Numerical and theoretical treatment based on the compact finite difference and spectral collocation algorithms of the space fractional-order Fisher's equation

M. Adel[1,2] and M. M. Khader[3,4]

[1]Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt

[2]Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madina, Saudi Arabia

[3]Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

[4]Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh: 11566, Saudi Arabia

mohamed.khader@fsc.bu.edu.eg, mohamedmbd@yahoo.com, adel@sci.cu.edu.eg

Abstract

This paper present an accurate numerical algorithm to solve the space fractional-order Fisher's equation where the derivative operator is described in the Caputo derivative sense. In the presented discretization process, first we use the compact finite difference (CFD) to occur a semi-discrete in time derivative, and implement the Chebyshev spectral collocation method (CSCM) of the third-kind to discretize the spatial fractional derivative. The presented method converts the studied problem to be a system of algebraic equations which can be easily solved. To study the convergence and stability analysis, some theorems are given with their profs. A numerical simulation is given to test the accuracy and the applicability of our presented algorithm.

Keywords: The fractional-order Fisher's equation; The compact finite difference method; Chebyshev-spectral collocation method; Convergence and stability analysis.

MSC 2010: 65N20; 41A30.

1. Introduction

In the past decades, fractional analysis [1] was developed rapidly. Its applications exist in many scientific areas, such as engineering, viscoelasticity, physics, diffusion processes, rheology, etc [2]. Fractional analysis is considered as a useful substitution model to classical calculus theory. Fractional differential equations (FDEs) including fixed, variable, distributed, tempered order have been used as perfect models for our real life problems including earthquake analysis, bio-chemical, electric circuits, signal processing, etc. ([3]-[5]). And so, the urgent necessity to find either the exact solutions or merely the approximate solutions to these problems that described many applications has emerged. Most FDEs do not have an exact solution; therefore numerical techniques must be used ([6]-[9]). For more details about fractional analysis, see ([10]-[14],).
Spectral methods are useful tools to develop the approximate solutions of FDEs. The main

advantage of these spectral methods is their capability to achieve accurate outcomes with a small degrees of freedom ([15], [16]). Orthogonal polynomials as Chebyshev polynomials have been utilized in the approximation of functions on the interval $[-1,1]$. These polynomials have an essential role in spectral methods for FDEs ([17]-[28]).

2. Preliminaries and notation

In this section, some definitions and notation about the fractional derivatives, the shifted Chebyshev polynomials of the third-kind, and the fractional-order Fisher's equation are presented to use them in the derivation of our presented scheme.

2.1 Some definitions of fractional derivatives

Definition 1.

The two-dimension integral operator of the Riemann-Liouville I^{α} of order $\alpha \in \mathbb{R}^{+}$ is defined as follows:

$$(I^{\alpha} f)(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau,t)\, d\tau, & \alpha > 0, \\ f(x,t), & \alpha = 0. \end{cases}$$

This integral operator satisfies the following properties [1]:

1. $I^{\alpha} I^{\beta} f(x,t) = I^{\beta} I^{\alpha} f(x,t) = I^{\alpha+\beta} f(x,t)$.
2. $I^{\alpha}(x-a)^{r} = \frac{\Gamma(r+1)}{\Gamma(\alpha+r+1)}(x-a)^{r+\alpha}$.
3. $I^{\alpha}\left(\sigma_1 f(x,t) + \sigma_2 g(x,t)\right) = \sigma_1 I^{\alpha} f(x,t) + \sigma_2 I^{\alpha} g(x,t)$, where σ_1 and σ_2 are constants.

Definition 2.

The Caputo fractional derivative D_x^{ν} of order $\nu \in \mathbb{R}^{+}$ of a function $\varphi(x,t)$ is defined as follows:

$$D_x^{\nu}\varphi(x,t) = \begin{cases} \frac{1}{\Gamma(n-\nu)} \int_0^x (x-\tau)^{n-\nu-1} \frac{\partial^n \varphi(\tau,t)}{\partial \tau^n}\, d\tau, & n-1 < \nu \le n \in \mathbb{N}, \\ \frac{\partial^n \varphi(x,t)}{\partial x^n}, & \nu = n. \end{cases}$$

This fractional derivative operator satisfies the following properties [1]:

1. $I^{\nu} D_x^{\nu} f(x,t) = f(x,t) - \sum_{j=0}^{n-1} \frac{1}{j!} \frac{\partial^j f(0^+,t)}{\partial x^j}, \qquad n-1 < \nu \le n \in \mathbb{N}.$
2. $\lim_{\nu \to n} D_x^{\nu} f(x,t) = \frac{\partial^n f(x,t)}{\partial x^n}, \qquad \lim_{\nu \to n-1} D_x^{\nu} f(x,t) = \frac{\partial^{n-1} f(x,t)}{\partial x^{n-1}} - \frac{\partial^{n-1} f(0,t)}{\partial x^{n-1}}.$
3. $D_x^{\nu}(x^r) = \frac{\Gamma(1+r)}{\Gamma(1+r-\nu)} x^{r-\nu}, \qquad 0 < \nu < r+1, \quad \nu > -1.$
4. $D_x^{\nu}\left(\sigma_1 f(x,t) + \sigma_2 g(x,t)\right) = \sigma_1 D_x^{\nu} f(x,t) + \sigma_2 D_x^{\nu} g(x,t)$, where σ_1 and σ_2 are constants.
5. $D_x^{\nu} D_x^{\mu} f(x,t) = D_x^{\nu+\mu} f(x,t) \neq D_x^{\nu} D_x^{\mu} f(x,t)$.
6. $D_x^{\nu}(C) = 0$, where C is a constant.

2.2 Shifted Chebyshev polynomials of the third-kind

The Chebyshev polynomials of the third-type of degree n are defined on the interval $[-1,1]$ with the help of the orthogonal Jacobi polynomials as follows [19]:

$$V_n(z) = \frac{(2^n\, n!)^2}{(2n)!} P_n^{(-\frac{1}{2},\frac{1}{2})}(z), \tag{1}$$

where,

$$P_n^{(l,m)}(z) = \frac{\Gamma(l+n+1)}{n!\Gamma(l+m+n+1)} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(l+m+n+k+1)}{\Gamma(l+k+1)} \left(\frac{z-1}{2}\right)^k. \tag{2}$$

These polynomials are orthogonal on $[-1,1]$ w.r.t. the weight function $w^C(z) = \sqrt{\frac{1+z}{1-z}}$ and satisfied the following condition of the inner product:

$$< V_r(z),\, V_s(z) >= \int_{-1}^{1} w^C(z) V_r(z) V_s(z) dz = \begin{cases} 0, & \text{if} \quad r \neq s; \\ \pi, & \text{if} \quad r = s. \end{cases}$$

The analytical form of $V_n(z)$ can be occurred by substitution from the formula (2) in (1), then we have:

$$V_n(z) = \frac{(2^{n-1}\,(n-1)!)^2\,\Gamma(n-\frac{1}{2})}{(2n-2)!\,(n-1)!\; n!} \sum_{j=0}^{n-1} \sum_{i=0}^{j} \frac{(-1)^i\, n!\Gamma(n+j)}{(n-j-1)!\, i!\,(j-i)!\,\Gamma(j+\frac{1}{2})\, 2^j}\; z^{j-i},$$

which can be rewritten in the following compact form:

$$V_n(z) = \Omega_n \sum_{j=0}^{n-1} \sum_{i=0}^{j} \Theta_n^{i,j}\, z^{j-i}, \qquad n = 1, 2, ..., N+1,$$

where,

$$\Omega_n = \frac{(2^{n-1}\,(n-1)!)^2 \Gamma(n-\frac{1}{2})}{(2n-2)!\,(n-1)!\,n!}, \qquad \Theta_n^{i,j} = \frac{(-1)^i\, n!\,\Gamma(n+j)}{(n-j-1)!\,i!\,(j-i)!\,\Gamma(j+\frac{1}{2})\,2^j}.$$

These functions will be used on $[0,1]$, so we should use the so-called shifted Chebyshev polynomials by implementing a linear transform as $z = 2x-1$. These polynomials will be expressed as follows: $\bar{\mathbb{V}}_n(x) = V_n(2x-1)$, where $\bar{\mathbb{V}}_0(x) = 1$, $\bar{\mathbb{V}}_1(x) = 4x-3$. The analytical form of the shifted Chebyshev polynomials $\bar{\mathbb{V}}_n(x)$ is given by the following formula:

$$\bar{\mathbb{V}}_n(x) = \Omega_n \sum_{j=0}^{n-1} \sum_{i=0}^{j} \Theta_n^{i,j}\, 2^j\; x^{j-i}, \qquad n = 1, 2, ..., N+1. \tag{3}$$

These resulting third-type shifted Chebyshev polynomials of degree n are orthogonal on $[0,1]$ in the following sense:

$$\int_0^1 \sqrt{\frac{x}{1-x}}\bar{\mathbb{V}}_r(x)\bar{\mathbb{V}}_s(x)\,dx = \begin{cases} 0, & \text{if} \quad r \neq s; \\ \frac{\pi}{2}, & \text{if} \quad r = s. \end{cases} \tag{4}$$

The common use of these polynomials is to express and approximate any function $f(x) \in L_2[0,1]$ as a sum of these polynomials as follows:

$$f_N(x) \simeq \sum_{n=0}^{N} a_n\,\bar{\mathbb{V}}_n(x), \qquad a_n = \frac{2}{\pi}\int_0^1 \sqrt{\frac{x}{1-x}} f(x)\,\bar{\mathbb{V}}_n(x)\,dx, \qquad n = 0, 1, ..., N. \tag{5}$$

In [20], Handan proved the uniformly convergent of the finite series (5) and derived a formula for the upper bound of the error if we approximate the function $f(x)$ by $f_N(x)$. Also, in this part, the analytical form of the shifted Chebyshev polynomials (3) and some properties of the Caputo fractional derivative D^ν-operator can be applied directly to derove an approximate formula of $D^\nu f_N(x)$, through the following theorem.

Theorem 1.

An approximate formula of $D^\nu\left(f_N(x)\right)$ can be give in the following analytical form:

$$D^\nu(f_N(x)) = \sum_{n=\lceil\nu\rceil}^{N}\sum_{j=0}^{n-\lceil\nu\rceil}\sum_{i=0}^{j} a_n \Pi_{i,j}^{n,\nu}\, x^{j-i-\nu+\lceil\nu\rceil}, \tag{6}$$

where,

$$\Pi_{i,j}^{n,\nu} = (-1)^i \left(\begin{array}{c} j+\lceil\nu\rceil \\ i \end{array}\right) \frac{2^{2n}\, n!\,\Gamma(n+j+\lceil\nu\rceil+1)\,\Gamma(n+\frac{1}{2})\,\Gamma(j-i+\lceil\nu\rceil+1)}{(2n)!\,(n-j-\lceil\nu\rceil)!\,(j+\lceil\nu\rceil)!\,\Gamma(j+\lceil\nu\rceil+\frac{1}{2})\,\Gamma(j-i-\nu+\lceil\nu\rceil+1)}.$$

2.3 The fractional Fisher's equation

The well known linear space fractional-order Fisher's equation (FFE) is given by:

$$u_t = \alpha\, D_x^\nu u + \beta(1-u(x,t)), \qquad 1 < \nu \leq 2, \tag{7}$$

where $0 \leq \alpha \leq 1$ is the diffusive constant and $0 \leq \beta \leq 1$ is the reactive constant, with the following boundary conditions are

$$u(0,t) = f(t), \quad u(1,t) = 0, \tag{8}$$

and the following initial condition:

$$u(x,0) = 0. \tag{9}$$

The analytical solution to this equation (in the case of $\nu = 2$) is given by:

$$u(x,t) = 1 - \frac{\cosh(x)}{\cosh(1)} - \frac{16}{\pi^2}\sum_{i=1}^{\infty}\frac{(-1)^i \cos(0.5\pi(2i-1)x)}{(2i-1)(\pi^2(2i-1)^2+4)}e^{-(1+0.25\pi^2(2i-1)^2)t}.$$

The applications in science and engineering fields of this type of equations are many, for example you can read ([21], [22]). Many researchers studied some meaningful generalization of this equation, one of these generalizations is called as one component reaction diffusion equation. Many reaction diffusion equations have travelling wave fronts which give a common role to study many real life problems in physics, chemistry, and biology ([23], [24]). Reaction-diffusion models are useful mathematical tools which describe how the concentration of one or more substances distributed in space varies under the influence of two processes, first one is local chemical reactions in which the substances are transformed into each other and second is the diffusion which is the reason of the substances to spread out over a surface in space. Many numerical algorithms are presented to study this these equations, for example, the Petrov-Galerkin finite element method [25], the Sinc collocation method [26], and others [27].

3. Derivation the numerical scheme using CFD-CSCM

In this section, a numerical scheme of the studied problem (7)-(9) is derived using CFD-CSCM. Assume that M and N are two positive integers such that $\tau_j = (j-1)\,\Delta\tau,\ j = 1, 2, ..., M+1$ where $\Delta\tau = \frac{Tf}{M}$, and let $\{x_p\}_{p=1}^{N+1-\lceil\nu\rceil}$ be the roots of the shifted Chebyshev polynomial of the third-kind $\bar{\mathbb{V}}_{N+1-\lceil\nu\rceil}(x)$.
A function $u(x,t) \in \mathbb{C}^3(0,1)$ and its Taylor's expansion, we have the following formula:

$$\frac{\partial u(x_p,t_j)}{\partial t} = \frac{u_p^j - u_p^{j-1}}{\Delta\tau} - \frac{\Delta\tau}{2}\frac{\partial^2 u(x_p,t_j)}{\partial t^2} + O((\Delta\tau)^2). \tag{10}$$

Evaluate the fractional Fisher's equation (7) at the grid-point (x_p, t_j) and apply the above discretization (10), we get

$$\frac{u_p^j - u_p^{j-1}}{\Delta\tau} - \frac{\Delta\tau}{2}\frac{\partial^2 u(x_p,t_j)}{\partial t^2} + O((\Delta\tau)^2) = \alpha\, D_x^{\nu} u(x_p,t_j) + \beta(1 - u(x_p,t_j)). \tag{11}$$

Now, differentiate the Eq.(7) with respect to t to get a discretization of $\frac{\partial^2 u(x_p,t_j)}{\partial t^2}$, then insert this discretization in the above equation (11) and after some simplifications be letting $u(x_p,t_j) = U_p^j$, one can get the following semi-discrete numerical scheme:

$$\frac{U_p^j - U_p^{j-1}}{\Delta\tau} = \alpha\, D_x^{\nu} U_p^j + \beta - \beta\, U_p^j - \frac{\Delta\tau}{2}\left[\alpha\,\frac{D_x^{\nu}U_p^j - D_x^{\nu}U_p^{j-1}}{\Delta\tau} - \beta\,\frac{U_p^j - U_p^{j-1}}{\Delta\tau}\right] + \Re^j(x)(\Delta\tau)^2.$$

Or,

$$c_1\, U_p^j - c_2\, D_x^{\nu} U_p^j = c_2\, D_x^{\nu} U_p^{j-1} + c_3\, U_p^{j-1} + \beta\Delta\tau + \Re^j(x)(\Delta\tau)^3, \tag{12}$$

where $\Re^j(x)$ is the resulting truncation term, and the constants c_1, c_2, and c_3 are given by:

$$c_1 = 1 + 0.5\beta\Delta\tau, \qquad c_2 = 0.5\alpha\Delta\tau, \qquad c_3 = 1 - 0.5\beta\Delta\tau.$$

In the following, we are going to occur the required full-discrete of the studied problem by evaluating the Caputo derivative $D_x^{\nu} U_p^j$ as defined in formula (6). Also, we approximate the solution $u(x,t)$ using the shifted Chebyshev collocation approach as follows:

$$u_N(x,t) = \sum_{n=0}^{N} u_n(t)\, \bar{\mathbb{V}}_n(x). \tag{13}$$

The connection between the formulae (6), (12), and (13) with taking into consideration that $U_p^j = u_N(x_p, t_j)$ and u_n^j is the coefficient in the point t_j, we can get:

$$\begin{aligned} c_1 \sum_{n=0}^{N} u_n^j \bar{\mathbb{V}}_n(x_p) - c_2 \sum_{n=\lceil \nu \rceil}^{N} \sum_{k=0}^{n-\lceil \nu \rceil} \sum_{i=0}^{k} u_n^j \Pi_{i,k}^{n,\nu}\, x_p^{k-i-\nu+\lceil \nu \rceil} = \beta\Delta\tau + \\ c_2 \sum_{n=\lceil \nu \rceil}^{N} \sum_{k=0}^{n-\lceil \nu \rceil} \sum_{i=0}^{k} u_n^{j-1} \Pi_{i,k}^{n,\nu}\, x_p^{k-i-\nu+\lceil \nu \rceil} + c_3 \sum_{n=0}^{N} u_n^{j-1} \bar{\mathbb{V}}_n(x_p). \end{aligned} \tag{14}$$

Apply the boundary conditions (8) together with Eq.(13), we get:

$$\sum_{n=0}^{N} (-1)^n (2n+1) u_n^j = f(t_j), \quad \text{and} \quad \sum_{n=0}^{N} u_n^j = 0. \tag{15}$$

Now, with $\lceil \nu \rceil$ equations (15) together with Eq. (14), we get $N+1$ of linear algebraic equations which can be solved numerically to obtain the unknowns u_n^j, $n = 0, 1, 2, ..., N$.
The above system of equations (14)-(15) can be written in a special case $N = 3$ and x_0, and x_1 are the roots of the shifted Chebyshev polynomial $\bar{\mathbb{V}}_2(x)$, i.e., $x_0 = 0.345492$, $\quad x_1 = 0.904508$, in a matrix form as follows:

$$\begin{aligned} \begin{pmatrix} c_1 & c_1\, k_1 & -c_2\, R_1 & c_1\, k_2 - c_2\, R_2 \\ c_1 & c_1\, k_{11} & -c_2\, R_{11} & c_1\, k_{22} - c_2\, R_{22} \\ 1 & -3 & 5 & -7 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{j} = \\ \begin{pmatrix} c_3 & c_3\, k_1 & c_2\, R_1 & c_3\, k_2 + c_2\, R_2 \\ c_3 & c_3\, k_{11} & c_2\, R_{11} & c_3\, k_{22} + c_2\, R_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}^{j-1} + \begin{pmatrix} \beta\Delta\tau \\ \beta\Delta\tau \\ f(t_j) \\ 0 \end{pmatrix}^{j}, \end{aligned} \tag{16}$$

where:

$$k_1 = \bar{\mathbb{V}}_1(x_0), \qquad k_2 = \bar{\mathbb{V}}_3(x_0), \qquad k_{11} = \bar{\mathbb{V}}_1(x_1), \qquad k_{22} = \bar{\mathbb{V}}_3(x_1),$$

$$R_1 = \Pi^{2,\nu}_{0,0}\, x_0^{2-\nu}, \qquad R_2 = \Pi^{3,\nu}_{0,0}\, x_0^{2-\nu} + \Pi^{3,\nu}_{1,0}\, x_0^{3-\nu} + \Pi^{3,\nu}_{1,1}\, x_0^{2-\nu},$$
$$R_{11} = \Pi^{2,\nu}_{0,0}\, x_1^{2-\nu}, \qquad R_{22} = \Pi^{3,\nu}_{0,0}\, x_1^{2-\nu} + \Pi^{3,\nu}_{1,0}\, x_1^{3-\nu} + \Pi^{3,\nu}_{1,1}\, x_1^{2-\nu}.$$

We will use the notation for the above system (16):

$$A\, U^j = B\, U^{j-1} + S^j, \qquad \text{or} \quad U^j = A^{-1} B\, U^{j-1} + A^{-1}\, S^j, \tag{17}$$

where: $U^j = (u_0^j,\, u_1^j,\, u_2^j,\, u_3^j)^T$, $S^j = (\beta\Delta\tau,\, \beta\Delta\tau,\, f^j, 0)^T$.
For $j = 1$, the initial solution U^0, can be occurred from the initial condition (9), $u(x, 0)$, i.e, combining with Eq.(5), we can get the initial solution U^0 of the linear system (17) and this allows us to compute the approximate solution at each step of the time.

4. Study the convergence and stability analysis

In this part, the convergence and the stability analysis of the given numerical scheme is studied. Assume that Ω denotes an open bounded region in $\mathbb{R}^2$ and let $L_2(\Omega)$ be a Hilbert space with the usual inner product and the norm are defined by:

$$\langle f(x),\, g(x)\rangle = \int_\Omega f(x)\, g(x)\, dx, \qquad \|f(x)\|_2 = \sqrt{\langle f(x),\, f(x)\rangle}.$$

Define,

$$H^s(\Omega) = \{f(x) \in L_2(\Omega),\;\; f^{(s)}(x) \in L_2(\Omega)\}.$$

To complete this study, the main numerical scheme (12) of the proposed problem can be rewritten in the following compact form:

$$U^k - \Lambda_1\, {}_aD_x^\nu U^k = \Lambda_2\, U^{k-1} + \Lambda_1\, {}_aD_x^\nu U^{k-1} + \Lambda_3, \tag{18}$$

where $k = 1, 2, ..., M+1$; and the constants Λ_1, Λ_2, and Λ_3 are defined as follows:

$$\Lambda_1 = \frac{\alpha\Delta\tau}{2+\beta\Delta\tau}, \qquad \Lambda_2 = \frac{2-\beta\Delta\tau}{2+\beta\Delta\tau}, \qquad \Lambda_3 = \frac{2\beta\Delta\tau}{2+\beta\Delta\tau}.$$

Lemma 1. *[29]*

Assume that $f(x)$ and $g(x)$ are two functions in the space $H^{\frac{\nu}{2}}(\Omega)$. Then for $1 < \nu < 2$, we have:

$$\langle {}_aD_x^\nu f,\, g\rangle = \langle {}_aD_x^{\frac{\nu}{2}} f,\, {}_xD_b^{\frac{\nu}{2}} g\rangle \quad \text{and} \quad \langle {}_xD_b^\nu f,\, g\rangle = \langle {}_xD_b^{\frac{\nu}{2}} f,\, {}_aD_x^{\frac{\nu}{2}} g\rangle.$$

Lemma 2. *[29]*

For a positive number ν, the following relation holds:

$$\langle {}_aD_x^\nu f,\, {}_xD_b^\nu f\rangle = \cos(\pi\nu)\, \|\, {}_aD_x^\nu f\, \|^2_{L_2(\Omega)} = \cos(\pi\nu)\, \|\, {}_xD_b^\nu f\, \|^2_{L_2(\Omega)}.$$

Lemma 3.

Assume that $f(x)$ and its fractional derivative ${}_aD_x^{\nu}f(x)$ belongs to the space $H^{\nu}(\Omega)$. Then for $1<\nu<2$, there is $\Delta\tau << 1$ such that the following inequality will occur:

$$\| f(x) + \Lambda_1\, {}_aD_x^{\nu}f(x) \| \leq \| f(x) \| . \tag{19}$$

Proof. Using the properties of the inner product and the norm, we can obtain:

$$\begin{aligned} \| f(x) + \Lambda_1\, {}_aD_x^{\nu}f(x) \|^2 &\leq \langle f(x) + \Lambda_1\, {}_aD_x^{\nu}f(x), f(x) + \Lambda_1\, {}_aD_x^{\nu}f(x)\rangle \\ &= \| f(x) \|^2 + 2\Lambda_1 \langle {}_aD_x^{\frac{\nu}{2}}f(x),\, {}_xD_b^{\frac{\nu}{2}}f(x)\rangle + \Lambda_1^2 \| \,{}_aD_x^{\nu}f(x) \|^2 . \end{aligned}$$

Using Lemma 2 and since $\cos(\frac{\nu}{2}\pi)<0$ (in a case $\nu\in(1,2)$), then $\langle {}_aD_x^{\frac{\nu}{2}}f(x),\, {}_xD_b^{\frac{\nu}{2}}f(x)\rangle$ is negative. Thus, if there exists $\Delta\tau << 1$, then it will confirm the following:

$$2\Lambda_1 \langle {}_aD_x^{\frac{\nu}{2}}f,\, {}_xD_b^{\frac{\nu}{2}}f\rangle + \Lambda_1^2 \| \,{}_aD_x^{\nu}f(x) \|^2 < 0,$$

then directly we can occur the desired inequality (19) and finish the proof of the lemma. □

Lemma 4.

Let $U^k \in H^1(\Omega)$, $k = 1, 2, ..., M+1$ be an approximate solution of the main numerical scheme (18), then the following holds:

$$\| U^k \| \leq \Lambda_2 \| U^{k-1} \| + \Lambda_3. \tag{20}$$

Proof. Apply the mathematical induction on k to prove this inequality (20) as follows:
For $k=1$, then the Eq.(18) takes the form:

$$U^1 - \Lambda_1\, {}_aD_x^{\nu}U^1 = \Lambda_2\, U^0 + \Lambda_1\, {}_aD_x^{\nu}U^0 + \Lambda_3, \tag{21}$$

multiply the above equation by U^1 and integrate over the region Ω, then we have:

$$\| U^1 \|^2 - \Lambda_1 \langle\, {}_aD_x^{\nu}U^1, U^1\rangle = \Lambda_2 \langle U^0, U^1\rangle - \Lambda_1 \langle\, {}_aD_x^{\nu}U^0, U^1\rangle + \langle \Lambda_1, U^1\rangle. \tag{22}$$

Since $\cos(\frac{\nu}{2}\pi)<0$ (in a case $\nu\in(1,2)$) and by applying the Cauchy-Schwarz inequality together with Lemmas 1 and 2, we can show that the quantity $\langle {}_aD_x^{\nu}U^1, U^1\rangle$ in the LHS of (22) is negative as follows:

$$\langle {}_aD_x^{\nu}U^1, U^1\rangle = \langle {}_aD_x^{\frac{\nu}{2}}U^1,\, {}_xD_b^{\frac{\nu}{2}}U^1\rangle = \cos\left(\frac{\nu}{2}\pi\right) \| \,{}_aD_x^{\frac{\nu}{2}}U^1 \|^2 < 0.$$

Also, using the conclusion of Lemma 3 for the RHS of Eq.(22), we can get the following inequality:

$$
\begin{aligned}
\| \Lambda_2\langle U^0, U^1\rangle + \Lambda_1\langle\, {}_aD_x^{\nu} U^0, U^1\rangle \| &= \| \langle \Lambda_2 U^0 + \Lambda_1\, {}_aD_x^{\nu} U^0, U^1\rangle \| \\
&\leq \| \Lambda_2 U^0 + \Lambda_1\, {}_aD_x^{\nu} U^0 \| \| U^1 \| \\
&\leq \Lambda_2 \| U^0 \| \| U^1 \| .
\end{aligned}
$$

The connection between the above inequalities confirms to us that the relation (20) is true for the case $k = 1$, and obtain:

$$\| U^1 \| \leq \Lambda_2 \| U^0 \| + \Lambda_3.$$

Assume that the inequality (20) is true for all $s = 1, 2, ..., k-1$, then we have:

$$\| U^s \| \leq \Lambda_2 \| U^{s-1} \| + \Lambda_3,$$

now, repeat the same processes that we applied in the first step of the induction ($k = 1$), with the help of the Cauchy-Schwarz inequality again and the fact that $\cos(\frac{\nu}{2}\pi) < 0$, we can show and prove that this inequality holds for all k and this completes the proof. □

Theorem 2.

The numerical scheme that was mentioned in Eq.(12) is un-conditionally stable.

Proof. To discuss the condition of the stability, let us introduce $\overline{U}_p^k$, $k = 1, 2, ...M+1$ as an approximate solution of the scheme (12), therefore, we can see that the error $\varepsilon^k = U_p^k - \overline{U}_p^k$ satisfies the same scheme (12), i.e,

$$c_1\, \varepsilon^k - c_2\, D_x^{\nu} \varepsilon^k - c_2\, D_x^{\nu} \varepsilon^{k-1} = c_3\, \varepsilon^{k-1}, \tag{23}$$

where the constants c_1, c_2, and c_3 are defined in (12). Now, directly with applying the same process of the proof for Lemma 4 on the error ε^k which was defined in (23), we can obtain the following:

$$\| \varepsilon^k \| \leq \Lambda_2 \| \varepsilon^0 \|, \quad \Lambda_2 < 1, \quad k = 1, 2, ..., M+1,$$

where $\Lambda_2 = \frac{c_3}{c_1}$; which means that the resulting scheme (12) by applying the proposed method is unconditionally stable, and this completes the proof. □

Theorem 3.

The resulting error $\varepsilon^k = u(x, t_k) - U^k$; $k = 1, 2, ..., M+1$, from the numerical scheme (12) satisfies the following error estimate:

$$\| \varepsilon^k \| \leq E_x (\Delta\tau)^2, \tag{24}$$

where E_x is a constant that dependent on x (the maximum of the truncation error term).

Proof. Using the mathematical induction on k to ensure the inequality (24) with the following three steps:

1. For $k = 1$:

   The multiplying Eq.(23) with $(k = 1)$ by ε^1 and integrating over the region Ω, will allow us to obtain the following form:

   $$c_1 \parallel \varepsilon^1 \parallel^2 - c_2 \langle\, {}_a D_x^\nu \varepsilon^1, \varepsilon^1 \rangle = c_2 \langle\, {}_a D_x^\nu \varepsilon^0, \varepsilon^1 \rangle + c_3 \langle \varepsilon^0, \varepsilon^1 \rangle + (\Delta\tau)^3 \langle \Re^1(x), \varepsilon^1 \rangle,$$

   where $\Re^1(x)$ is the truncation error term in the step k. Now, using Lemmas 2 and 3 together with the fact that $\varepsilon^0 = 0$ and after some simplifications, it gives that:

   $$\parallel \varepsilon^1 \parallel \leq E_x (\Delta\tau)^2.$$

2. Assume that the statement (24) is true for $k = 1, 2, ..., M$, which means that:

   $$\parallel \varepsilon^k \parallel \leq E_x (\Delta\tau)^2, \quad k = 1, ..., M.$$

3. Now the truth if the statement (24) when $k = M + 1$ completes the proof, for this aim, we multiply the Eq.(23) by ε^k and integrate over the region Ω, to get the following equation:

   $$c_1 \parallel \varepsilon^k \parallel^2 - c_2 \langle\, {}_a D_x^\nu \varepsilon^k, \varepsilon^k \rangle = c_2 \langle\, {}_a D_x^\nu \varepsilon^{k-1}, \varepsilon^k \rangle + c_3 \langle \varepsilon^{k-1}, \varepsilon^k \rangle + (\Delta\tau)^3 \langle \Re^k(x), \varepsilon^k \rangle.$$

   Now with the help of Lemma 3 together the above equation, we obtain that:

   $$\parallel \varepsilon^k \parallel^2 \leq (c_2/c_1) \parallel \varepsilon^k \parallel \parallel \varepsilon^{k-1} \parallel + (c_2/c_1) E (\Delta\tau)^2 \parallel \varepsilon^k \parallel,$$

   by dividing with $\parallel \varepsilon^k \parallel$ and repeating the resulting inequality, we can get the desired statement (24) which completes the proof of inequality of the error estimate (24).

□

5. Numerical simulation and comparison

In this section, we develop the aforesaid algorithm to treat and solve FFEs (7)-(9) with different values of the fractional-order, ν and the order of approximation, m; through the verified numerical results. Also, a comparison with the exact solution $\nu = 2$, is given. In addition, the effect of the diffusive constant; α and the reactive constant; β is given. Finally, the adequacy of the aforesaid algorithm is verified by computing the residual error function (REF) with different values of ν and m.

The Figures 1-4 plotted the numerical simulation for the problem under study by utilizing the aforesaid algorithm. In Figure 1, the behavior of the approximate and exact solutions is mentioned at $\nu = 2$, $m = 3, \Delta\tau = 0.002$, $\alpha = \beta = 1$, via distinct values of $Tf = 1(a)$ and $Tf = 1.5(b)$. Figure 2, is

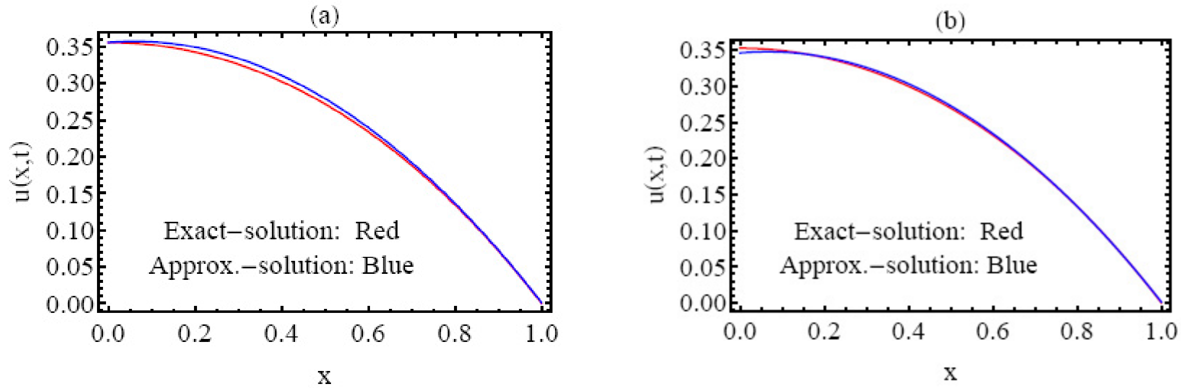

Figure 1. Graph of the approximate and exact solutions at $m = 3$ via $Tf = 1(a)$ and $Tf = 1.5(b)$.

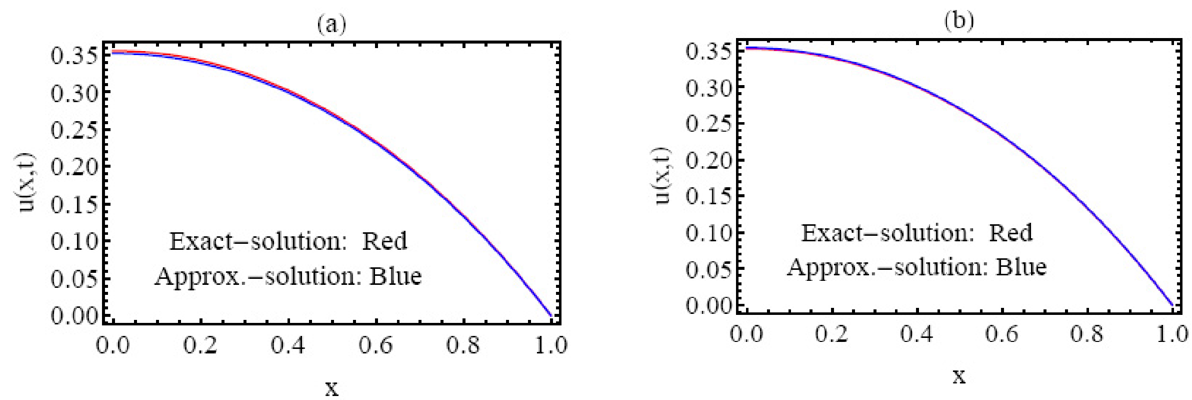

Figure 2. Graph of the approximate and exact solutions at $m = 6$ via $Tf = 1(a)$ and $Tf = 1.5(b)$.

the same Figure 1 but with $m = 6$. In Figure 3, the behavior of the approximate solution is plotted via distinct values of the fractional-order, $\nu = 2.0,\ 1.8,\ 1.6,\ 1.4,\ 1.2$ at $m = 6, \Delta\tau = 0.01$, $\alpha = \beta = 1$, with $Tf = 2(a)$ and $Tf = 5(b)$. Finally, in Figure 4, the behavior of the approximate solution is plotted at $m = 6$, $Tf = 5$, $\Delta\tau = 0.005$, via distinct values of the diffusive constant $\alpha = 0.25,\ 0.5,\ 0.75,\ 1(a)$ and the reactive constant $\beta = 0.25,\ 0.5,\ 0.75,\ 1(b)$. From these figures, we can note that the behavior of the numerical solution depends on the values of ν and m, and this confirms that the proposed numerical method is implemented in a good way for solving the proposed problem under study in the case of fractional derivatives. In addition, we found that the diffusive constant and the reactive constant are affected clearly by the behavior of the obtained solution. Likewise, these results bring to light the reasonability convergence of the proposed method for given problem and consistent with which was predicted in the theoretical study through the proved lemmas and theorems. In addition, to validate our numerical solutions with $\tau = 0.05$, $Tf = 0.5$, $\alpha = \beta = 1$, we compute the residual error function (REF) via distinct values of $\nu = 1.8,\ 1.85,\ 1.9$, and $m = 3, 6, 9$. From the Table 1 and all Figures 1-4, it is evident that the overall errors can be decreased by adding new terms from the series (13).

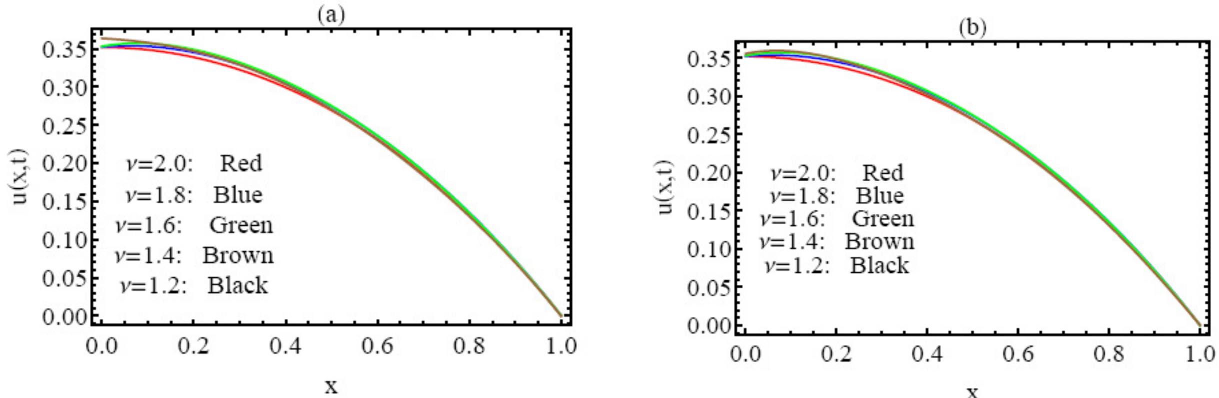

Figure 3. The approximate solution at $m = 6$ via different values of ν with $Tf = 2(a)$ and $Tf = 4(b)$.

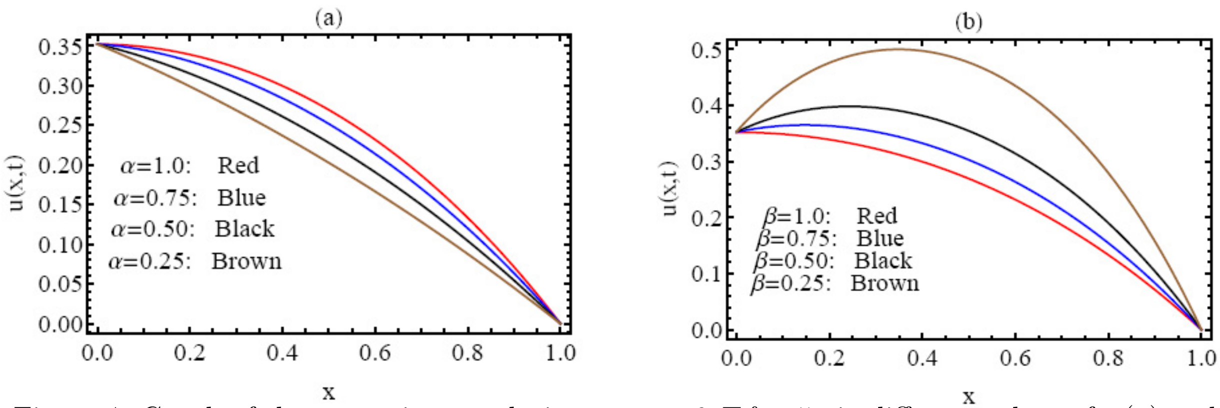

Figure 4. Graph of the approximate solution at $m = 6, Tf = 5$ via different values of $\alpha(a)$ and $\beta(b)$.

Table 1. A comparison of REF at $Tf = 0.5$ via different values of ν, m.

	Current Method–REF at:			Current Method–REF at:		
x	ν=1.8	ν=1.85	ν=1.9	m=3	m=6	m=9
0.0	4.75394E-05	7.85420E-07	8.96321E-08	1.25841E-04	0.02341E-06	4.65478E-08
0.2	3.75385E-06	7.69851E-09	0.05411E-09	6.02154E-05	1.95487E-07	4.96541E-08
0.4	1.85241E-06	9.32541E-09	2.96542E-10	0.65217E-05	8.96541E-07	7.35214E-09
0.6	1.65421E-07	0.85230E-09	9.60450E-10	8.66587E-05	1.98210E-07	4.95421E-10
0.8	4.65237E-07	3.12054E-09	8.32047E-11	0.65402E-05	1.85201E-08	7.96541E-10
1.0	3.85214E-07	4.65421E-10	6.96321E-11	1.74123E-05	7.95412E-08	3.65412E-10

6. Conclusion

The properties of the Chebyshev polynomials of the third-kind and the compact finite difference method with order $O(\Delta\tau)^2$ are addressed to reduce the fractional Fisher's equation in the Caputo sense of the solution of a linear system of algebraic equations. Here we employed the Caputo derivative because it only needs initial condition described in terms of integer-order derivatives. The unconditional stability of the proposed scheme is discussed and proved in an appropriate Sobolev space. From the solutions obtained using the suggested method, we can confirm that these solutions are in excellent agreement with the already existing ones and explain that this method is efficient and applicable to solve the aforesaid problem effectively. It is evident that the overall errors can be decreased by adding new terms from the series (13). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of our numerical method.

References

[1] Podlubny, I., Fractional Differential Equations. Academic Press, New York (1999).

[2] Heydaria, M. H., and Atangana, A., A cardinal approach for nonlinear variable-order time fractional Schrödinger equation defined by Atangana-Baleanu-Caputo derivative. Chaos, Solitons & Fractals 128 (2019) 339-348.

[3] 15. N. H. Sweilam, M. M. Khader and M. Adel, Weighted average finite difference methods for fractional reaction-subdiffusion equations, *Walailak Journal of Science and Technology*, 11(4), P. 361-377, 2014.

[4] Amabili, M., Balasubramanian, P., and Breslavsky, I., Anisotropic fractional viscoelastic constitutive models for humandescending thoracic aortas. Journal of the Mechanical Behavior of Biomedical Materials 99 (2019) 186-197.

[5] Li, X., Xue, Z., and Tian, X., A modified fractional order generalized bio-thermoelastic theory with temperature-dependent thermal material properties. Inter. J. of Thermal Sciences 132 (2018) 249-256.

[6] Khader, M. M., On the numerical solutions for the fractional diffusion equation. Communications in Nonlinear Science and Numerical Simulations 16 (2011) 2535-2542.

[7] Khader, M. M., and Saad, K. M., On the numerical evaluation for studying the fractional KdV, KdV-Burger's, and Burger's equations. European Physical Journal Plus 133 (2018) 1-13.

[8] Khader, M. M., and Mohammed M. Babatin, Numerical treatment for solving fractional SIRC model and influenza A. Computational and Applied Mathematics 33(3) (2014) 543-556.

[9] Khader, M. M., The numerical solution for BVP of the liquid film flow over an unsteady stretching sheet with thermal radiation and magnetic field using FEM. International Journal of Modern Physics C 15(3) (2019) 190422b.

[10] M. M. Khader and M. Adel, Introducing the windowed Fourier frames technique for obtaining the approximate solution of coupled system of differential equations, *Journal of Pseudo-Differential Operators and Applications*, 2019(10), P. 241-256, 2019.

[11] M. M. Khader and M. Adel, Numerical solutions of fractional wave equations using an efficient class of FDM based on Hermite formula, *Advances in Difference Equations*, 12/2016, 2016 (1), 34, 2016.

[12] Kilbas, S. G., Kilbas, A. A., and Marichev, O. I., Fractional Integrals and Derivatives: Theory and Applications. Gordon & Breach, Yverdon (1993).

[13] Abdeljawad, T., and Baleanu, D., Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. Nonlin. Sci. Appl. 9 (2017) 1098-1107.

[14] Atangana, A., and Baleanu, D., New fractional derivative with non-local and non-singular kernel. Therm. Sci. 20(2) (2016) 736-769.

[15] Mason J., Chebyshev polynomials of the second, third and fourth kinds in approximation, indefinite integration, and integral transforms. Journal of Computational and Applied Mathematics 49(1-3) (1993) 169-178.

[16] Mason, J. C., and Handscomb, D. C., Chebyshev Polynomials. Chapman and Hall, CRC, New York, NY, Boca Raton (2003).

[17] Saad, K. M., Khader, M. M., Gomez-Aguilar, J. F., and Dumitru Baleanu, Numerical solutions of the fractional Fisher's type equations with Atangana-Baleanu fractional derivative by using spectral collocation methods. Chaos 29 (2019) 1-5.

[18] Khader, M. M., and Saad, K. M., A numerical study using Chebyshev collocation method for a problem of biological invasion: fractional Fisher equation. International Journal of Biomathematics 11(8) (2018) 1-15.

[19] Snyder, M. A., Chebyshev Methods in Numerical Approximation. Prentice-Hall, Inc. Englewood Cliffs, N. J. (1966).

[20] Handan, C. Y., Numerical solution of fractional Riccati differential equation via shifted Chebyshev polynomials of the third kind. J. of Eng. Technology and Applied Sciences 28 (2017) 1-11.

[21] Canosa, J. C., On a nonlinear diffusion equation describing population growth. IBM Journal of Research Development 17 (1973) 307-313.

[22] Newman, W. I., Some exact solutions to a non-linear diffusion problem in population genetics and combustion. Journal of Theoretical Biology 85 (1980) 325-334.

[23] M. M. Khader and M. Adel, Numerical study of the fractional modeling on SIR equations with constant vaccination rate using GEM, *International Journal of Nonlinear Sciences and Numerical Simulation*, 20(1), P. 69-76, 2019.

[24] N. H. Sweilam, M. M. Khader and M. Adel, On the fundamental equations for modeling neuronal dynamics, *Journal of Advanced Research (JAR)*, 5(2), 253-259, 2014.

[25] Tang, S. T., and Weber, R. O., Numerical study of Fishers equation by a Petrov-Galerkin finite element method. Journal of the Australian Mathematical Society Series B 33 (1991) 27-38.

[26] Khaled, K. A., Numerical study of Fishers diffusion reaction equation by the sinc collocation method. Journal of Computational and Applied Mathematics 137 (2001) 245-255.

[27] Khader, M. M., and Saad, K. M., A numerical approach for solving the problem of biological invasion (fractional Fisher equation) using Chebyshev spectral collocation method. Chaos, Solitons & Fractals 110 (2018) 169-177.

[28] M. M. Khader and M. Adel, Chebyshev wavelet procedure for solving FLDEs, *Acta Applicandae Mathematicae*, P. 158(1), P. 1-10, 2018.

[29] Ervin, V. J., Heuer, N., and Roop, J. P., Numerical approximation of a time dependent, nonlinear, space fractional diffusion equation. SIAM Journal on Numerical Analysis 45(2) (2007) 572-591.