[9],[10], Dang etc[11],[12], Gao[13], Grimes etc[14], García-Sánchez etc[15], Hetnarski etc[16], Higuchi etc[17], Dunn[18], Li etc[19], Li etc[20], Marder[21], Morita etc[22], Nan[23], Nan etc[24], Pak[25], Qin[26], Ryu etc[27], Silva etc[28], Straughan[29], Tauchert etc[30], Uchino[31], Van Run etc[32], Wang etc[33],[34],[35], Wang etc[36], Yang[37], Zohdi[38], Wei[39], Suo etc[40], Harshe etc[41], Fabrizio etc[42], Hench[43], Natroshvili etc[44], Natroshvili[45], and the references therein).

In spite of the fact that there are a huge number of papers devoted to numerical solutions of the above mentioned problems for particular cases (see, e.g., a survey paper Benjeddou[46], Kaltenbacher[47], Rojas-Dáz etc[48], and the references therein), the three-dimensional mixed initial-boundary-transmission problems for multi-component composed bodies with interior and interfacial cracks, in the scientific literature have not been systematically treated from the point of view of rigorous mathematical study.

In the present paper, we consider the generalized Green-Lindsay's thermo-electro-magneto elasticity (GTEME) model describing physical processes with a finite speed of heat propagation in contrast to the conventional thermoelasticity theory (for details see, e.g., Straughan[29]). We consider the case of composed layered solids and analyse mixed boundary transmission problems for the system of elliptic partial differential equations ("pseudo-oscillation equations"), which are obtained from the corresponding dynamical equations by the Laplace transform. The mathematical model contains a fully coupled system of six second order partial differential equations (PDE) with respect to six unknowns: three components of the displacement vector, the electric and magnetic potentials, and the temperature function. These equations are equipped with appropriate mixed boundary-transmission conditions.

The main questions of our investigation are the existence and uniqueness of solutions to the essentially mixed boundary-transmission problems under consideration and analysis of smoothness properties of solutions at the exceptional curves - interfacial crack edges and curves where different type boundary conditions collide. Near the exceptional curves singularity zones appear usually and in practice it is very important to analyse and calculate the stress singularity exponents explicitly.

In our study, the main tools are the generalized potential method and the theory of pseudodifferential equations on manifolds with boundary. By the same approach, similar problems for simpler piezo-elastic models were considered in the references Buchukuri etc[49],[50], Buchukuri etc[51]. The basic boundary-transmission problems for the system of thermo-electro-magneto elasticity theory for composed structures without interfacial cracks were studied in Buchukuri etc[52] and Buchukuri etc[53].

It should be mentioned that in the case of interfacial crack, the investigation of the corresponding mixed boundary-transmission problem becomes very complicated theoretically and technically in comparison with the case of interior crack problem.

The paper is organized as follows.

In the second section, we introduce the basic differential operators of the generalized thermo-electro-magneto elasticity model and derive the corresponding Green formulas. In the third section, we derive representation of solutions in the form of single layer potentials for the problem when the Dirichlet condition is given on the exterior boundary of the layered composite solid and the rigid transmission conditions are prescribed on the interface surface. This representation is then essentially applied in the fourth section, where we investigate the problems with mixed boundary conditions on the exterior boundary of the composed solid and with mixed transmission conditions on the interface surface. This problem covers the case when the composed solid contains interfacial cracks. We reduce the problem to the system of pseudodifferential equations which live on the interface crack surface and on the Neumann part of the exterior boundary. We establish invertibility of the corresponding pseudodifferential operators in appropriate function spaces and establish unique solvability of the original mixed boundary-transmission problem in Sobolev-Slobodetskii spaces. Finally, we analyse regularity properties of solutions near the exceptional curves and show that solutions are Hölder continuous functions under some reasonable restrictions on boundary and transmission data. From our analysis it follows that in general the Hölder smoothness exponents are less then $\frac{1}{2}$ and they essentially depend on the material parameters of the composed solid. It is shown that this smoothness exponent can be calculated explicitly by the principal homogeneous symbol matrix of the above mentioned pseudodifferential operator.

2 | FIELD EQUATIONS OF THE GTEME MODEL AND GREEN'S FORMULAS

The basic linear system of pseudo-oscillation equations for the thermo-electro-magneto-elasticity theory associated with Green-Lindsay's model for homogeneous solids in matrix form reads as (see Buchukuri etc[53])

$$A(\partial_x, \tau)\, U(x, \tau) = \Phi(x, \tau)\,,$$

where $U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top := (u, \varphi, \psi, \vartheta)^\top$ is the sought for complex-valued vector function, $\Phi = (\Phi_1, ..., \Phi_6)^\top$ is a given vector-function, and $A(\partial_x, \tau)$ is a matrix differential operator

$$A(\partial_x, \tau) = \big[A_{pq}(\partial_x, \tau)\big]_{6\times 6} := \begin{bmatrix} [c_{rjkl}\partial_j\partial_l - \varrho\tau^2\delta_{rk}]_{3\times 3} & [e_{lrj}\partial_j\partial_l]_{3\times 1} & [q_{lrj}\partial_j\partial_l]_{3\times 1} & [-(1+\nu_0\tau)\lambda_{rj}\partial_j]_{3\times 1} \\ [-e_{jkl}\partial_j\partial_l]_{1\times 3} & \varkappa_{jl}\partial_j\partial_l & a_{jl}\partial_j\partial_l & -(1+\nu_0\tau)\, p_j\partial_j \\ [-q_{jkl}\partial_j\partial_l]_{1\times 3} & a_{jl}\,\partial_j\partial_l & \mu_{jl}\partial_j\partial_l & -(1+\nu_0\tau) m_j\partial_j \\ [-\tau\lambda_{kl}\partial_l]_{1\times 3} & \tau p_l\partial_l & \tau m_l\partial_l & \eta_{jl}\partial_j\partial_l - \tau^2 h_0 - \tau d_0 \end{bmatrix}_{6\times 6}. \tag{1}$$

The superscript $(\cdot)^\top$ denotes transposition operation, $\tau = \sigma + i\omega$ is a complex parameter, the summation over the repeated indices is meant from 1 to 3; $\partial = \partial_x = (\partial_1, \partial_2, \partial_3)$, $\partial_j = \partial/\partial x_j$. The components of the vector function U have the following physical sense: the first three components correspond to the elastic displacement vector $u = (u_1, u_2, u_3)^\top$, the forth and fifth ones, φ and ψ are respectively the electric and magnetic potentials, and the sixth component ϑ stands for the temperature distribution; c_{rjkl} are the elastic constants, e_{jkl} are the piezoelectric constants, q_{jkl} are the piezomagnetic constants, $\varkappa_{jk}$ are the dielectric (permittivity) constants, μ_{jk} are the magnetic permeability constants, a_{jk} are the coupling coefficients connecting electric and magnetic fields, p_j and m_j are constants characterizing the relation between thermodynamic processes and electromagnetic effects, λ_{rj} are the thermal strain constants, η_{jk} are the heat conductivity coefficients, ϱ denotes the mass density, ν_0 and h_0 are two relaxation times, d_0 is a constitutive coefficient. These constants satisfy the symmetry conditions:

$$\begin{gathered} c_{rjkl} = c_{jrkl} = c_{klrj}, \quad e_{klj} = e_{kjl}, \quad q_{klj} = q_{kjl}, \\ \varkappa_{kj} = \varkappa_{jk}, \quad \lambda_{kj} = \lambda_{jk}, \quad \mu_{kj} = \mu_{jk}, \quad a_{kj} = a_{jk}, \quad \eta_{kj} = \eta_{jk}, \quad r, j, k, l = 1, 2, 3. \end{gathered} \tag{2}$$

From physical considerations it follows that (see, e.g., Aouadi[4], Straughan[29], Green etc[54]) for all $\xi_{kj} = \xi_{jk} \in \mathbb{R}$ and for all $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$:

$$\begin{gathered} c_{rjkl}\,\xi_{rj}\,\xi_{kl} \ge \delta_0\,\xi_{kl}\,\xi_{kl}, \quad \varkappa_{kj}\,\xi_k\,\xi_j \ge \delta_1\,|\xi|^2, \quad \mu_{kj}\,\xi_k\,\xi_j \ge \delta_2\,|\xi|^2, \quad \eta_{kj}\,\xi_k\,\xi_j \ge \delta_3\,|\xi|^2, \\ \nu_0 > 0, \quad h_0 > 0, \quad d_0\nu_0 - h_0 > 0, \end{gathered} \tag{3}$$

where δ_0, δ_1, δ_2, and δ_3 are positive constants depending on material parameters.

Due to the symmetry conditions (2), with the help of (3) one can easily derive the inequalities:

$$\begin{gathered} c_{rjkl}\,\zeta_{rj}\,\overline{\zeta_{kl}} \ge \delta_0\,\zeta_{kl}\,\overline{\zeta_{kl}}, \quad \varkappa_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_1\,|\zeta|^2, \quad \mu_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_2\,|\zeta|^2, \quad \eta_{kj}\,\zeta_k\,\overline{\zeta_j} \ge \delta_3\,|\,\zeta|^2, \\ \text{for all } \ \zeta_{kj} = \zeta_{jk} \in \mathbb{C} \text{ and for all } \ \zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3, \end{gathered}$$

where the over bar denotes complex conjugation. The positive definiteness of the potential energy and the laws of thermodynamics imply that the following 8×8 matrix

$$M = [M_{kj}]_{8\times 8} := \begin{bmatrix} [\varkappa_{jl}]_{3\times 3} & [a_{jl}]_{3\times 3} & [p_j]_{3\times 1} & [\nu_0 p_j]_{3\times 1} \\ [a_{jl}]_{3\times 3} & [\mu_{jl}]_{3\times 3} & [m_j]_{3\times 1} & [\nu_0 m_j]_{3\times 1} \\ [p_j]_{1\times 3} & [m_j]_{1\times 3} & d_0 & h_0 \\ [\nu_0 p_j]_{1\times 3} & [\nu_0 m_j]_{1\times 3} & h_0 & \nu_0 h_0 \end{bmatrix}_{8\times 8}$$

is positive definite. Moreover, it follows that the matrices

$$\Lambda^{(1)} := \begin{bmatrix} [\varkappa_{kj}]_{3\times 3} & [a_{kj}]_{3\times 3} \\ [a_{kj}]_{3\times 3} & [\mu_{kj}]_{3\times 3} \end{bmatrix}_{6\times 6}, \qquad \Lambda^{(2)} := \begin{bmatrix} d_0 & h_0 \\ h_0 & \nu_0 h_0 \end{bmatrix}_{2\times 2} \tag{4}$$

are positive definite as well, i.e.,

$$\begin{gathered} \varkappa_{kj}\,\zeta'_k\,\overline{\zeta'_j} + a_{kj}\,(\zeta'_k\,\overline{\zeta''_j} + \overline{\zeta'_k}\,\zeta''_j) + \mu_{kj}\,\zeta''_k\,\overline{\zeta''_j} \ge \kappa_1\,(|\zeta'|^2 + |\zeta''|^2) \qquad \forall \zeta', \zeta'' \in \mathbb{C}^3, \\ d_0\,|z_1|^2 + h_0\,(z_1\,\overline{z_2} + \overline{z_1}\,z_2) + \nu_0 h_0\,|z_2|^2 \ge \kappa_2\,(|z_1|^2 + |z_2|^2) \qquad \forall z_1, z_2 \in \mathbb{C}, \end{gathered}$$

with some positive constants κ_1 and κ_2 depending on the material parameters involved in matrices (4).
Further, let us introduce the generalized stress operator $\mathcal{T}(\partial_x, n, \tau)$ associated with the pseudo-oscillation operator $A(\partial_x, \tau)$,

$$\mathcal{T}(\partial_x, n, \tau) = \big[\, \mathcal{T}_{pq}(\partial_x, n, \tau)\, \big]_{6\times 6} := \begin{bmatrix} [c_{rjkl}\, n_j \partial_l]_{3\times3} & [e_{lrj}\, n_j \partial_l]_{3\times1} & [q_{lrj}\, n_j \partial_l]_{3\times1} & [-(1+\nu_0\tau)\lambda_{rj} n_j]_{3\times1} \\ [-e_{jkl}\, n_j \partial_l]_{1\times3} & \varkappa_{jl}\, n_j \partial_l & a_{jl}\, n_j \partial_l & -(1+\nu_0\tau) p_j n_j \\ [-q_{jkl}\, n_j \partial_l]_{1\times3} & a_{jl}\, n_j \partial_l & \mu_{jl}\, n_j \partial_l & -(1+\nu_0\tau) m_j n_j \\ [0]_{1\times3} & 0 & 0 & \eta_{jl} n_j \partial_l \end{bmatrix}_{6\times6}. \tag{5}$$

For a six vector $U = (u, \varphi, \psi, \vartheta)^\top$ we can calculate the so-called generalized stress vector $\mathcal{T}\, U$,

$$\begin{aligned} \mathcal{T}(\partial_x, n, \tau)\, U(x,\tau) = \big(&\sigma_{1j}(x,\tau) n_j(x),\ \sigma_{2j}(x,\tau) n_j(x),\ \sigma_{3j}(x,\tau) n_j(x), \\ &- D_j(x,\tau) n_j(x),\ -B_j(x,\tau) n_j(x),\ -T_0^{-1} q_j(x,\tau) n_j(x)\big)^\top. \end{aligned} \tag{6}$$

Due to (6), the components of the stress vector have the following physical sense: the first three components correspond to the mechanical stress vector in the theory of generalized thermo-electro-magneto-elasticity, the forth and the fifth components correspond to the normal components of the electric displacement vector and the magnetic induction vector respectively with opposite sign, and finally the sixth component is $(-T_0^{-1})$ times the normal component of the heat flux vector; here $n = (n_1, n_2, n_3)$ stands for the unit normal vector to the corresponding surface element, σ_{ij} are the components of the mechanical stress tensor, T_0 is the initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields, $D = (D_1, D_2, D_3)^\top$ is the electric displacement vector and $B = (B_1, B_2, B_3)^\top$ is the magnetic induction vector.

Recall that $E = (E_1, E_2, E_3)^\top = -\operatorname{grad}\varphi$ and $H = (H_1, H_2, H_3)^\top = -\operatorname{grad}\psi$ are electric and magnetic fields respectively, $\varepsilon_{kj} = 2^{-1}(\partial_k u_j + \partial_j u_k)$ are the components of the mechanical strain tensor, $q = (q_1, q_2, q_3)^\top$ is the heat flux vector, and the corresponding constitutive equations read as

$$\begin{aligned} \sigma_{rj}(x,\tau) &= c_{rjkl}\varepsilon_{kl}(x,\tau) + e_{lrj}\partial_l\varphi(x,\tau) + q_{lrj}\partial_l\psi(x,\tau) - (1+\nu_0\tau)\lambda_{rj}\vartheta(x,\tau), \\ D_j(x,\tau) &= e_{jkl}\,\varepsilon_{kl}(x,\tau) - \varkappa_{jl}\,\partial_l\varphi(x,\tau) - a_{jl}\,\partial_l\psi(x,\tau) + (1+\nu_0\tau)p_j\vartheta(x,\tau), \\ B_j(x,\tau) &= q_{jkl}\,\varepsilon_{kl}(x,\tau) - a_{jl}\,\partial_l\varphi(x,\tau) - \mu_{jl}\partial_l\psi(x,\tau) + (1+\nu_0\tau)m_j\vartheta(x,\tau), \\ q_j(x,\tau) &= -\,T_0\,\eta_{jl}\partial_l\vartheta(x,\tau)\,. \end{aligned}$$

Let Ω be a bounded domain of $\mathbb{R}^3$ with sufficiently smooth boundary $S = \partial\Omega$.

By $C^k(\overline{\Omega})$ we denote the subspace of functions from $C^k(\Omega)$ whose derivatives up to the order k are continuously extendable to S from Ω; $C^{k,\alpha}(\overline{\Omega})$ denotes the subspace of functions from $C^k(\overline{\Omega})$ whose kth order derivatives are Hölder continuous in Ω with exponent $\alpha \in (0,1]$. By L_p, $L_{p,loc}$, W^r_p, $W^r_{p,loc}$, H^s_p, $H^s_{p,loc}$, $B^s_{p,q}$, and $B^s_{p,q,loc}$ (with $r \ge 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., Triebel[55]). Recall that $H^r_2 = W^r_2 = B^r_{2,2}$, $H^s_2 = B^s_{2,2}$, $W^t_p = B^t_{p,p}$, and $H^k_p = W^k_p$, for any $r \ge 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. Let us introduce also the following spaces:

$$\begin{aligned} \widetilde{H}^s_p(\mathcal{M}) &= \{f \ :\ f \in H^s_p(\mathcal{M}_0),\ \operatorname{supp} f \subset \overline{\mathcal{M}}\}, \quad & H^s_p(\mathcal{M}) &= \{r_{\mathcal{M}} f \ :\ f \in H^s_p(\mathcal{M}_0)\}, \\ \widetilde{B}^s_{p,q}(\mathcal{M}) &= \{f \ :\ f \in B^s_{p,q}(\mathcal{M}_0),\ \operatorname{supp} f \subset \overline{\mathcal{M}}\}, \quad & B^s_{p,q}(\mathcal{M}) &= \{r_{\mathcal{M}} f \ :\ f \in B^s_{p,q}(\mathcal{M}_0)\,\}, \end{aligned}$$

where $\mathcal{M}$ is a proper submanifold of a manifold $\mathcal{M}_0$ and $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

For arbitrary vector functions

$$U = (u_1, u_2, u_3, \varphi, \psi, \vartheta)^\top \in \big[C^2(\overline{\Omega})\big]^6 \text{ and } U' = (u_1', u_2', u_3', \varphi', \psi', \vartheta')^\top \in \big[C^2(\overline{\Omega})\big]^6,$$

the following first Green identity holds (see Buchukuri etc[53])

$$\int\limits_{\Omega} \Big[A(\partial_x, \tau)\, U \cdot U' + \mathcal{E}_\tau(U, \overline{U'})\Big] dx = \int\limits_{\partial\Omega} \{\mathcal{T}(\partial_x, n, \tau) U\}^+ \cdot \{U'\}^+ dS\,, \tag{7}$$

where the central dot denotes the scalar product of two vectors in the complex vector space $\mathbb{C}^N$, i.e., $a \cdot b \equiv (a,b) := \sum_{j=1}^N a_j\, \overline{b_j}$ for $a, b \in \mathbb{C}^N$, the symbol $\{\cdot\}^+$ denotes the one sided limit (the trace operator) on $\partial\Omega$, the operators $A(\partial_x, \tau)$ and $\mathcal{T}(\partial_x, n, \tau)$ are given by (1) and (5) respectively and

$$\begin{aligned}\mathcal{E}_\tau(U,\overline{U'}) := {} & c_{rjkl}\,\partial_l u_k\,\overline{\partial_j u_r'} + \varrho\,\tau^2\,u_r\,\overline{u_r'} + e_{lrj}\,(\partial_l\varphi\,\overline{\partial_j u_r'} - \partial_j u_r\,\overline{\partial_l\varphi'})\\
& + q_{lrj}\,(\partial_l\psi\,\overline{\partial_j u_r'} - \partial_j u_r\,\overline{\partial_l\psi'}) + \varkappa_{jl}\,\partial_l\varphi\,\overline{\partial_j\varphi'} + a_{jl}\,(\partial_l\varphi\,\overline{\partial_j\psi'} + \partial_j\psi\,\overline{\partial_l\varphi'})\\
& + \mu_{jl}\partial_l\psi\overline{\partial_j\psi'} + \lambda_{kj}[\tau\overline{\vartheta'}\partial_j u_k - (1+\nu_0\tau)\vartheta\overline{\partial_j u_k'}] - p_l[\tau\overline{\vartheta'}\partial_l\varphi + (1+\nu_0\tau)\vartheta\overline{\partial_l\varphi'}]\\
& - m_l\,[\tau\,\overline{\vartheta'}\,\partial_l\psi + (1+\nu_0\tau)\vartheta\,\overline{\partial_l\psi'}] + \eta_{jl}\,\partial_l\vartheta\,\overline{\partial_j\vartheta'} + \tau\,(h_0\tau + d_0)\,\vartheta\,\overline{\vartheta'}\,.\end{aligned}$$

Note that Green's formula (7) by standard limiting procedure can be generalized to Lipschitz domains and to vector functions $U \in [W_p^1(\Omega)]^6$ with $A(\partial_x,\tau)U \in \big[L_p(\Omega)\big]^6$ and $U' \in [W_q^1(\Omega)]^6$, $\frac{1}{p}+\frac{1}{q}=1$. Using Green's first identity we can correctly determine a *generalized trace of the stress vector* $\{\mathcal{T}(\partial_x,n,\tau)U\}^+ \in [B_{p,p}^{-\frac{1}{p}}(\partial\Omega)]^6$ for a function $U \in [W_p^1(\Omega)]^6$ with $A(\partial_x,\tau)U \in [L_p(\Omega)]^6$ by the following duality relation (cf. McLean[56], Buchukuri etc[53])

$$\Big\langle\,\{\mathcal{T}(\partial_x,n,\tau)U\}^+\,,\;\{U'\}^+\Big\rangle_{\partial\Omega} := \int\limits_{\Omega}[\,A(\partial_x,\tau)\,U\cdot U' + \mathcal{E}_\tau(U,\overline{U'})\,]\,dx,$$

where $U' \in [W_q^1(\Omega)]^6$ is an arbitrary vector function. Here the symbol $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes the duality pairing of $[B_{p,p}^{-\frac{1}{p}}(\partial\Omega)]^6$ with $[B_{q,q}^{\frac{1}{p}}(\partial\Omega)]^6$ which extends the standard L_2 inner product for complex-valued vector functions,

$$\langle\,f\,,\,g\,\rangle_{\partial\Omega} = \int\limits_{\partial\Omega}\sum_{j=1}^{6} f_j(x)\,\overline{g_j(x)}\,dS \quad\text{for}\quad f,g \in [L_2(\partial\Omega)]^6.$$

In particular, for $p=2$ we have the inclusion $\{\mathcal{T}(\partial_x,n,\tau)U\}^+ \in [B_{2,2}^{-\frac{1}{2}}(\partial\Omega)]^6 = [H_2^{-\frac{1}{2}}(\partial\Omega)]^6$.

3 | TRANSMISSION PROBLEM WITH THE DIRICHLET CONDITION

3.1 | Formulation of the problem and uniqueness theorem

Let $\Omega^{(1)}$ be a bounded simply connected domain with the simply connected boundary S_1 and $\Omega^{(2)}$ be an adjacent bounded simply connected domain with interior boundary S_1 and simply connected exterior boundary S_2, $S_1 \cap S_2 = \varnothing$. We assume that the domains $\Omega^{(1)}$ and $\Omega^{(2)}$ are occupied by anisotropic homogeneous materials possessing different thermo-electro-magneto-elastic properties. This means that their material parameters are different constants in different domains, in general, and we have a composed layered elastic solid $\overline{\Omega^{(1)}}\bigcup\overline{\Omega^{(2)}}$ with interfacial surface S_1 and with exterior boundary S_2.

Throughout the paper n stands for the exterior normal vector to the surfaces S_1 and S_2, and, for simplicity, we assume that S_1 and S_2 are infinitely smooth manifolds if not otherwise stated.

We equip with the superscript (β), $\beta = 1,2$, the thermo-mechanical and electro-magnetic characteristics, differential and generalized stress operators associated with the domain $\Omega^{(\beta)}$. Later, we use the same notation for the fundamental solutions, layer potentials, and the corresponding boundary integral operators.

First we consider an auxiliary boundary-transmission problem for a layered composed solid with the Dirichlet type condition on the exterior boundary. Representation formula for this problem plays a crucial role in our further analysis.

Problem (TD). *Find solutions*

$$U^{(1)} \in [W_p^1(\Omega^{(1)})]^6, \quad U^{(2)} \in [W_p^1(\Omega^{(2)})]^6, \quad p>1,$$

to the pseudo-oscillation equations of the GTEME theory

$$A^{(\beta)}(\partial_x,\tau)\,U^{(\beta)}(x) = 0\,, \qquad x \in \Omega^{(\beta)}, \quad \beta = 1,2, \tag{8}$$

satisfying the transmission conditions on the interface S_1

$$\{U^{(1)}(x)\}^+ - \{U^{(2)}(x)\}^- = f^{(1)}(x), \quad x \in S_1, \tag{9}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}^+ - \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^- = F^{(1)}(x), \quad x \in S_1, \tag{10}$$

and the Dirichlet condition on S_2

$$\{U^{(2)}(x)\}^{+} = f^{(2)}(x), \quad x \in S_2, \tag{11}$$

where the symbols $\{\cdot\}^{\pm}$ *denote the one sided interior and exterior limits (the trace operators) on* S_1 *and* S_2 *and*

$$f^{(1)} \in [B_{p,p}^{1-\frac{1}{p}}(S_1)]^6, \quad f^{(2)} \in [B_{p,p}^{1-\frac{1}{p}}(S_2)]^6, \quad F^{(1)} \in [B_{p,p}^{-\frac{1}{p}}(S_1)]^6. \tag{12}$$

Here the differential equations (8) of pseudo-oscillations are understood in the distributional sense, the Dirichlet type boundary-transmission conditions (9) and (11) in the usual trace sense, and the Neumann type transmission condition (10) in the generalized functional sense defined by the corresponding Green formulas.

Theorem 1. Let S_1 and S_2 be Lipschitz, conditions (12) be satisfied with $p = 2$, and the time relaxation parameters $\nu_0^{(1)}$ and $\nu_0^{(2)}$ be the same,

$$\nu_0^{(1)} = \nu_0^{(2)} =: \nu_0. \tag{13}$$

Then the transmission problem (TD) (8)-(11) possesses at most one solution in the space $[W_2^1(\Omega^{(1)})]^6 \times [W_2^1(\Omega^{(2)})]^6$.

Proof. Actually, the proof of the theorem is quite similar to the proof of Theorem 8.1 in the reference Buchukuri etc[53]. □

3.2 | Representation formulas of solutions to the problem (TD)

Let us look for solution vectors $U^{(1)}$ and $U^{(2)}$ of the boundary-transmission problem (TD) in the form of single layer potentials associated with the operator $A^{(\beta)}(\partial_x, \tau)$, $\beta = 1, 2$, constructed by the corresponding fundamental matrix $\Gamma^{(\beta)}(x-y,\tau)$ (see Appendix, formula (70))

$$U^{(1)}(x) = V_{S_1}^{(1)}(g^{(1)})(x) \ \text{ in } \ \Omega^{(1)}, \tag{14}$$

$$U^{(2)}(x) = V_{S_1}^{(2)}(h^{(1)})(x) + V_{S_2}^{(2)}(h^{(2)})(x) \ \text{ in } \ \Omega^{(2)}, \tag{15}$$

where $g^{(1)} \in [B_{p,p}^{-1/p}(S_1)]^6$, $h^{(1)} \in [B_{p,p}^{-1/p}(S_1)]^6$, and $h^{(2)} \in [B_{p,p}^{-1/p}(S_2)]^6$ are unknown density vectors. The boundary-transmission conditions (9), (10), and (11), and properties of the single layer potentials (see Theorems 9-10) lead then to the following system of pseudodifferential equations for $g^{(1)}$, $h^{(1)}$, and $h^{(2)}$:

$$\mathcal{H}_{S_1}^{(1)} g^{(1)} - \mathcal{H}_{S_1}^{(2)} h^{(1)} - \{V_{S_2}^{(2)}(h^{(2)})\}_{S_1}^{-} = f^{(1)} \quad \text{on} \quad S_1, \tag{16}$$

$$\Big(-\frac{1}{2}\, I_6 + \mathcal{K}_{S_1}^{(1)}\Big) g^{(1)} - \Big(\frac{1}{2}\, I_6 + \mathcal{K}_{S_1}^{(2)}\Big) h^{(1)} - \{\mathcal{T}^{(2)} V_{S_2}^{(2)}(h^{(2)})\}_{S_1}^{-} = F^{(1)} \quad \text{on} \quad S_1, \tag{17}$$

$$\{V_{S_1}^{(2)}(h^{(1)})\}_{S_2}^{+} + \mathcal{H}_{S_2}^{(2)} h^{(2)} = f^{(2)} \quad \text{on} \quad S_2. \tag{18}$$

The integral operators $\mathcal{H}_{S_l}^{(\beta)}$ and $\mathcal{K}_{S_l}^{(\beta)}$, $\beta = 1, 2$, $l = 1, 2$, are defined in Appendix, see formulas (73) and (74). Denote the operator generated by the left hand side expressions of system (16)-(18) by $\mathbf{M} = [\mathbf{M}_{kj}]_{18\times 18}$,

$$\mathbf{M} = \mathbf{N} + \mathbf{K},$$

where

$$\mathbf{N} = [\mathbf{N}_{kj}]_{18\times 18} = \begin{bmatrix} \mathcal{H}_{S_1}^{(1)} & -\mathcal{H}_{S_1}^{(2)} & \mathbf{0} \\ -\frac{1}{2}\, I_6 + \mathcal{K}_{S_1}^{(1)} & -\frac{1}{2}\, I_6 - \mathcal{K}_{S_1}^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{H}_{S_2}^{(2)} \end{bmatrix}_{18\times 18}, \tag{19}$$

$$\mathbf{K} = [\mathbf{K}_{kj}]_{18\times 18} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -r_{_{S_1}} V_{S_2}^{(2)} \\ \mathbf{0} & \mathbf{0} & -r_{_{S_1}} \mathcal{T}^{(2)} V_{S_2}^{(2)} \\ \mathbf{0} & r_{_{S_2}} V_{S_1}^{(2)} & \mathbf{0} \end{bmatrix}_{18\times 18}, \tag{20}$$

where $r_{_{S_j}}$ is the restriction operator to S_j and the boldface zero $\mathbf{0}$ stands for the 6×6 null-matrix, $\mathbf{0} = [0]_{6\times 6}$. Note that due to the properties of the single layer potentials $\{V_{S_2}^{(2)}(h^{(2)})\}_{S_1}^{-} = r_{_{S_1}} V_{S_2}^{(2)}(h^{(2)})$, $\{V_{S_1}^{(2)}(h^{(1)})\}_{S_2}^{+} = r_{_{S_2}} V_{S_1}^{(2)}(h^{(1)})$ and $\{\mathcal{T}^{(2)} V_{S_2}^{(2)}\}_{S_1}^{-} = r_{_{S_1}} \mathcal{T}^{(2)} V_{S_2}^{(2)}$.

System (16)-(18) can be rewritten then as

$$\mathbf{M}\Phi = \Psi, \tag{21}$$

with an unknown vector-function $\Phi = (g^{(1)}, h^{(1)}, h^{(2)})^\top$ and a given right hand side vector-function $\Psi = (f^{(1)}, F^{(1)}, f^{(2)})^\top$.

Let us introduce the function spaces:

$$\mathbb{X}^t_{p,q} := \big[B^t_{p,q}(S_1)\big]^6 \times \big[B^t_{p,q}(S_1)\big]^6 \times \big[B^t_{p,q}(S_2)\big]^6,$$
$$\mathbb{Y}^t_{p,q} := \big[B^{t+1}_{p,q}(S_1)\big]^6 \times \big[B^t_{p,q}(S_1)\big]^6 \times \big[B^{t+1}_{p,q}(S_2)\big]^6.$$

Evidently, the above introduced operators possess the following mapping properties (see Theorem 10 in Appendix):

$$\mathbf{M} \; : \; \mathbb{X}^t_{p,q} \to \mathbb{Y}^t_{p,q}, \tag{22}$$

$$\mathbf{N} \; : \; \mathbb{X}^t_{p,q} \to \mathbb{Y}^t_{p,q}, \tag{23}$$

$$\mathbf{K} \; : \; \mathbb{X}^t_{p,q} \to \mathbb{Y}^t_{p,q}, \tag{24}$$

for $p > 1$, $q \geqslant 1$, and $t \in \mathbb{R}$. In view of (19), it is easy to see that operator (24) is infinitely smoothing compact operator since S_1 and S_2 are disjoint $C^\infty-$regular manifolds. Therefore, the operator $\mathbf{N}$ defined in (23) is a compact perturbation of the operator $\mathbf{M}$ defined in (22).

First we prove the following assertion.

Theorem 2. Let $S_j \in C^\infty$, $j = 1, 2$. Then operator (23) is invertible for all $p > 1$, $q \geqslant 1$, and $t \in \mathbb{R}$.

Proof. Denote

$$\mathbb{A}_{S_1} := \begin{bmatrix} \mathcal{H}^{(1)}_{S_1} & -\mathcal{H}^{(2)}_{S_1} \\ -\frac{1}{2} I_6 + \mathcal{K}^{(1)}_{S_1} & -\frac{1}{2} I_6 - \mathcal{K}^{(2)}_{S_1} \end{bmatrix}_{12\times 12}.$$

The following invertibility result is proved in Buchukuri etc[53] as Theorem 8.2: The operator

$$\mathbb{A}_{S_1} \; : \; \big[B^{-\frac{1}{p}}_{p,p}(S_1)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_1)\big]^6 \to \big[B^{1-\frac{1}{p}}_{p,p}(S_1)\big]^6 \times \big[B^{-\frac{1}{p}}_{p,p}(S_1)\big]^6$$

is invertible for all $p > 1$. Due to the general theory of pseudodifferential operators, this implies that the operator

$$\mathbb{A}_{S_1} \; : \; \big[B^t_{p,q}(S_1)\big]^6 \times \big[B^t_{p,q}(S_1)\big]^6 \to \big[B^{t+1}_{p,q}(S_1)\big]^6 \times \big[B^t_{p,q}(S_1)\big]^6$$

is invertible for all $p > 1$, $q \geqslant 1$, and $t \in \mathbb{R}$.

The invertibility of the operator (see Theorem 10 in Appendix)

$$\mathcal{H}^{(2)}_{S_2} \; : \; \big[B^t_{p,q}(S_2)\big]^6 \to \big[B^{t+1}_{p,q}(S_2)\big]^6$$

and the relation (19) complete the proof. □

Remark 1. Consider the operator

$$\widetilde{\mathbb{A}}_{S_1} := \mathbb{A}_{S_1}\mathbb{H}_{S_1} = \begin{bmatrix} I_6 & -I_6 \\ \mathcal{A}^{(1)}_{S_1} & -\mathcal{A}^{(2)}_{S_1} \end{bmatrix}_{12\times 12}, \tag{25}$$

where

$$\mathbb{H}_{S_1} = \begin{bmatrix} [\mathcal{H}^{(1)}_{S_1}]^{-1} & \mathbf{0} \\ \mathbf{0} & [\mathcal{H}^{(2)}_{S_1}]^{-1} \end{bmatrix}_{12\times 12}$$

and

$$\mathcal{A}^{(1)}_{S_1} := \Big(-\frac{1}{2} I_6 + \mathcal{K}^{(1)}_{S_1}\Big)[\mathcal{H}^{(1)}_{S_1}]^{-1}, \qquad \mathcal{A}^{(2)}_{S_1} := \Big(\frac{1}{2} I_6 + \mathcal{K}^{(2)}_{S_1}\Big)[\mathcal{H}^{(2)}_{S_1}]^{-1} \tag{26}$$

are the Steklov-Poincaré type operators associated with the surface S_1 (see Appendix). Note that (26) are strongly elliptic pseudodifferential operators of order 1 (see Buchukuri etc[53]).

From Theorem 2 and relation (25) it follows that the strongly elliptic pseudodifferential operator

$$\widetilde{\mathbb{A}}_{S_1} \; : \; \big[B^{t+1}_{p,q}(S_1)\big]^6 \times \big[B^{t+1}_{p,q}(S_1)\big]^6 \to \big[B^{t+1}_{p,q}(S_1)\big]^6 \times \big[B^t_{p,q}(S_1)\big]^6$$

is invertible for all $p > 1$, $q \geqslant 1$, and $t \in \mathbb{R}$. Moreover, in Subsection 8.2 in Buchukuri etc[53], it is also shown that for all $p > 1$, $q \geqslant 1$, and $t \in \mathbb{R}$, the operator

$$\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1} \ : \ \big[B^{t+1}_{p,q}(S_1)\big]^6 \to \big[B^{t}_{p,q}(S_1)\big]^6$$

is invertible as well (see Appendix, Theorem 13).

It can be shown that the inverse operator to the operator (23)

$$\mathbf{N}^{-1} \ : \ \mathbb{Y}^t_{p,q} \to \mathbb{X}^t_{p,q}$$

has the form

$$\mathbf{N}^{-1} \equiv \mathbb{N} = [\mathbb{N}_{kj}]_{18\times 18} = \begin{bmatrix} -[\mathcal{H}^{(1)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1}\mathcal{A}^{(2)}_{S_1} & [\mathcal{H}^{(1)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1} & \mathbf{0} \\ -[\mathcal{H}^{(2)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1}\mathcal{A}^{(1)}_{S_1} & [\mathcal{H}^{(2)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\mathcal{H}^{(2)}_{S_2}]^{-1} \end{bmatrix}_{18\times 18}. \tag{27}$$

Due to Theorem 2 and the compactness of the operator (24), the operator (22) is Fredholm with zero index. Let us show that the null-space of the operator (22) is trivial. Indeed, let $\Phi = (g^{(1)}, h^{(1)}, h^{(2)})^\top \in \mathbb{X}^{-\frac{1}{2}}_{2,2} = \big[H_2^{-\frac{1}{2}}(S_1)\big]^6 \times \big[H_2^{-\frac{1}{2}}(S_1)\big]^6 \times \big[H_2^{-\frac{1}{2}}(S_2)\big]^6$ be a solution to the homogenous equation $\mathbf{M}\Phi = 0$ and construct vectors $U^{(1)}$ and $U^{(2)}$ by formulas (14) and (15). It can easily be shown that the pair $\big(U^{(1)}, U^{(2)}\big) \in [H^1_2(\Omega^{(1)})]^6 \times [H^1_2(\Omega^{(2)})]^6$ solves the homogeneous boundary-transmission Problem (TD). Therefore, $U^{(1)} = 0$ in $\Omega^{(1)}$ and $U^{(2)} = 0$ in $\Omega^{(2)}$ due to the uniqueness Theorem 1. In view of continuity property of the single layer potentials and uniqueness theorems for the interior and exterior Dirichlet boundary value problems for the operators $A^{(\beta)}(\partial_x, \tau)$, $\beta = 1, 2$, (see, e.g., Theorems 2.25-2.26 in Buchukuri etc[53]), we conclude that

$$V^{(1)}_{S_1}(g^{(1)})(x) = 0 \ \text{ in } \ \mathbb{R}^3,$$

$$V^{(2)}_{S_1}(h^{(1)})(x) + V^{(2)}_{S_2}(h^{(2)})(x) = 0 \ \text{ in } \ \mathbb{R}^3,$$

which imply $g^{(1)} = h^{(1)} = 0$ on S_1 and $h^{(2)} = 0$ on S_2. Whence, it follows that the null space of the operator (22) is trivial for $p = q = 2$, and therefore it is also trivial for all $p > 1$, $q \geqslant 1$, and $t \in \mathbb{R}$. Consequently, we have the following invertibility result.

Theorem 3. Operator (22) is invertible for all $p > 1$, $q \geqslant 1$, and $t \in \mathbb{R}$, when $S_j \in C^\infty$, $j = 1, 2$.

This theorem implies the corresponding existence result.

Theorem 4. Let $S_j \in C^\infty$, $j = 1, 2$, $p > 1$, $q \geqslant 1$, $t \geqslant 1$, and the following conditions hold

$$f^{(1)} \in [B^{t-\frac{1}{p}}_{p,p}(S_1)]^6, \quad f^{(2)} \in [B^{t-\frac{1}{p}}_{p,p}(S_2)]^6, \quad F^{(1)} \in [B^{t-1-\frac{1}{p}}_{p,p}(S_1)]^6.$$

Then the boundary-transmission problem (TD) is uniquely solvable in the space $[W^t_p(\Omega^{(1)})]^6 \times [W^t_p(\Omega^{(2)})]^6$ and the solution pair is representable in the form of single layer potentials (14)-(15), where the densities $g^{(1)}$, $h^{(1)}$, and $h^{(2)}$ are defined by the system of pseudodifferential equations (16)-(18).

Proof. Solvability of the problem follows from Theorem 3. We have uniqueness for $p = 2$, thanks to Theorem 1. To show the unique solvability for arbitrary $p > 1$, because invertibility of the operator (22), it is enough to show that an arbitrary solution $U^{(\beta)} \in [W^1_p(\Omega^{(\beta)})]^6$ of equation (8) is uniquely representable in the form of single layer potentials, (14) for $\beta = 1$ and (15) for $\beta = 2$. Indeed, let $U^{(2)} \in [W^1_p(\Omega^{(2)})]^6$ be a solution of the equation $A^{(2)}(\partial_x, \tau)\, U^{(2)}(x) = 0$, $x \in \Omega^{(2)}$. It is evident that

$$f^{(1)} := \{U^{(2)}\}^-_{S_1} \in [B^{1-\frac{1}{p}}_{p,p}(S_1)]^6, \quad f^{(2)} := \{U^{(2)}\}^+_{S_2} \in [B^{1-\frac{1}{p}}_{p,p}(S_2)]^6. \tag{28}$$

Thus, the vector function $U^{(2)} \in [W^1_p(\Omega^{(2)})]^6$ can be considered as a solution to the Dirichlet problem with conditions (28). On the other hand, let us consider the vector function

$$U^*(x) = V^{(2)}_{S_1}(h^{(1)})(x) + V^{(2)}_{S_2}(h^{(2)})(x) \ \text{ in } \ \Omega^{(2)},$$

where $h^{(1)}$ and $h^{(2)}$ solve the system of pseudodifferential equations

$$\mathcal{H}^{(2)}_{S_1} h^{(1)} + \{V^{(2)}_{S_2}(h^{(2)})\}^-_{S_1} = f^{(1)} \quad \text{on} \quad S_1, \tag{29}$$

$$\{V^{(2)}_{S_1}(h^{(1)})\}^+_{S_2} + \mathcal{H}^{(2)}_{S_2} h^{(2)} = f^{(2)} \quad \text{on} \quad S_2, \tag{30}$$

where $f^{(1)}$ and $f^{(2)}$ are defined by (28).

Put

$$\mathbf{H} := \begin{bmatrix} \mathcal{H}^{(2)}_{S_1} & \{V^{(2)}_{S_2}\}^-_{S_1} \\ \{V^{(2)}_{S_1}\}^+_{S_2} & \mathcal{H}^{(2)}_{S_2} \end{bmatrix}.$$

With the help of Theorem 10 and taking into consideration unique solvability of the Dirichlet problem in the space $[W^1_2(\Omega^{(2)})]^6$ it is easy to shown that the strongly elliptic pseudodifferential operator

$$\mathbf{H} \,:\, [B^{-\frac{1}{2}}_{2,2}(S_1)]^6 \times [B^{-\frac{1}{2}}_{2,2}(S_2)]^6 \to [B^{\frac{1}{2}}_{2,2}(S_1)]^6 \times [B^{\frac{1}{2}}_{2,2}(S_2)]^6$$

is invertible. This implies that the operator

$$\mathbf{H} \,:\, [B^{t}_{p,p}(S_1)]^6 \times [B^{t}_{p,p}(S_2)]^6 \to [B^{t+1}_{p,p}(S_1)]^6 \times [B^{t+1}_{p,p}(S_2)]^6 \tag{31}$$

is invertible for all $t \in \mathbb{R}$ and $p > 1$. Therefore system (29)-(30) is uniquely solvable and for the solution pair we have the inclusion $(h^{(1)}, h^{(2)}) \in [B^{-\frac{1}{p}}_{p,p}(S_1)]^6 \times [B^{-\frac{1}{p}}_{p,p}(S_2)]^6$ implying by Theorem 10 the inclusion $U^* \in [W^1_p(\Omega^{(2)})]^6$.

Further, let $\widetilde{U} := U^{(2)} - U^*$. Evidently $\widetilde{U} \in [W^1_p(\Omega^{(2)})]^6$ solves the homogeneous Dirichlet problem. Due to the homogeneous Dirichlet conditions, $\{\widetilde{U}\}^-_{S_1} = 0$ and $\{\widetilde{U}\}^+_{S_2} = 0$, the general integral representation of the vector function $\widetilde{U}$ reads as follows (see Theorem 3.5 in Buchukuri etc[53])

$$\widetilde{U}(x) = V^{(2)}_{S_1}\big(\{\mathcal{T}^{(2)}\widetilde{U}\}^-_{S_1}\big)(x) - V^{(2)}_{S_2}\big(\{\mathcal{T}^{(2)}\widetilde{U}\}^+_{S_2}\big)(x) \ \text{ in } \ \Omega^{(2)}.$$

Introducing the notation

$$\widetilde{h}^{(1)} := \{\mathcal{T}^{(2)}\widetilde{U}\}^-_{S_1} \in [B^{-\frac{1}{p}}_{p,p}(S_1)]^6, \quad \widetilde{h}^{(2)} := -\{\mathcal{T}^{(2)}\widetilde{U}\}^+_{S_2} \in [B^{-\frac{1}{p}}_{p,p}(S_2)]^6,$$

and keeping in mind that the vector function $\widetilde{U}$ satisfies the homogeneous Dirichlet conditions on S_1 and S_2, we find that the vector function $\widetilde{\Phi} = (\widetilde{h}^{(1)}, \widetilde{h}^{(2)})^\top \in [B^{-\frac{1}{p}}_{p,p}(S_1)]^6 \times [B^{-\frac{1}{p}}_{p,p}(S_2)]^6$ solves the homogeneous equation $\mathbf{H}\widetilde{\Phi} = 0$. In view of invertibility of the operator (31), we conclude that $\widetilde{\Phi} = 0$, which implies that $\widetilde{U} = 0$ in $\Omega^{(2)}$, i.e. $U^{(2)} = U^*$ in $\Omega^{(2)}$ which proves that an arbitrary solution $U^{(2)} \in [W^1_p(\Omega^{(2)})]^6$ of equation (8) for $\beta = 2$ is uniquely representable by formula (15) as the sum of single layer potentials.

We can show the similar result for a vector function $U^{(1)} \in [W^1_p(\Omega^{(1)})]^6$ quite analogously.

This completes the proof. □

Remark 2. The inverse operator to operator (22),

$$\mathbf{M}^{-1} \equiv \mathbb{M} = [\mathbb{M}_{kj}]_{18\times 18} \,:\, \mathbb{Y}^t_{p,q} \to \mathbb{X}^t_{p,q}, \tag{32}$$

can be represented in the following form

$$\mathbf{M}^{-1} = \mathbf{N}^{-1} + \mathbb{L}, \ \text{ i.e., } \ \mathbb{M} = \mathbb{N} + \mathbb{L}, \tag{33}$$

where $\mathbf{N}^{-1} \equiv \mathbb{N}$ is defined in (27), while $\mathbb{L} = [\mathbb{L}_{kj}]_{18\times 18} \,:\, \mathbb{Y}^t_{p,q} \to \mathbb{X}^t_{p,q}$ is an infinitely smoothing compact integral operator. This follows from the relation $\mathbf{M}^{-1} = \mathbf{N}^{-1}[\mathbf{I} + \mathbf{K}\mathbf{N}^{-1}]^{-1}$, where $\mathbf{I}$ is the identity operator. Note that $[\mathbf{I} + \mathbf{K}\mathbf{N}^{-1}]^{-1} = \mathbf{I} + \mathbf{R}$, where $\mathbf{R} \,:\, \mathbb{X}^t_{p,q} \to \mathbb{X}^t_{p,q}$ is an infinitely smoothing compact operator due to the properties of the operator $\mathbf{K}$ defined by (20).

With the help of relations (33) and (27) we find

$$\mathbf{M}^{-1} \equiv \mathbb{M} = \begin{bmatrix} \mathbb{M}^{11} & \mathbb{M}^{12} & \mathbb{M}^{13} \\ \mathbb{M}^{21} & \mathbb{M}^{22} & \mathbb{M}^{23} \\ \mathbb{M}^{31} & \mathbb{M}^{32} & \mathbb{M}^{33} \end{bmatrix}_{18\times 18},$$

where

$$\mathbb{M}^{11} = [\mathbb{M}^{11}_{kj}]_{6\times 6} = -[\mathcal{H}^{(1)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1}\mathcal{A}^{(2)}_{S_1} + \mathbb{L}^{11} \,:\, \big[B^{t+1}_{p,q}(S_1)\big]^6 \to \big[B^{t}_{p,q}(S_1)\big]^6, \tag{34}$$

$$\mathbb{M}^{12} = [\mathbb{M}^{12}_{kj}]_{6\times 6} = [\mathcal{H}^{(1)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1} + \mathbb{L}^{12} \,:\, \big[B^{t}_{p,q}(S_1)\big]^6 \to \big[B^{t}_{p,q}(S_1)\big]^6, \tag{35}$$

$$\mathbb{M}^{13} = [\mathbb{M}^{13}_{kj}]_{6\times 6} = \mathbb{L}^{13} \,:\, \big[B^{t+1}_{p,q}(S_2)\big]^6 \to \big[C^{\infty}(S_1)\big]^6, \tag{36}$$

$$\mathbb{M}^{21} = [\mathbb{M}^{21}_{kj}]_{6\times 6} = -[\mathcal{H}^{(2)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1}\mathcal{A}^{(1)}_{S_1} + \mathbb{L}^{21} \,:\, \big[B^{t+1}_{p,q}(S_1)\big]^6 \to \big[B^{t}_{p,q}(S_1)\big]^6, \tag{37}$$

$$\mathbb{M}^{22} = [\mathbb{M}^{22}_{kj}]_{6\times 6} = [\mathcal{H}^{(2)}_{S_1}]^{-1}[\mathcal{A}^{(1)}_{S_1} - \mathcal{A}^{(2)}_{S_1}]^{-1} + \mathbb{L}^{22} \,:\, \big[B^{t}_{p,q}(S_1)\big]^6 \to \big[B^{t}_{p,q}(S_1)\big]^6, \tag{38}$$

$$\mathbb{M}^{23} = [\mathbb{M}^{23}_{kj}]_{6\times 6} = \mathbb{L}^{23} \,:\, \big[B^{t+1}_{p,q}(S_2)\big]^6 \to \big[C^{\infty}(S_1)\big]^6, \tag{39}$$

$$\mathbb{M}^{31} = [\mathbb{M}^{31}_{kj}]_{6\times 6} = \mathbb{L}^{31} \,:\, \big[B^{t+1}_{p,q}(S_1)\big]^6 \to \big[C^{\infty}(S_2)\big]^6, \tag{40}$$

$$\mathbb{M}^{32} = [\mathbb{M}^{32}_{kj}]_{6\times 6} = \mathbb{L}^{32} \,:\, \big[B^{t}_{p,q}(S_1)\big]^6 \to \big[C^{\infty}(S_2)\big]^6, \tag{41}$$

$$\mathbb{M}^{33} = [\mathbb{M}^{33}_{kj}]_{6\times 6} = [\mathcal{H}^{(2)}_{S_2}]^{-1} + \mathbb{L}^{33} \,:\, \big[B^{t+1}_{p,q}(S_2)\big]^6 \to \big[B^{t}_{p,q}(S_2)\big]^6, \tag{42}$$

with $\mathbb{L}^{lm}$ being the corresponding 6×6 block matrices of the matrix operator $\mathbb{L}$ involved in (33),

$$\mathbb{L} = \begin{bmatrix} \mathbb{L}^{11} & \mathbb{L}^{12} & \mathbb{L}^{13} \\ \mathbb{L}^{21} & \mathbb{L}^{22} & \mathbb{L}^{23} \\ \mathbb{L}^{31} & \mathbb{L}^{32} & \mathbb{L}^{33} \end{bmatrix}_{18\times 18}, \quad \mathbb{L}^{lm} = [\,\mathbb{L}^{lm}_{kj}\,]_{6\times 6},\ l,m = 1,2,3.$$

Notice that

(i) the pseudodifferential operators $\mathbb{M}^{11}$, $\mathbb{M}^{12}$, $\mathbb{M}^{21}$, $\mathbb{M}^{22}$, $\mathbb{N}^{11}$, $\mathbb{N}^{12}$, $\mathbb{N}^{21}$, $\mathbb{N}^{22}$, $\mathbb{L}^{11}$, $\mathbb{L}^{12}$, $\mathbb{L}^{21}$, and $\mathbb{L}^{22}$ map a space of vector functions defined on the surface S_1 into a space of vector functions defined on the same surface S_1;

(ii) the operators $\mathbb{M}^{31} = \mathbb{L}^{31}$ and $\mathbb{M}^{32} = \mathbb{L}^{32}$ map a space of vector functions defined on the surface S_1 into a space of vector functions defined on the surface S_2 and generate infinitely smoothing compact operators;

(iii) the operators $\mathbb{M}^{13} = \mathbb{L}^{13}$ and $\mathbb{M}^{23} = \mathbb{L}^{23}$ map a space of vector functions defined on the surface S_2 into a space of vector functions defined on the surface S_1 and generate infinitely smoothing compact operators;

(iv) the pseudodifferential operators $\mathbb{M}^{33}$, $\mathbb{N}^{33}$, and $\mathbb{L}^{33}$ map a space of vector functions defined on the surface S_2 into a space of vector functions defined on the same surface S_2.

Therefore the solution of the pseudodifferential equation (21) can be written as

$$\Phi = \mathbb{M}\Psi,$$

i.e.,

$$\begin{bmatrix} g^{(1)} \\ h^{(1)} \\ h^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbb{M}^{11} & \mathbb{M}^{12} & \mathbb{M}^{13} \\ \mathbb{M}^{21} & \mathbb{M}^{22} & \mathbb{M}^{23} \\ \mathbb{M}^{31} & \mathbb{M}^{32} & \mathbb{M}^{33} \end{bmatrix} \begin{bmatrix} f^{(1)} \\ F^{(1)} \\ f^{(2)} \end{bmatrix}.$$

Using the above relations, we can uniquely represent the solution vectors (14)-(15) of the transmission problem (TD) with the Dirichlet data on the exterior boundary of the composed solid in the following form

$$U^{(1)}(x) = V^{(1)}_{S_1}\big(\mathbb{M}^{11} f^{(1)} + \mathbb{M}^{12} F^{(1)} + \mathbb{M}^{13} f^{(2)}\big)(x) \ \text{ in } \ \Omega^{(1)}, \tag{43}$$

$$\begin{aligned} U^{(2)}(x) = {} & V^{(2)}_{S_1}\big(\mathbb{M}^{21} f^{(1)} + \mathbb{M}^{22} F^{(1)} + \mathbb{M}^{23} f^{(2)}\big)(x) \\ & + V^{(2)}_{S_2}\big(\mathbb{M}^{31} f^{(1)} + \mathbb{M}^{32} F^{(1)} + \mathbb{M}^{33} f^{(2)}\big)(x) \ \text{ in } \ \Omega^{(2)}, \end{aligned} \tag{44}$$

where $f^{(1)}$, $F^{(1)}$, and $f^{(2)}$ are transmission and boundary data in the formulation of the problem (8)-(12).

Note that due to the mapping properties of the operators involved in (43)-(44), we can conclude that if

$$f^{(1)} \in [B^{1-\frac{1}{p}}_{p,q}(S_1)]^6, \quad f^{(2)} \in [B^{1-\frac{1}{p}}_{p,q}(S_2)]^6, \quad F^{(1)} \in [B^{-\frac{1}{p}}_{p,q}(S_1)]^6, \ \ p > 1, \ \ q \geqslant 1,$$

then

$$U^{(1)} \in [B^{1}_{p,q}(\Omega^{(1)})]^6, \qquad U^{(2)} \in [B^{1}_{p,q}(\Omega^{(2)})]^6. \tag{45}$$

Invertibility of the operator $\mathbb{M}$ in (32) implies that any pair of solutions to the Problem (TD) satisfying the inclusion (45) is representable in the form (43)-(44) and this representation is unique.

4 | GENERAL MIXED BOUNDARY-TRANSMISSION PROBLEM

Let us consider the most general mixed boundary-transmission problem for the composed solid $\overline{\Omega^{(1)}}\bigcup\overline{\Omega^{(2)}}$. To this end, we assume that the interface surface S_1 of the composed body under consideration contains an interfacial crack submanifold S_{1C} and the transmission submanifold S_{1T}, $S_1 = \overline{S}_{1C}\bigcup\overline{S}_{1T}$ with $S_{1C}\bigcap S_{1T} = \varnothing$, while the exterior boundary is divided into two parts, the Dirichlet and Neumann parts, $S_2 = \overline{S}_{2D}\bigcup\overline{S}_{2N}$ with $S_{2D}\bigcap S_{2N} = \varnothing$. Further, let $\ell_1 = \partial S_{1C} = \partial S_{1T}$ and $\ell_2 = \partial S_{2D} = \partial S_{2N}$. Again, for simplicity, we assume that S_1, S_2, ℓ_1, and ℓ_2 are infinitely smooth if not otherwise stated.

Mixed Boundary-Transmission Problem (MBT). *Find solutions*

$$U^{(1)} \in [W_p^1(\Omega^{(1)})]^6, \quad U^{(2)} \in [W_p^1(\Omega^{(2)})]^6, \quad p>1,$$

to the pseudo-oscillation equations of the GTEME theory

$$A^{(\beta)}(\partial_x,\tau)\,U^{(\beta)}(x)=0\,, \qquad x\in\Omega^{(\beta)}, \quad \beta=1,2, \tag{46}$$

satisfying the transmission conditions on the interface part S_{1T}

$$\{U^{(1)}(x)\}^+ - \{U^{(2)}(x)\}^- = f^{(T)}(x), \quad x\in S_{1T}, \tag{47}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}^+ - \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^- = F^{(T)}(x), \quad x\in S_{1T}, \tag{48}$$

the interfacial crack conditions on S_{1C}

$$\begin{aligned}
&\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}^+ = F^{(C+)}(x), \quad x\in S_{1C},\\
&\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^- = F^{(C-)}(x), \quad x\in S_{1C},
\end{aligned} \tag{49}$$

the Dirichlet condition on S_{2D}

$$\{U^{(2)}(x)\}^+ = f^{(D)}(x), \quad x\in S_{2D}, \tag{50}$$

and the Neumann condition on S_{2N}

$$\{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^+ = F^{(N)}(x), \quad x\in S_{2N}. \tag{51}$$

The transmission and boundary data belong to the natural function spaces,

$$\begin{aligned}
&f^{(T)} \in [B_{p,p}^{1-\frac{1}{p}}(S_{1T})]^6, \quad f^{(D)} \in [B_{p,p}^{1-\frac{1}{p}}(S_{2D})]^6, \quad F^{(T)} \in [B_{p,p}^{-\frac{1}{p}}(S_{1T})]^6,\\
&F^{(C+)}, F^{(C-)} \in [B_{p,p}^{-\frac{1}{p}}(S_{1C})]^6, \quad F^{(N)} \in [B_{p,p}^{-\frac{1}{p}}(S_{2N})]^6.
\end{aligned} \tag{52}$$

Theorem 5. Let S_1, S_2, ℓ_1, and ℓ_2 be Lipschitz, conditions (52) be satisfied with $p=2$, and the relation (13) hold. Then the mixed boundary-transmission problem (MBT) (46)-(51) possesses at most one solution in the space $[W_2^1(\Omega^{(1)})]^6\times[W_2^1(\Omega^{(2)})]^6$.

Proof. Actually, the proof of the theorem is quite similar to the proof of Theorem 8.1 in the reference Buchukuri etc[53] and can be verbatim performed. □

Note that the transmission and interfacial crack conditions (48) and (49) can be rewritten in the following equivalent form

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}^+ - \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^- = F^{(1)}(x), \ \ x\in S_1, \tag{53}$$

$$\{\mathcal{T}^{(1)}(\partial_x,n,\tau)U^{(1)}(x)\}^+ + \{\mathcal{T}^{(2)}(\partial_x,n,\tau)U^{(2)}(x)\}^- = F^{(C+)}(x)+F^{(C-)}(x), \ x\in S_{1C}, \tag{54}$$

where

$$F^{(1)}(x)=\begin{cases} F^{(T)}(x), & x\in S_{1T},\\ F^{(C+)}(x)-F^{(C-)}(x), & x\in S_{1C}.\end{cases} \tag{55}$$

Therefore, in the above formulation of the problem (MBT), instead of conditions (48) and (49) we consider the relations (53) and (54).

We assume that the following necessary condition is satisfied

$$F^{(1)} \in [B_{p,p}^{-\frac{1}{p}}(S_1)]^6. \tag{56}$$

Further, let $\widetilde{f}^{(T)}$ and $\widetilde{f}^{(D)}$ be some fixed extensions of the vector functions $f^{(T)} \in [B_{p,p}^{1-\frac{1}{p}}(S_{1T})]^6$ and $f^{(D)} \in [B_{p,p}^{1-\frac{1}{p}}(S_{2D})]^6$ respectively preserving the spaces. Then we can assume that

$$\{U^{(1)}(x)\}^+ - \{U^{(2)}(x)\}^- = \widetilde{f}^{(T)}(x) + \widetilde{g}(x) \quad \text{on} \quad S_1,$$
$$\{U^{(2)}(x)\}^+ = \widetilde{f}^{(D)}(x) + \widetilde{h}(x) \quad \text{on} \quad S_2,$$

where

$$\widetilde{f}^{(T)} \in [B_{p,p}^{1-\frac{1}{p}}(S_1)]^6, \qquad \widetilde{f}^{(D)} \in [B_{p,p}^{1-\frac{1}{p}}(S_2)]^6 \tag{57}$$

are known vector functions, while

$$\widetilde{g} \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_{1C})]^6, \qquad \widetilde{h} \in [\widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_{2N})]^6 \tag{58}$$

are unknown vector functions.

Motivated by the results obtained in Section 3, we look for solution pair to the problem (MBT) in the form (see Remark 2):

$$U^{(1)}(x) = V_{S_1}^{(1)}\Big(\mathbb{M}^{11}[\widetilde{f}^{(T)} + \widetilde{g}] + \mathbb{M}^{12}F^{(1)} + \mathbb{M}^{13}[\widetilde{f}^{(D)} + \widetilde{h}]\Big)(x) \quad \text{in} \quad \Omega^{(1)}, \tag{59}$$

$$\begin{aligned} U^{(2)}(x) = {} & V_{S_1}^{(2)}\Big(\mathbb{M}^{21}[\widetilde{f}^{(T)} + \widetilde{g}] + \mathbb{M}^{22}F^{(1)} + \mathbb{M}^{23}[\widetilde{f}^{(D)} + \widetilde{h}]\Big)(x) \\ & + V_{S_2}^{(2)}\Big(\mathbb{M}^{31}[\widetilde{f}^{(T)} + \widetilde{g}] + \mathbb{M}^{32}F^{(1)} + \mathbb{M}^{33}[\widetilde{f}^{(D)} + \widetilde{h}]\Big)(x) \quad \text{in} \quad \Omega^{(2)}, \end{aligned} \tag{60}$$

where the operators $\mathbb{M}^{lm}$ are defined in Remark 2, $\widetilde{f}^{(T)}$, $\widetilde{f}^{(D)}$, and $F^{(1)}$ are above introduced known vector functions, while $\widetilde{g}$ and $\widetilde{h}$ are unknown vector functions.

In accordance with formulas (43)-(44), derived in Remark 2, the vectors (59) and (60) under conditions (52), (55), (56), (57), and (58) belong to the spaces $[W_p^1(\Omega^{(1)})]^6$ and $[W_p^1(\Omega^{(2)})]^6$ respectively and automatically satisfy conditions (46), (47), (50), and (53). It remains to satisfy conditions (51) and (54) which lead to the following system of pseudodifferential equations with respect to the unknown vectors $\widetilde{g}$ and $\widetilde{h}$:

$$\begin{aligned} r_{S_{1C}}\Big[& \Big(-\frac{1}{2}I_6 + \mathcal{K}_{S_1}^{(1)}\Big)\Big(\mathbb{M}^{11}\widetilde{g} + \mathbb{M}^{13}\widetilde{h}\Big) + \Big(\frac{1}{2}I_6 + \mathcal{K}_{S_1}^{(2)}\Big)\Big(\mathbb{M}^{21}\widetilde{g} + \mathbb{M}^{23}\widetilde{h}\Big) \\ & + \Big\{\mathcal{T}^{(2)}V_{S_2}^{(2)}\Big(\mathbb{M}^{31}\widetilde{g} + \mathbb{M}^{33}\widetilde{h}\Big)\Big\}^-\Big] = G^{(1)} \quad \text{on} \quad S_{1C}, \end{aligned}$$

$$r_{S_{2N}}\Big[\Big\{\mathcal{T}^{(2)}V_{S_1}^{(2)}\Big(\mathbb{M}^{21}\widetilde{g} + \mathbb{M}^{23}\widetilde{h}\Big)\Big\}^+ + \Big(-\frac{1}{2}I_6 + \mathcal{K}_{S_2}^{(2)}\Big)\Big(\mathbb{M}^{31}\widetilde{g} + \mathbb{M}^{33}\widetilde{h}\Big)\Big] = G^{(2)} \quad \text{on} \quad S_{2N},$$

where the right hand sides are known vector functions

$$\begin{aligned} G^{(1)} = {} & F^{(C+)} + F^{(C-)} - r_{S_{1C}}\Big[\Big(-\frac{1}{2}I_6 + \mathcal{K}_{S_1}^{(1)}\Big)\Big(\mathbb{M}^{11}\widetilde{f}^{(T)} + \mathbb{M}^{12}F^{(1)} + \mathbb{M}^{13}\widetilde{f}^{(D)}\Big) \\ & + \Big(\frac{1}{2}I_6 + \mathcal{K}_{S_1}^{(2)}\Big)\Big(\mathbb{M}^{21}\widetilde{f}^{(T)} + \mathbb{M}^{22}F^{(1)} + \mathbb{M}^{23}\widetilde{f}^{(D)}\Big) \\ & + \Big\{\mathcal{T}^{(2)}V_{S_2}^{(2)}\Big(\mathbb{M}^{31}\widetilde{f}^{(T)} + \mathbb{M}^{32}F^{(1)} + \mathbb{M}^{33}\widetilde{f}^{(D)}\Big)\Big\}_{S_1}^-\Big] \in [B_{p,p}^{-\frac{1}{p}}(S_{1C})]^6, \end{aligned}$$

$$\begin{aligned} G^{(2)} = {} & F^{(N)} - r_{S_{2N}}\Big[\Big\{\mathcal{T}^{(2)}V_{S_1}^{(2)}\Big(\mathbb{M}^{21}\widetilde{f}^{(T)} + \mathbb{M}^{22}F^{(1)} + \mathbb{M}^{23}\widetilde{f}^{(D)}\Big)\Big\}_{S_2}^+ \\ & + \Big(-\frac{1}{2}I_6 + \mathcal{K}_{S_2}^{(2)}\Big)\Big(\mathbb{M}^{31}\widetilde{f}^{(T)} + \mathbb{M}^{32}F^{(1)} + \mathbb{M}^{33}\widetilde{f}^{(D)}\Big)\Big] \in [B_{p,p}^{-\frac{1}{p}}(S_{2N})]^6. \end{aligned}$$

Thanks to the relations (26) and (34)-(42), and taking into account the equality (cf., Buchukuri etc[53], formula (8.67))

$$\mathcal{A}_{S_1}^{(1)}\big[\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(2)}\big]^{-1}\mathcal{A}_{S_1}^{(2)} = \mathcal{A}_{S_1}^{(2)}\big[\mathcal{A}_{S_1}^{(1)} - \mathcal{A}_{S_1}^{(2)}\big]^{-1}\mathcal{A}_{S_1}^{(1)},$$

we can rewrite the above system as follows

$$r_{_{S_{1C}}}\Big[-2\mathcal{A}^{(1)}_{S_1}\big[\mathcal{A}^{(1)}_{S_1}-\mathcal{A}^{(2)}_{S_1}\big]^{-1}\mathcal{A}^{(2)}_{S_1}\widetilde{g}+\Lambda^{11}\widetilde{g}+\Lambda^{12}\widetilde{h}\Big]=G^{(1)}\quad\text{on}\quad S_{1C},\tag{61}$$

$$r_{_{S_{2N}}}\Big[\mathcal{B}^{(2)}_{S_2}\widetilde{h}+\Lambda^{21}\widetilde{g}+\Lambda^{22}\widetilde{h}\Big]=G^{(2)}\quad\text{on}\quad S_{2N},\tag{62}$$

where

$$\begin{aligned}
&\mathcal{B}^{(2)}_{S_2}:=\Big(-\frac{1}{2}I_6+\mathcal{K}^{(2)}_{S_2}\Big)[\mathcal{H}^{(2)}_{S_2}]^{-1},\\
&\Lambda^{11}=\Big(-\frac{1}{2}I_6+\mathcal{K}^{(1)}_{S_1}\Big)\mathbb{L}^{11}+\Big(\frac{1}{2}I_6+\mathcal{K}^{(2)}_{S_1}\Big)\mathbb{L}^{21}+r_{_{S_1}}\mathcal{T}^{(2)}V^{(2)}_{S_2}\mathbb{L}^{31},\\
&\Lambda^{12}=\Big(-\frac{1}{2}I_6+\mathcal{K}^{(1)}_{S_1}\Big)\mathbb{L}^{13}+\Big(\frac{1}{2}I_6+\mathcal{K}^{(2)}_{S_1}\Big)\mathbb{L}^{23}+r_{_{S_1}}\mathcal{T}^{(2)}V^{(2)}_{S_2}\mathbb{M}^{33},\\
&\Lambda^{21}=r_{_{S_2}}\mathcal{T}^{(2)}V^{(2)}_{S_1}\mathbb{M}^{21}+\Big(-\frac{1}{2}I_6+\mathcal{K}^{(2)}_{S_2}\Big)\mathbb{L}^{31},\\
&\Lambda^{22}=r_{_{S_2}}\mathcal{T}^{(2)}V^{(2)}_{S_1}\mathbb{M}^{23}+\Big(-\frac{1}{2}I_6+\mathcal{K}^{(2)}_{S_2}\Big)\mathbb{L}^{33}.
\end{aligned}$$

Note that $\mathcal{B}^{(2)}_{S_2}$ is the Steklov-Poincré type pseudodifferential operator of order 1, while the operators Λ^{kj}, $k,j=1,2$, are infinitely smoothing operators.

Further, let us introduce the operators

$$\mathbb{T}=\begin{bmatrix} r_{_{S_{1C}}}\Big[-2\mathcal{A}^{(1)}_{S_1}\big[\mathcal{A}^{(1)}_{S_1}-\mathcal{A}^{(2)}_{S_1}\big]^{-1}\mathcal{A}^{(2)}_{S_1}\Big] & \mathbf{0}\\ \mathbf{0} & r_{_{S_{2N}}}\mathcal{B}^{(2)}_{S_2}\end{bmatrix},\qquad \mathbb{Q}=\begin{bmatrix} r_{_{S_{1C}}}\Lambda^{11} & r_{_{S_{1C}}}\Lambda^{12}\\ r_{_{S_{2N}}}\Lambda^{21} & r_{_{S_{2N}}}\Lambda^{22}\end{bmatrix},$$

and rewrite system (61)-(62) in matrix form

$$(\mathbb{T}+\mathbb{Q})\widetilde{\Phi}=\widetilde{G}$$

with

$$\begin{aligned}
\widetilde{\Phi}&:=(\widetilde{g},\widetilde{h})^\top\in[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_{1C})]^6\times[\widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_{2N})]^6,\\
\widetilde{G}&:=(G^{(1)},G^{(2)})^\top\in[B^{-\frac{1}{p}}_{p,p}(S_{1C})]^6\times[B^{-\frac{1}{p}}_{p,p}(S_{2N})]^6.
\end{aligned}$$

Due to Theorems 10-13 in Appendix, we have the following mapping property

$$\begin{gathered}\mathbb{T}:[\widetilde{B}^{s+1}_{p,q}(S_{1C})]^6\times[\widetilde{B}^{s+1}_{p,q}(S_{2N})]^6\to[B^{s}_{p,q}(S_{1C})]^6\times[B^{s}_{p,q}(S_{2N})]^6,\\ s\in\mathbb{R},\ p>1,\ q\geq 1.\end{gathered}\tag{63}$$

Note that the operator $\mathbb{Q}$ considered between the function spaces involved in (63),

$$\begin{gathered}\mathbb{Q}:[\widetilde{B}^{s+1}_{p,q}(S_{1C})]^6\times[\widetilde{B}^{s+1}_{p,q}(S_{2N})]^6\to[B^{s}_{p,q}(S_{1C})]^6\times[B^{s}_{p,q}(S_{2N})]^6,\\ s\in\mathbb{R},\ p>1,\ q\geq 1,\end{gathered}\tag{64}$$

is an infinitely smoothing compact operator (see Remark 2). Therefore the Fredholm properties of the operator

$$\begin{gathered}\mathbb{T}+\mathbb{Q}:[\widetilde{B}^{s+1}_{p,q}(S_{1C})]^6\times[\widetilde{B}^{s+1}_{p,q}(S_{2N})]^6\to[B^{s}_{p,q}(S_{1C})]^6\times[B^{s}_{p,q}(S_{2N})]^6,\\ s\in\mathbb{R},\ p>1,\ q\geq 1,\end{gathered}\tag{65}$$

coincides with the Fredholm properties of the operator (63).

In what follows, we apply the theory of pseudodifferential operators on manifolds with boundary (see, e.g., Eskin[57], Shargorodsky[58]) and show that operator (65) is invertible under some restrictions on the parameters s and p.

Denote by $\mathfrak{S}(\mathcal{A}_{S_1};x,\xi_1,\xi_2)$ the principal homogeneous symbol matrix of the operator

$$\mathcal{A}_{S_1}:=\mathcal{A}^{(1)}_{S_1}\big[\mathcal{A}^{(1)}_{S_1}-\mathcal{A}^{(2)}_{S_1}\big]^{-1}\mathcal{A}^{(2)}_{S_1}$$

and let $\lambda^{(1)}_j(x)$ $(j=\overline{1,6})$ be the eigenvalues of the matrix

$$\mathcal{D}_1(x):=[\mathfrak{S}(\mathcal{A}_{S_1};x,0,+1)]^{-1}\,\mathfrak{S}(\mathcal{A}_{S_1};x,0,-1),\quad x\in\ell_1=\partial S_{1C}.$$

Similarly, let $\mathfrak{S}(\mathcal{B}^{(2)}_{S_2}; x, \xi_1, \xi_2)$ be the principal homogeneous symbol matrix of the operator $\mathcal{B}^{(2)}_{S_2}$ and let $\lambda^{(2)}_j(x)$ $(j = \overline{1,6})$ be the eigenvalues of the matrix

$$\mathcal{D}_2(x) := [\mathfrak{S}(\mathcal{B}^{(2)}_{S_2}; x, 0, +1)]^{-1}\, \mathfrak{S}(\mathcal{B}^{(2)}_{S_2}; x, 0, -1), \quad x \in \ell_2 = \partial S_{2N}.$$

Further, let

$$\gamma'_1 := \inf_{x\in\ell_1,\, 1\le j\le 6} \frac{1}{2\pi} \arg \lambda^{(1)}_j(x), \qquad \gamma''_1 := \sup_{x\in\ell_1,\, 1\le j\le 6} \frac{1}{2\pi} \arg \lambda^{(1)}_j(x),$$
$$\gamma'_2 := \inf_{x\in\ell_2,\, 1\le j\le 6} \frac{1}{2\pi} \arg \lambda^{(2)}_j(x), \qquad \gamma''_2 := \sup_{x\in\ell_2,\, 1\le j\le 6} \frac{1}{2\pi} \arg \lambda^{(2)}_j(x),$$
$$\gamma' := \min\{\gamma'_1, \gamma'_2\}, \qquad \gamma'' := \max\{\gamma''_1, \gamma''_2\}. \tag{66}$$

By the same arguments, as in Buchukuri etc[53], Subsection 5.7, we can show that one of the eigenvalues of the matrix $\mathcal{D}_2(x)$ equals to 1, which leads to the inequalities

$$-\frac{1}{2} < \gamma' \le 0 \le \gamma'' < \frac{1}{2}.$$

Theorem 6. Let the following inequalities hold

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma'. \tag{67}$$

Then operator (65) is invertible.

Proof. If inequalities (67) are satisfied, then the operators

$$r_{S_{1C}} \mathcal{A}_{S_1} : [\widetilde{B}^{s+1}_{p,q}(S_{1C})]^6 \to [B^s_{p,q}(S_{1C})]^6,$$
$$r_{S_{2N}} \mathcal{B}^{(2)}_{S_2} : [\widetilde{B}^{s+1}_{p,q}(S_{2N})]^6 \to [B^s_{p,q}(S_{2N})]^6,$$

are invertible (cf. Lemma 5.20 and Theorems 8.9 in Buchukuri etc[53]). Therefore operator (63) is invertible for s and p satisfying the above inequalities and consequently (65) is Fredholm with zero index in view of compactness of the operator $\mathbb{Q}$ in (64). Note that $s = -\frac{1}{2}$ and $p = 2$ satisfy inequality (67). Now we show that the null space of operator (65) is trivial for $s = -\frac{1}{2}$ and $p = q = 2$.

Indeed, let a pair $\widetilde{\Phi} = (\widetilde{g}, \widetilde{h}) \in [\widetilde{B}^{\frac{1}{2}}_{2,2}(S_{1C})]^6 \times [\widetilde{B}^{\frac{1}{2}}_{2,2}(S_{2N})]^6 = [\widetilde{H}^{\frac{1}{2}}_2(S_{1C})]^6 \times [\widetilde{H}^{\frac{1}{2}}_2(S_{2N})]^6$ be a solution to the homogeneous equation

$$(\mathbb{T} + \mathbb{Q})\widetilde{\Phi} = 0 \tag{68}$$

and construct the vectors

$$\widetilde{U}^{(1)}(x) = V^{(1)}_{S_1}\left(\mathbb{M}^{11}\widetilde{g} + \mathbb{M}^{13}\widetilde{h}\right)(x) \quad \text{in} \quad \Omega^{(1)},$$

$$\widetilde{U}^{(2)}(x) = V^{(2)}_{S_1}\left(\mathbb{M}^{21}\widetilde{g} + \mathbb{M}^{23}\widetilde{h}\right)(x) + V^{(2)}_{S_2}\left(\mathbb{M}^{31}\widetilde{g} + \mathbb{M}^{33}\widetilde{h}\right)(x) \quad \text{in} \quad \Omega^{(2)}.$$

Evidently, $\widetilde{U}^{(1)} \in [W^1_2(\Omega^{(1)})]^6$ and $\widetilde{U}^{(2)} \in [W^1_2(\Omega^{(2)})]^6$. Using the results presented in Remark 2 and taking into account equation (68), we conclude that the vectors $\widetilde{U}^{(1)}$ and $\widetilde{U}^{(2)}$ solve the homogeneous Problem (MBT). Therefore $\widetilde{U}^{(1)} = 0$ in $\Omega^{(1)}$ and $\widetilde{U}^{(2)} = 0$ in $\Omega^{(2)}$ in view of the uniqueness Theorem 5. These equations imply that the densities of the potentials vanish on the corresponding manifolds, i.e.,

$$\mathbb{M}^{11}\widetilde{g} + \mathbb{M}^{13}\widetilde{h} = 0 \;\text{ on }\; S_1,$$
$$\mathbb{M}^{21}\widetilde{g} + \mathbb{M}^{23}\widetilde{h} = 0 \;\text{ on }\; S_1,$$
$$\mathbb{M}^{31}\widetilde{g} + \mathbb{M}^{33}\widetilde{h} = 0 \;\text{ on }\; S_2,$$

which can be rewritten as

$$\begin{bmatrix} \mathbb{M}^{11} & \mathbb{M}^{12} & \mathbb{M}^{13} \\ \mathbb{M}^{21} & \mathbb{M}^{22} & \mathbb{M}^{23} \\ \mathbb{M}^{31} & \mathbb{M}^{32} & \mathbb{M}^{33} \end{bmatrix} \begin{bmatrix} \widetilde{g} \\ \widetilde{h}_0 \\ \widetilde{h} \end{bmatrix} = 0 \tag{69}$$

with $\widetilde{h}_0 = (0,0,0)^\top$ on S_1.

In accordance with Theorem 3 and Remark 2 the operator $\mathbb{M} = \big[\mathbb{M}^{kj}\big]_{18\times 18}$ in the left hand side of (69) is invertible from the space $\mathbb{Y}_{2,2}^{-\frac{1}{2}} = \big[B_{2,2}^{\frac{1}{2}}(S_1)\big]^6\times\big[B_{2,2}^{-\frac{1}{2}}(S_1)\big]^6\times\big[B_{2,2}^{\frac{1}{2}}(S_2)\big]^6$ into the space $\mathbb{X}_{2,2}^{-\frac{1}{2}} = \big[B_{2,2}^{-\frac{1}{2}}(S_1)\big]^6\times\big[B_{2,2}^{-\frac{1}{2}}(S_1)\big]^6\times\big[B_{2,2}^{-\frac{1}{2}}(S_2)\big]^6$. Consequently, we deduce $\widetilde{g} = 0$ and $\widetilde{h} = 0$. Thus, the null space of operator (65) is trivial for $s = -\frac{1}{2}$ and $p = q = 2$ and consequently it is invertible for $s = -\frac{1}{2}$ and $p = q = 2$. Then by the general theory of pseudodifferential equations on manifolds with boundary we conclude that operator (65) is invertible for all s and p satisfying inequality (67) for all $q \geqslant 1$ (see, e.g., Theorem B1 in Buchukuri etc [53]). This completes the proof of the theorem. □

This invertibility theorem leads to the following existence result.

Theorem 7. Let the following inequality hold

$$\frac{4}{3-2\gamma''} < p < \frac{4}{1-2\gamma'}$$

with γ' and γ'' defined in (66).
Then the mixed transmission problem (MBT) has a unique solution

$$(U^{(1)}, U^{(2)}) \in [W_p^1(\Omega^{(1)})]^6 \times [W_p^1(\Omega^{(2)})]^6,$$

which can be represented by the single layer potentials (59)-(60).

Proof. It directly follows from Theorem 6 and Remark 2. □

With the help of the arguments applied in the proof of Theorem 5.22 in Buchukuri etc [53], we can deduce the following regularity result.

Theorem 8. Let $\alpha > 0$ and

$$\begin{aligned} &f^{(D)} \in [C^{\alpha}(S_{2D})]^6, && F^{(N)} \in [B^{\alpha-1}_{\infty,\infty}(S_{2N})]^6,\\ &f^{(T)} \in [C^{\alpha}(S_{1T})]^6, && F^{(T)} \in [B^{\alpha-1}_{\infty,\infty}(S_{1T})]^6,\\ &F^{(C+)},\ F^{(C-)} \in [B^{\alpha-1}_{\infty,\infty}(S_{1C})]^6. && \end{aligned}$$

Then

$$U^{(\beta)} \in \bigcap_{\alpha'<\kappa} [\,C^{\alpha'}(\,\overline{\Omega^{(\beta)}}\,)]^6, \qquad \beta = 1,2,$$

where $0 < \kappa = \min\{\alpha,\ \gamma' + \frac{1}{2}\} \leqslant \frac{1}{2}$.

It is evident that the smoothness exponent κ essentially depends on the material parameters and it can be explicitly determined by the principal homogeneous symbol matrix of the pseudodifferential operator $\mathbb{T}$.

5 | APPENDIX

Here we collect some results which are employed in the main text of the paper. Proofs of the theorems presented in this Appendix can be found in the reference Buchukuri etc [53].

Let S be a closed simply connected surface surrounding a bounded region $\Omega^+ = \Omega^{(1)}$ and $\Omega^- = \Omega^{(2)} = \mathbb{R}^3 \setminus \overline{\Omega^+}$. Assume that the domains $\Omega^{(1)}$ and $\Omega^{(2)}$ are occupied by anisotropic homogeneous materials possessing different thermo-electro-magneto-elastic properties described in the main text (see Section 3).

Fundamental matrices $\Gamma^{(\beta)}(x,\tau)$ of the operators $A^{(\beta)}(\partial_x,\tau)$, $\beta = 1,2$, for $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$, read as

$$\Gamma^{(\beta)}(x,\tau) = \mathcal{F}^{-1}_{\xi\to x}\big[\big(A^{(\beta)}(-i\,\xi,\tau)\big)^{-1}\big],$$

where $\big(A^{(\beta)}(-i\,\xi,\tau)\big)^{-1}$ is the matrix inverse to $A^{(\beta)}(-i\,\xi,\tau)$, $\mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse distributional Fourier transforms in the space of tempered distributions which for regular summable functions f and g read as follows

$$\mathcal{F}_{x\to\xi}[f] = \int\limits_{\mathbb{R}^3} f(x)\,e^{i\,x\cdot\xi}dx, \qquad \mathcal{F}^{-1}_{\xi\to x}[g\,] = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{R}^3} g(\xi)\,e^{-i\,x\cdot\xi}d\xi.$$

These fundamental matrices solve the following distributional equations

$$A^{(\beta)}(\partial_x,\tau)\Gamma^{(\beta)}(x,\tau) = I_6\,\delta(x),$$

where I_6 is 6×6 unit matrix and $\delta(x)$ is Dirac's distribution. The entries of the matrix $\Gamma^{(\beta)}(x,\tau)$ in a vicinity of the origin have the property

$$\Gamma^{(\beta)}(x,\tau) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{5\times 5} & [\mathcal{O}(1)]_{5\times 1} \\ [\mathcal{O}(1)]_{1\times 5} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{6\times 6},$$

while at infinity they have the following asymptotic behaviour

$$\Gamma^{(\beta)}(x,\tau) = \begin{bmatrix} \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\ \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\ \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-5}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-4}) \\ \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2}) \\ \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-3}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-1}) & \mathcal{O}(|x|^{-2}) \\ \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-4}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-2}) & \mathcal{O}(|x|^{-3}) \end{bmatrix}_{6\times 6}.$$

Introduce the generalized single layer potential

$$V_S^{(\beta)}(g)(x) = \int\limits_S \Gamma^{(\beta)}(x-y,\tau)\,g(y)\,dS_y, \quad \beta = 1,2, \quad x\in\mathbb{R}^3\setminus S, \tag{70}$$

where $g = (g_1,\cdots,g_6)^\top$ is a density vector function defined on the integration surface S.

Theorem 9. Let $S\in C^{m,\alpha'}$, $0<\alpha<\alpha'\leqslant 1$, and let $m\geqslant 1$ and $k\leqslant m-1$ be nonnegative integers. Then the operator

$$V_S^{(\beta)} \;:\; [C^{k,\alpha}(S)]^6 \to [C^{k+1,\alpha}(\overline{\Omega^\pm})]^6$$

is continuous.

For any $g\in[C^{0,\alpha}(S)]^6$ and for any $x\in S$ the following jump relations hold

$$\{V_S^{(\beta)}(g)(x)\}^\pm = \mathcal{H}_S^{(\beta)}\,g(x), \tag{71}$$

$$\{\mathcal{T}^{(\beta)}(\partial_x,n(x),\tau)\,V_S^{(\beta)}(g)(x)\}^\pm = [\,\mp 2^{-1} I_6 + \mathcal{K}_S^{(\beta)}\,]\,g(x), \tag{72}$$

where $\mathcal{H}_S^{(\beta)}$ is a weakly singular integral operator

$$\mathcal{H}_S^{(\beta)}\,g(x) := \int\limits_S \Gamma^{(\beta)}(x-y,\tau)\,g(y)\,dS_y\,, \quad x\in S, \tag{73}$$

while $\mathcal{K}_S^{(\beta)}$ is a singular integral operator

$$\mathcal{K}_S^{(\beta)}\,g(x) := \int\limits_S [\,\mathcal{T}^{(\beta)}(\partial_x,n(x),\tau)\,\Gamma^{(\beta)}(x-y,\tau)\,]\,g(y)\,dS_y\,, \quad x\in S. \tag{74}$$

The following operators are continuous

$$\begin{aligned} \mathcal{H}_S^{(\beta)} &\;:\; [C^{k,\alpha}(S)]^6 \to [C^{k+1,\alpha}(S)]^6\,, \\ \mathcal{K}_S^{(\beta)} &\;:\; [C^{k,\alpha}(S)]^6 \to [C^{k,\alpha}(S)]^6\,. \end{aligned} \tag{75}$$

Moreover, operator (75) is invertible.

Theorem 10. Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The operators $V_S^{(\beta)}$, $\mathcal{H}_S^{(\beta)}$, and $\mathcal{K}_S^{(\beta)}$, can be extended to the following continuous operators

$$\begin{aligned}
V_S^{(\beta)} &: [B^s_{p,p}(S)]^6 \to [H_p^{s+1+\frac{1}{p}}(\Omega^+)]^6 \ \Big[\,[B^s_{p,p}(S)]^6 \to [H_{p,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6\,\Big],\\
V_S^{(\beta)} &: [B^s_{p,q}(S)]^6 \to [B_{p,q}^{s+1+\frac{1}{p}}(\Omega^+)]^6 \ \Big[\,[B^s_{p,q}(S)]^6 \to [B_{p,q,loc}^{s+1+\frac{1}{p}}(\Omega^-)]^6\,\Big],\\
\mathcal{H}_S^{(\beta)} &: [H^s_p(S)]^6 \to [H^{s+1}_p(S)]^6 \qquad \Big[\,[B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6\,\Big],\\
\mathcal{K}_S^{(\beta)} &: [H^s_p(S)]^6 \to [H^{s}_p(S)]^6 \qquad \Big[\,[B^s_{p,q}(S)]^6 \to [B^{s}_{p,q}(S)]^6\,\Big].
\end{aligned}$$

For $s > -1$ the jump relations (71)-(72) remain valid in appropriate function spaces.
The operator

$$\mathcal{H}_S^{(\beta)} : [B^s_{p,q}(S)]^6 \to [B^{s+1}_{p,q}(S)]^6, \qquad s \in \mathbb{R}, \ \ p > 1, \ \ q \ge 1,$$

is invertible.

Theorem 11. Let $S = \partial\Omega \in C^{2,\alpha}$ with $\alpha > 0$. An arbitrary solution to the homogeneous equation

$$A^{(\beta)}(\partial,\tau)U^{(\beta)} = 0 \ \text{ in } \ \Omega, \quad U^{(\beta)} \in [W^1_p(\Omega)]^6, \ \ p > 1,$$

is uniquely representable in the form

$$U^{(\beta)} = V_S^{(\beta)}\big(\big[\mathcal{H}_S^{(\beta)}\big]^{-1} g\big) \ \text{ with } \ g = \{U^{(\beta)}\}_S^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^6.$$

The Steklov-Poincaré type operators are defined by the following formulas

$$\mathcal{A}_S^{(1)} := \big(-2^{-1} I_6 + \mathcal{K}_S^{(1)}\big)\big[\mathcal{H}_S^{(1)}\big]^{-1},$$

$$\mathcal{A}_S^{(2)} := \big(2^{-1} I_6 + \mathcal{K}_S^{(2)}\big)\big[\mathcal{H}_S^{(2)}\big]^{-1},$$

and they are related to the single layer potentials by the relations

$$\mathcal{A}_S^{(1)} g = \Big\{\mathcal{T}^{(1)}(\partial_x, n(x), \tau)\, V_S^{(1)}\big(\big[\mathcal{H}_S^{(1)}\big]^{-1} g\big)\Big\}_S^+,$$

$$\mathcal{A}_S^{(2)} g = \Big\{\mathcal{T}^{(2)}(\partial_x, n(x), \tau)\, V_S^{(2)}\big(\big[\mathcal{H}_S^{(2)}\big]^{-1} g\big)\Big\}_S^-.$$

Note that for arbitrary solution $U^{(\beta)} \in [W^1_p(\Omega^{(\beta)})]^6$ of the equation $A^{(\beta)}(\partial,\tau)U^{(\beta)} = 0$ in $\Omega^{(\beta)}$, $\beta = 1,2$, the Steklov-Poincaré type operators relate the Dirichlet data $\{U^{(\beta)}\}_S^\pm$ with the Neumann data $\{\mathcal{T}^{(\beta)}U^{(\beta)}\}_S^\pm$,

$$\{\mathcal{T}^{(1)}U^{(1)}\}_S^+ = \mathcal{A}_S^{(1)}\{U^{(1)}\}_S^+, \qquad \{\mathcal{T}^{(2)}U^{(2)}\}_S^- = \mathcal{A}_S^{(2)}\{U^{(2)}\}_S^-.$$

Theorem 12. Let $\tau = \sigma + i\omega$ with $\sigma > 0$ and $\omega \in \mathbb{R}$. Then for all $g \in [H_2^{\frac{1}{2}}(S)]^6$ there hold the coercivity inequalities

$$\mathrm{Re}\,\langle(\mathcal{A}_S^{(1)} + C^{(1)})g\,,\, g\rangle_S \ge C_1\,\|g\|^2_{[H_2^{\frac{1}{2}}(S)]^6},$$

$$\mathrm{Re}\,\langle(-\mathcal{A}_S^{(2)} + C^{(2)})g\,,\, g\rangle_S \ge C_2\,\|g\|^2_{[H_2^{\frac{1}{2}}(S)]^6},$$

where

$$C^{(1)} : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6 \ \text{ and } \ C^{(2)} : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6$$

are compact operators and C_j, $j = 1,2$, are positive constants.
The operator

$$\mathcal{A}_S^{(2)} : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6$$

is invertible, while

$$\mathcal{A}_S^{(1)} : [H_2^{\frac{1}{2}}(S)]^6 \to [H_2^{-\frac{1}{2}}(S)]^6$$

is a Fredholm operator of index zero with the null space spanned over the vectors

$$\Psi^{(1)} = (0,0,0,1,0,0)^\top, \quad \Psi^{(2)} = (0,0,0,0,1,0)^\top. \tag{76}$$

Theorem 13. Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. The operator

$$\mathcal{A}_S^{(2)} : [B_{p,q}^{s+1}(S)]^6 \to [B_{p,q}^{s}(S)]^6$$

is invertible, while the operator

$$\mathcal{A}_S^{(1)} : [B_{p,q}^{s+1}(S)]^6 \to [B_{p,q}^{s}(S)]^6$$

is Fredholm of zero index with a two-dimensional null-space spanned over the vectors (76).

The operator

$$\mathcal{A}_S^{(2)} - \mathcal{A}_S^{(1)} : [B_{p,q}^{s+1}(S)]^6 \to [B_{p,q}^{s}(S)]^6$$

is invertible.

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