Study of electrical R-L circuits composed of resistors and inductors and driven by a voltage of the current source: Simulation with implementing an accurate method

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Abstract

In this study, we propose to derive an accurate numerical procedure to solve the mathematical model which describes the electrical R-L circuit composed of resistors and inductors driven by a voltage of current source, which is the fractional-order model for the electrical RL-circuit. Our study depends on the spectral collocation method via the useful properties of the Chebyshev polynomials of the third-kind. Some theorems about the convergence analysis are given. The study concludes by comparing the resulting approximate solutions of the proposed model with the exact solution in the classical case. Illustrative graphical and numerical analysis of the derived results are also included in this study.

Key-words: Fractional RL-circuit model; Chebyshev-spectral collocation method; Convergent analysis.

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1. Introduction

Many researchers have researched the simulation of fractional differential equations (FDEs) using various types of estimation and computational methods ([8], [10], [15], [22]). This is due to the importance of this type of equation in many applications, such as fluid mechanics, biology, engineering, and many other applications ([13], [14], [18]-[20], [23]).

Definition 1.

The Liouville-Caputo fractional derivative of order ν, D^{ν} of a function φ can be defined by [14]:

$$D^{\nu}\varphi(t)=\frac{1}{\Gamma(\ell-\nu)}\int_{0}^{t}\frac{\varphi^{(\ell)}(\tau)}{(t-\tau)^{\nu-\ell+1}}d\tau,\quad \nu>0,\quad \ell-1<\nu\leq\ell,\ \ell\in\mathbb{N},\ t>0,$$

where $\Gamma(.)$ is the well-known gamma function.

The Spectral Collocation Method (SCM) is one of the most precise estimation analytical methods for the analysis of FDEs simulations ([2], [7], [16], [21]). This type of numerical techniques has many advantages such as in addressing this group of significant problems. The Chebyshev coefficients for the solution can be calculated very quickly by using the numerical programs provided. As a result, SCM is faster and better than any other numerical methods. The orthogonal Chebyshev polynomials are well known polynomials for many (or all) mathematicians and these polynomials are one of the most famous families of the orthogonal polynomials on the closed interval $[-1,1]$ that have many important applications as they are commonly used in the approximation of functions because of their excellent properties ([3], [6], [11], [12]).

In this paper, we plan to use one of these important techniques to numerically simulate the fractional RL-circuit model (SCM with the aid of third-kind Chebyshev polynomials); where the letters L and R signify inductance and resistance respectively, and they are constant.
The electromagnetic field that is generated around the conductor is created from the electrical charges as waves and electrons that are due to the conductor. An electrical circuit is a simple electrical device interconnection that has at least one closed path for the current to be owed, and by changing the applied source or changing the circuit components, the circuit may be changed from one state to another state. The transition time is temporary, during which new values are substituted for the former branch current and voltage values. And the circuit is said to be in a steady state after passing the transient one ([1], [17], [25]).

The electrical circuit is called the RL-circuit in the event that the electrical resistance is set, and the electrical current is opposed. Where "L" indicates an inductor which multiplies the electromotive force (EMF) produced by the current by many wire rotations. The RL circuit is a circuit that has a combination of inductors and resistors. These circuits, such as chokes of luminescent tubes, are widely used in our real lives because of their great significance and applications.

2. The Studied Problem

In this part of the paper, we will derive the formulation of the studied model of the electrical RL-circuit in its classical formula in the following steps:

Bt taking in considerations these three important and famous laws in the electric circuits which are Ohm's Law, Law of Inductance, and Kirchhoff's Voltage Law. Let the current which is flow through a resistor is denoted by $J(t)$, the resistance is denoted by R in Ohm, V_R be the voltage drop through a resistor, let V_L denotes the voltage across the inductor, and let the voltage added in a closed loop is denoted by V_E. Then, one can get:

$$V_R = RJ,\ V_L = L\frac{dJ}{dt} \text{ and } V_E = V_R + V_L.$$

Here in our case, we consider the RL-series circuit, then We can derive the corresponding differential equation of the proposed model from all these elements as follows:

$$L\frac{dJ(t)}{dt} + RJ(t) = V_E(t), \qquad J(0) = J_0, \qquad t \in [0,1]. \tag{1}$$

In this paper and in a more general case, the fractional order model [1] will be studied, and we will provide a discussion about the basic differential equation of electrical circuits that involve a resistor with a resistance R and an inductor with an inductance L. The following form is taken from this equation: [4]

$$LD^{\alpha}J(t) + RJ(t) = V_E(t), \qquad t \in [0,1], \qquad 0 < \alpha \leq 1, \tag{2}$$

where α denotes the fractional order of the Liouville-Caputo derivative. With the following initial condition:

$$J(0) = J_0. \tag{3}$$

Note that at $\alpha = 1$, Eq.(2) is the classical equation (1).
The purpose of this study is to use the third kind of Chebyshev polynomials to implement the spectral collocation method to greatly simplify the proposed problem to a simple algebraic system of equations that can be solved by using one of the well-known appropriate numerical methods.

3. Solutions for the fractional electrical RL-circuit model

3.1. Some properties and concepts of the Chebyshev polynomials

The third type of orthogonal Chebyshev polynomials of degree n can be defined as follows [17]:

$$V_n(z) = \frac{\cos((n+0.5)\varphi)}{\cos(0.5\varphi)}, \qquad z = \cos(\varphi), \qquad 0 \leq \varphi \leq \pi, \qquad -1 \leq z \leq 1.$$

In this paper, we will use these polynomials on $[0,1]$, so by adding the linear transform $z = 2x - 1$, we can construct the so-called shifted Chebyshev polynomials. This function form is denoted and defined as $\bar{\mathbb{T}}_n(x) = V_n(2x-1)$. The most used and useful formula of $\bar{\mathbb{T}}_n(x)$ is in the following analytic form [24]:

$$\bar{\mathbb{T}}_n(x) = \sum_{k=0}^{n} (-1)^k 2^{2n-2k} \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\,\Gamma(2n-2k+2)}\, x^{n-k}, \qquad n = 0, 1, \ldots . \tag{4}$$

It is easy to find that $\bar{\mathbb{T}}_i(0) = (-1)^i(2i+1)$, $\bar{\mathbb{T}}_i(1) = 1$, $i = 0, 1, 2\ldots$. The function $\Psi(x) \in L_2[0,1]$ may be defined as an infinite series sum of $\{\bar{\mathbb{T}}_0(x), \bar{\mathbb{T}}_1(x), \bar{\mathbb{T}}_2(x), \ldots\}$ as:

$$\Psi(x) = \sum_{\ell=0}^{\infty} c_\ell\, \bar{\mathbb{T}}_\ell(x), \qquad c_\ell = \frac{2}{\pi} \int_0^1 \sqrt{\frac{x}{1-x}}\Psi(x)\, \bar{\mathbb{T}}_\ell(x)\, dx, \qquad \ell = 0, 1, \ldots . \tag{5}$$

We take the first $(m+1)$-terms of (5) to obtain the following approximation form

$$\Psi_m(x) = \sum_{\ell=0}^{m} c_\ell\, \bar{\mathbb{T}}_\ell(x). \tag{6}$$

In this section, the formula (4) and some properties of the fractional derivative can be used to give an approximate formula of $D^\nu \Psi_m(x)$ directly through the following theorem.

Theorem 1. *([5], [24])*

Suppose that we approximate the function $\Psi(x)$ in the form (6) then $D^\nu(\Psi_m(x))$ can be defined as:

$$D^\nu(\Psi_m(x)) = \sum_{i=\lceil\nu\rceil}^{m} \sum_{k=0}^{i-\lceil\nu\rceil} c_i\, \Upsilon_{i,k,\nu}\, x^{i-k-\lceil\nu\rceil}, \quad \Upsilon_{i,k,\nu} = \frac{(-1)^k 2^{2i-2k}(2n+1)\,(2i-k)!(i-k)!}{(k!)\,\Gamma(2i-2k+2)\,\Gamma(i-k+1-\nu)}. \tag{7}$$

For more details about this method and its convergence analysis, see ([9], [10]).

3.2. Implementation of the proposed method

We will implement the Chebyshev collocation method to solve numerically the equation (2). We approximate $J(t)$ by $J_m(t)$ as follows:

$$J_m(t) = \sum_{i=0}^{m} c_i\, \bar{\mathbb{T}}_i(t). \tag{8}$$

By using Eqs.(2), (8) and the formula (7) we can obtain:

$$L \sum_{i=1}^{m} \sum_{k=0}^{i-1} c_i\, \Upsilon_{i,k,\alpha}\, t^{i-k-1} + R\left(\sum_{i=0}^{m} c_i\, \bar{\mathbb{T}}_i(t)\right) = V_E(t), \tag{9}$$

the previous equation (9) will be collocated at m of nodes t_r as follows:

$$L\sum_{i=1}^{m}\sum_{k=0}^{i-1} c_i\,\Upsilon_{i,k,\alpha}\,t_r^{i-k-1} + R\left(\sum_{i=0}^{m} c_i\,\bar{\mathbb{T}}_i(t_r)\right) = V_E(t_r). \tag{10}$$

In addition, the initial condition (3) can be expressed by substituting from Eq.(8) in (3) to obtain the equation:

$$\sum_{i=0}^{m}(-1)^i(2i+1)\,c_i = J_0. \tag{11}$$

Eqs.(10), together with equation (11), give $(m+1)$ of algebraic equations.

Here, we will use the suitable collocation points which are obtained from the roots of $\bar{\mathbb{T}}_m(t)$. In our computation, we take $m = 6$ (for example) therefore the required roots t_r of $\bar{\mathbb{T}}_6(t)$ are:

$$t_1 = 0.057272,\quad t_2 = 0.215968,\quad t_3 = 0.439732,\quad t_4 = 0.677302,\quad t_5 = 0.874255,\quad t_6 = 0.985471.$$

Hence we define the residual error function (REF) as follows [15]:

$$\mathrm{REF}(m,\alpha,t) = L\sum_{i=1}^{m}\sum_{k=0}^{i-1} c_i\,\Upsilon_{i,k,\alpha}\,t^{i-k-1} + R\left(\sum_{i=0}^{m} c_i\,\bar{\mathbb{T}}_i(t)\right) - V_E(t). \tag{12}$$

4. Numerical simulation

In this part of the paper, we solve numerically two models of the proposed FDE (2) that are dependent on the voltage source (V_E); by applying the proposed method.

Model 1. (Non-variant voltage source):

In this case, we are considering the equation (2) with a constant voltage source (V) with different values of $\alpha, m, R, V,\ L$ [4]. The classical solution for equation (2) at $\alpha = 1$ is:

$$J(t) = \frac{V}{R} + \left(J_0 - \frac{V}{R}\right)e^{-\frac{R}{L}t}.$$

We present a numerical simulation of this model by means of the proposed method through the figures 1-5. In Figure 1, we present the approximate solution at $R = V = 6,\ L = 3,\ J_0 = 0.4,\ m = 6$; with different values of $\alpha = 0.6,\ 0.7,\ 0.9,\ 1.0$. In Figures 2, we compare the exact solution (Continuous line) and the approximate solution (Discrete line) by using our proposed method at $R = V = 6,\ L = 3,\ J_0 = 0.4,\ \alpha = 1$; with different values of $m = 4$ (a) and $m = 8$ (b). In Figures 3, we present the approximate solution at $L = 3,\ J_0 = 0.4,\ \alpha = 0.98, m = 6$; with different values of $R = 3, 6, 9, 12$ (a) and $V = 3, 6, 9, 12$ (b). But in Figures 4, we give the REF at $R = V = 6,\ L = 3,\ J_0 = 0.4,\ \alpha = 0.97$ with different values of $m = 9$ (a) and $m = 14$ (b). Also, in Figures 5, we give the REF at $R = V = 6,\ L = 3,\ J_0 = 0.4,\ m = 12$ with different values of $\alpha = 0.93$ (a) and $\alpha = 0.99$ (b). From these figures, we can see that the behavior of the numerical solution is dependent on the values of α and this confirms that the proposed numerical method is well implemented for solving the proposed problem in case of fractional derivatives.

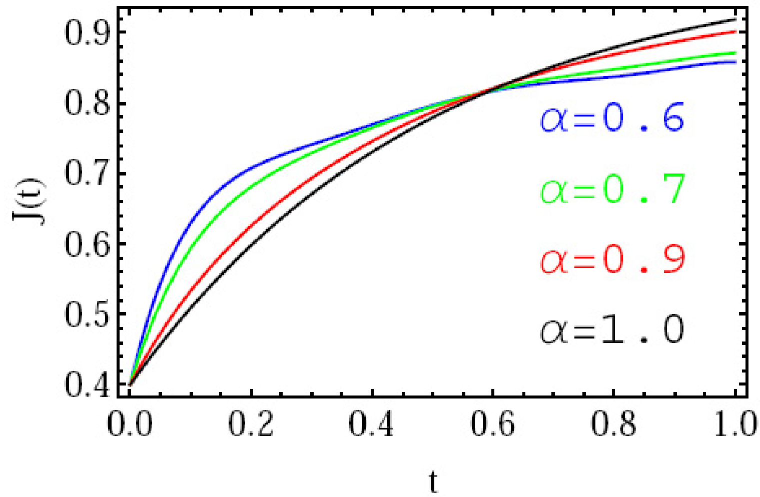

Figure 1. The approximate solution at $R = V = 6, L = 3, J_0 = 0.4, m = 6$; with different values of α.

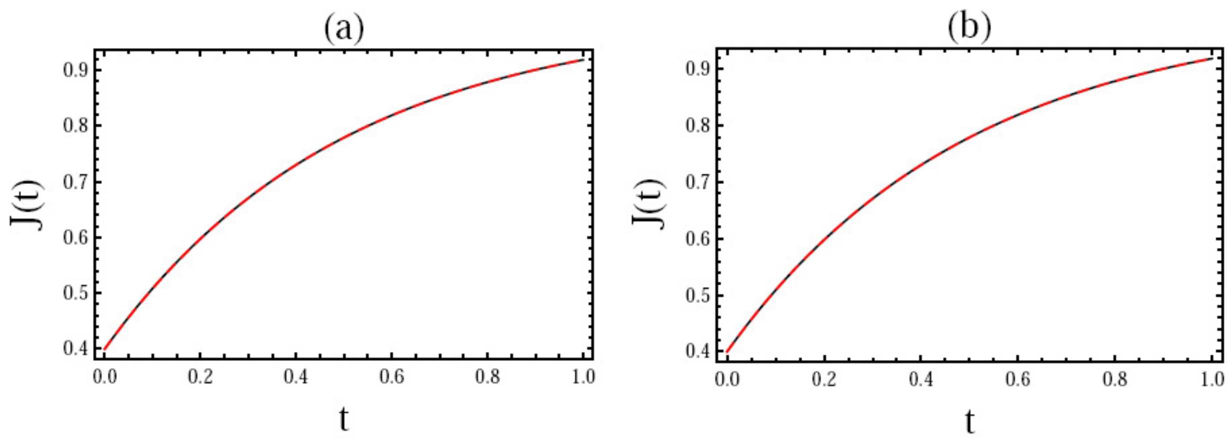

Figure 2: The approximate and exact solutions, with $R = V = 6$, $L = 3$, $J_0 = 0.4$, $\alpha = 1.0$; at $m = 4$ (a) and $m = 8$ (b).

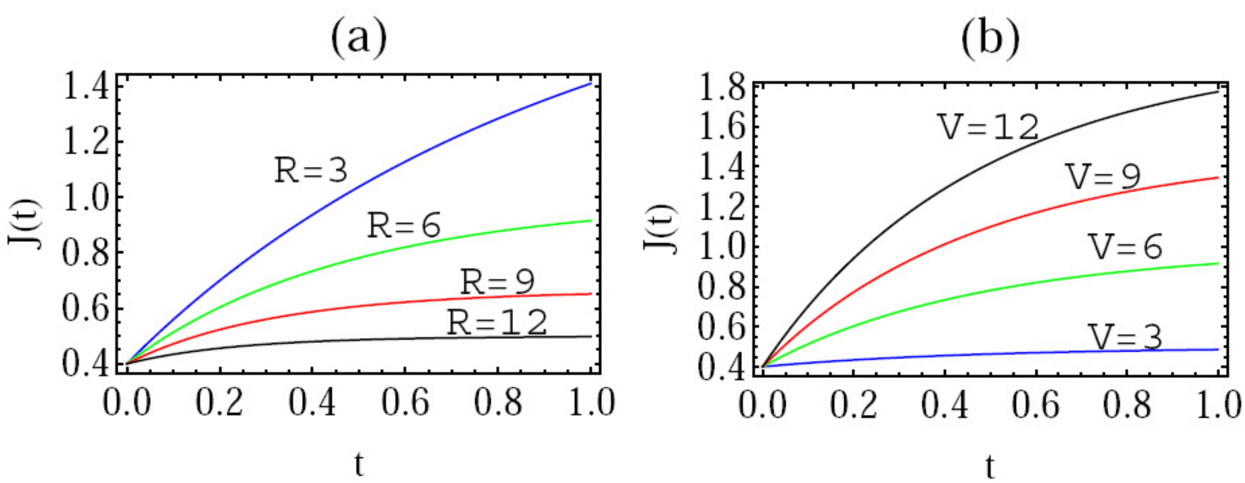

Figure 3. The approximate solution, with $L = 3$, $J_0 = 0.4$, $m = 6$, $\alpha = 0.98$; with different values of R (a) and V (b).

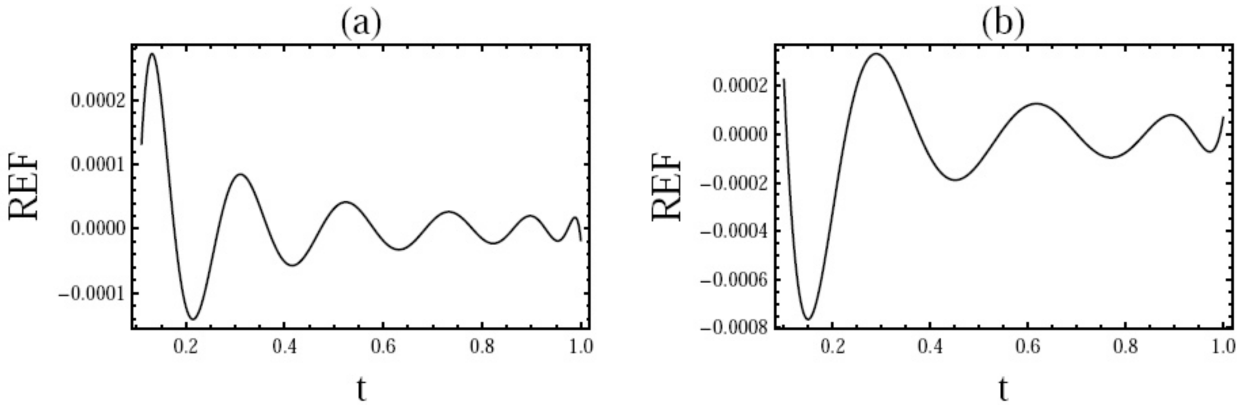

Figure 4. The REF with $R = V = 6,\ L = 3,\ J_0 = 0.4,\ \alpha = 0.97$;
at $m = 9$ (a) and $m = 14$ (b).

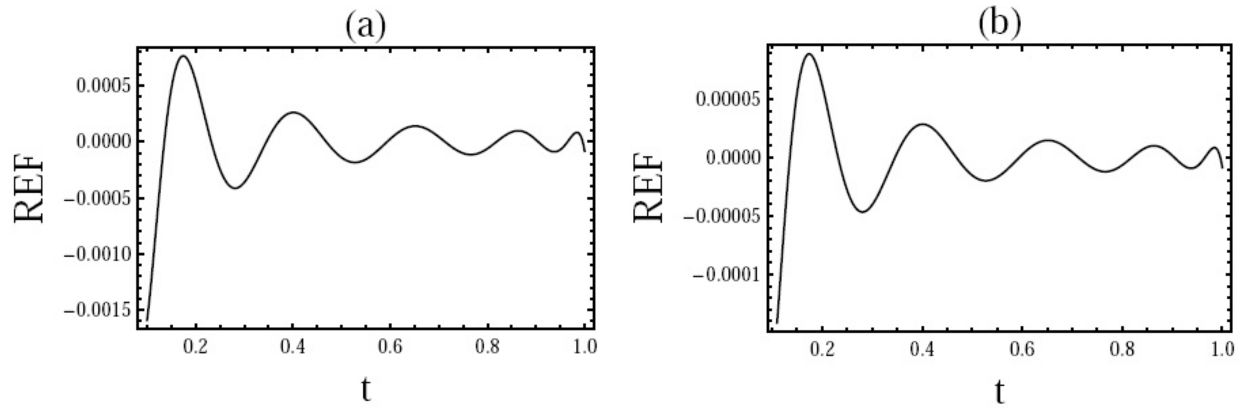

Figure 5. The REF with $R = V = 6,\ L = 3,\ J_0 = 0.4,\ m = 12$;
at $\alpha = 0.93$ (a) and $\alpha = 0.99$ (b).

Model 2. (Variant voltage source):

In this case, we are considering the equation (2) with a variant voltage source; $V(t) = e^{-(R/L)t}$ with different values of α, m, R, L. The classical solution for equation (2) at $\alpha = 1$ is:

$$J(t) = \left(\frac{t}{L} + J_0\right) e^{-\frac{R}{L}t}.$$

We present a numerical simulation of this model by means of the proposed method through the figures 6-10. In Figure 6, we present the approximate solution at $R = 6,\ L = 3,\ J_0 = 0.4,\ m = 6$; with different values of $\alpha = 0.6,\ 0.7,\ 0.9,\ 1.0$. In Figures 7, we compare the exact solution (Continuous line) and the approximate solution (Discrete line) by using our proposed method at $R = 6,\ L = 3,\ J_0 = 0.4,\ \alpha = 1$; with different values of $m = 5$ (a) and $m = 7$ (b). In Figures 8, we present the approximate solution at $J_0 = 0.4,\ \alpha = 0.97, m = 6$; with different values of $R = 3, 6, 9, 12$ (a) and $L = 0.3, 0.6, 0.9, 1.2$ (b). But in Figures 9, we give the REF at

$R = 6$, $L = 3$, $J_0 = 0.4$, $\alpha = 0.98$ with different values of $m = 8$ (a) and $m = 14$ (b). Also, in Figures 10, we give the REF at $R = 6$, $L = 3$, $J_0 = 0.4$, $m = 12$ with different values of $\alpha = 0.93$ (a) and $\alpha = 0.99$ (b). From these figures, we can see that the behavior of the numerical solution is dependent on the values of α and this confirms that the proposed numerical method is well implemented for solving the proposed problem in case of fractional derivatives.

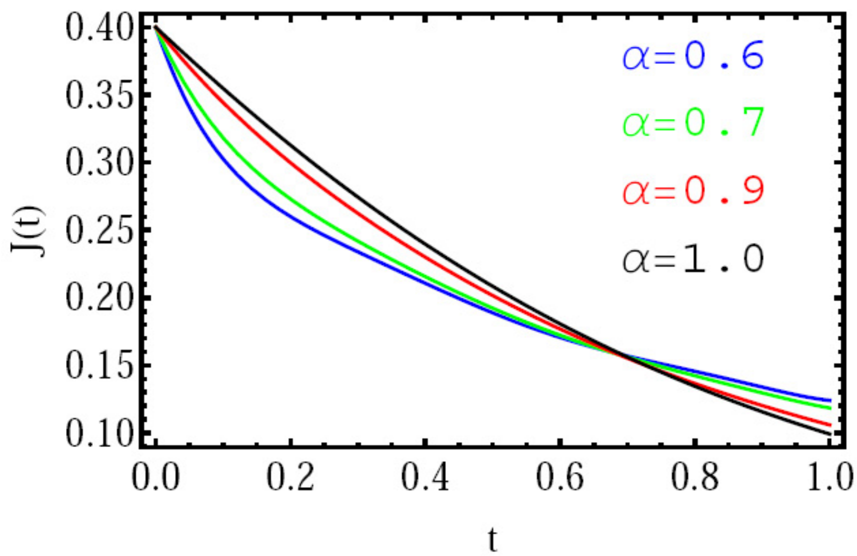

Figure 6. The approximate solution at $R = 6$, $L = 3$, $J_0 = 0.4$, $m = 6$; with different values of α.

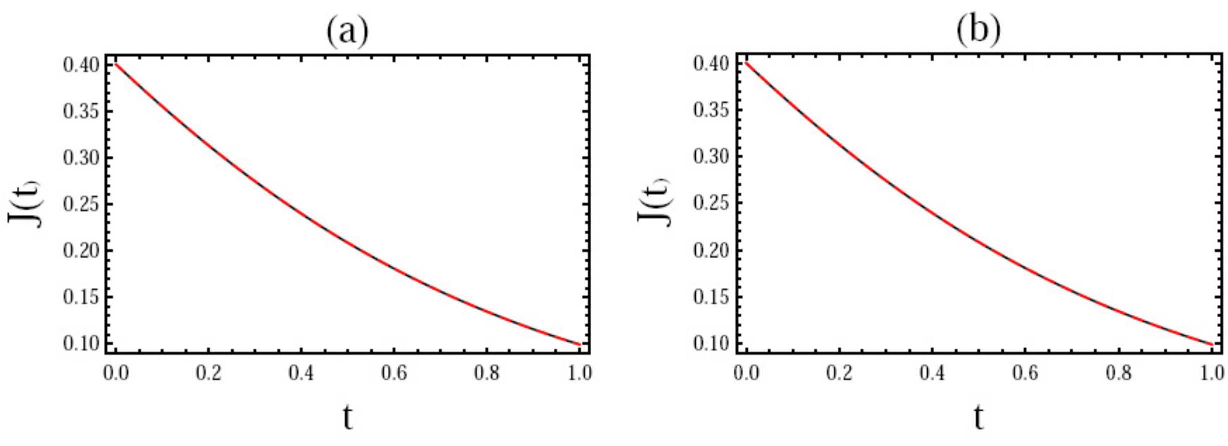

Figure 7: The approximate and exact solutions, with $R = 6$, $L = 3$, $J_0 = 0.4$, $\alpha = 1.0$; at $m = 5$ (a) and $m = 7$ (b).

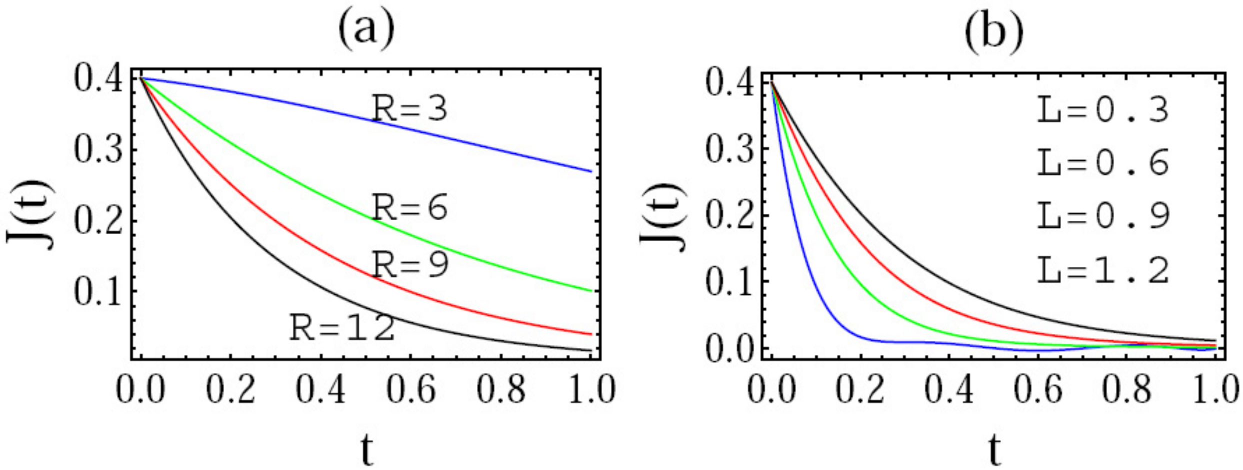

Figure 8: The approximate solution, with $J_0 = 0.4$, $m = 6$, $\alpha = 0.97$;
with different values of R (a) and L (b).

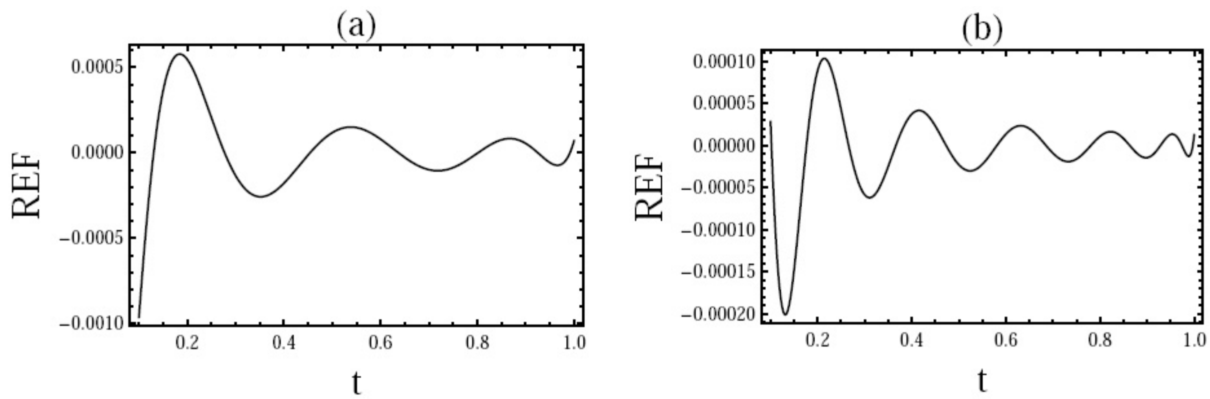

Figure 9. The REF with $R = 6$, $L = 3$, $J_0 = 0.4$, $\alpha = 0.98$;
at $m = 8$ (a) and $m = 14$ (b).

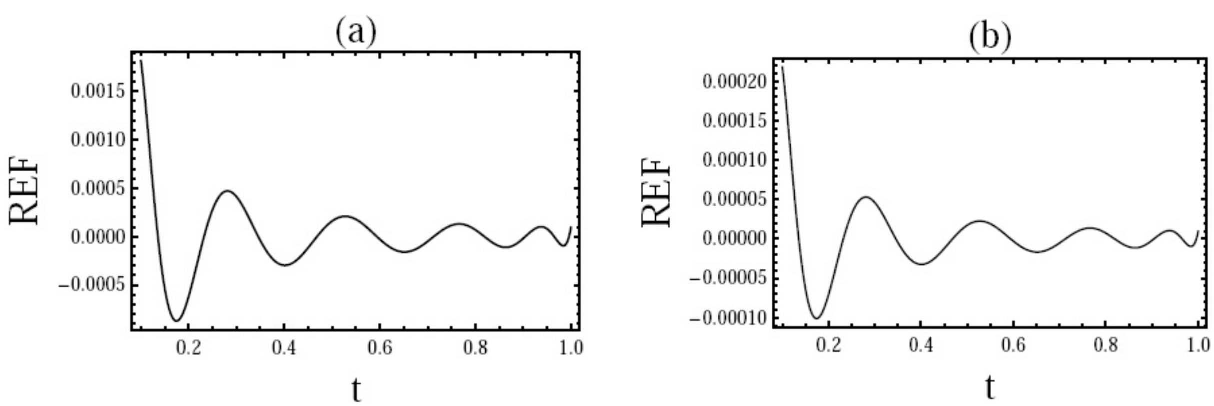

Figure 10. The REF with $R = 6$, $L = 3$, $J_0 = 0.4$, $m = 12$;
at $\alpha = 0.93$ (a) and $\alpha = 0.99$ (b).

5. Conclusion

In this paper, we used the proposed numerical method to convert the problem under study to a system of algebraic equations and to help solving it. In view of the obtained numerical solutions, we can confirm that there is an excellent agreement with the already existing solutions, and this shows that this approach can be implemented effectively to solve the proposed problems. Also, we can confirm that the overall errors can be decreased by adding new terms from the series of the solution. The given examples showed that the proposed method is more stable and more effective. Our method is much more effective and stable by comparing it to the exact solution.

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