An Efficient Approach for Solving the two Dimensional Variable Order Linear and Nonlinear Reaction Sub Diffusion Equation

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Abstract

We can not list the applications or the fields that use the anomalous sub-diffusion equations due to their wide area, one of these important applications are in the chemical reactions when a single substance tends to move from an area of high concentration to an area of low concentration until the concentration is equal across space. The mathematical model that describes these physical-chemical phenomena is called the reaction subdiffusion equation. In our study, we try to solve the 2D variable order version of these equations (2DVORSE) (linear and nonlinear) using an accurate numerical technique which is the weighted average finite difference method (WAFDM). We will study the stability of the resulting scheme using the fractional version of the John von Neumann stability analysis procedure. An accurate specific stability condition that is valid for some parameters in the resulting schemes is derived and checked. At the end of the study, we present some numerical examples to demonstrate the accuracy of the proposed technique.

Keywords: Weighted average finite difference approximations; variable order 2D reaction-subdiffusion equation; stability analysis; numerical treatments.

MSC 2010: : 65N12, 65M60, 41A30.

1. Introduction

The very important mathematical tool which allow us to deal with the real life phenomena of integrals and derivatives of any arbitrary order is the fractional analysis. In the last studies, many real life phenomena became easier to study and imagine by applying this important new topic, you can see for example ([1], [2], [8], [10]-[12], [18], [20]), and the other references cited therein. Actually, no one can list all applications of this important branch of mathematics, but in fact there are many examples that studied using this important topic, for example in the viscoelastic materials [4], control theory [9], advection and dispersion of solutes in fractured media [14] and image processing [8].

The so-called "anomalous" physical behaviours has a special attention through the fractional analysis, where scaling power law of fractional arbitrary order appears universally as an empirical description of such complicated phenomena [23]. There is a very useful mathematical tool, which is the fractional diffusion equation, to study anomalous transport processes which are processes in which $< x^2 >\sim K_\gamma t^\gamma$, where γ is the anomalous diffusion exponent, $< x^2 >$ is the mean square displacement, and K is the anomalous diffusion coefficient. When $0 < \gamma < 1$, the diffusion will be slower than normal which is called the anomalous sub-diffusion equations, and is called the anomalous super diffusion equations if the diffusion is faster than normal and this happens when $\gamma > 1$.

To explain, mathematically, how the concentration of one or more substances distributed in space changes , we can use the reaction diffusion systems and Figure 1 shows, chemically the process. Under two processes, these changes are obtained, the first one is the chemical reactions, where the second one is the transport of the substance in space. Using this fact Seki et al. [26] and Yuste et al. [27] derived the fractional version of the reaction sub-diffusion equation.

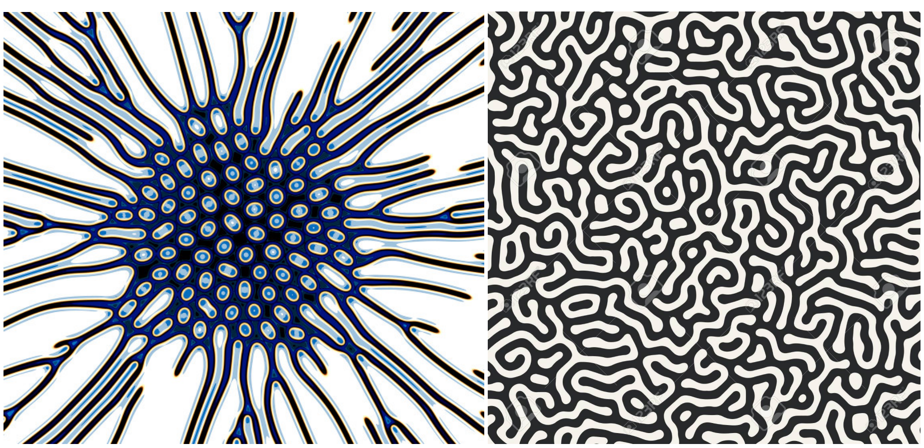

Figure 1: Diffusion reaction seamless pattern. Black and white organic shapes, lines pattern .

There are many papers that studied the fractional reaction sub-diffusion equation as ([20], [30], [31]).

Few researchers have considered the two-dimensional anomalous sub-diffusion equation ([22], [29], [30], [32]).

In many applications, the order of the derivative or the integral is fixed, but this may not be the best choice due to the dynamic process of the trajectories, so we may assume that the order γ of the integrals and derivatives is not constant during the process, and to be a function $\gamma(t)$ depending on time. In that case. To develop a theory where both fractional (right and left) operators are taken into account. Hence, we introduce some combined fractional operators like the combined Caputo fractional derivative that consists in a linear combination of the right and the left operators.

Recently, there are many researchers studied many applications using mathematical models describe by the variable order derivatives, for example you can see ([3], [5]-[7], [13], [19], [25]). We are sure that there are many researchers are working on this topic due to the huge number of applications that described using these important tools.

In our study we will solve the 2D variable order version of these equation (2DVORSE) which can be formulated in the following IBVP:

$$\frac{\partial u(x,y,t)}{\partial t} = D_t^{1-\gamma(x,y,t)}[\Delta u(x,y,t) - \varsigma u(x,y,t)] + v(u,x,y,t), \quad (x,y) \in D, \quad 0 \leq t < \tau, \tag{1}$$

,

where $D_t^{1-\gamma(x,y,t)}$ is the variable-order fractional derivative defined by the Riemann-Liouville operator of order $1-\gamma(x,y,t)$, where $\gamma(x,y,t)$ is the variable order such that and its maximum and minimum values are bounded by 0 and 1, $\varsigma > 0$ is a positive constant, and Δ is the two-dimensional Laplacian operator, with initial condition

$$u(x,y,0) = u_0(x,y), (x,y) \in D, \tag{2}$$

and the Dirichlet boundary conditions

$$\begin{aligned} u(0,y,t) &= u_1(y,t), u(l,y,t) = u_2(y,t), \\ u(x,0,t) &= v_1(x,t), u(x,h,t) = v_2(x,t), \end{aligned} \tag{3}$$

where $D = \{(x,y) | 0 \leq x \leq l, 0 \leq y \leq h\}$.

The definitions of Riemann-Liouville variable-order fractional derivatives and Grünwald-Letnikov variable-order fractional derivatives, which will be used in the studied model, are defined as follow:

Definition 1. [17] The left RiemannLiouville derivative of variable-order $\gamma(t)$ for the function $f(t)$ is defined by

$$D_x^{\gamma(t)} f(t) = \frac{1}{\Gamma(n-\gamma(t))} (\frac{d^n}{dt})^n \int_0^x \frac{f(s)}{(t-s)^{\gamma(t)-n+1}} ds, \tag{4}$$

where $n - 1 < \gamma_{min} < \gamma(t) < \gamma_{max} < n, n \in \mathbb{N}$ for all $t \in [0, \tau]$.

Definition 2. [17] The Grünwald-Letnikov variable order derivative of variable-order $\gamma(t)$ for the function $f(x)$ is defined as

$$D_x^{\gamma(x)} f(x) = lim_{h\longrightarrow 0} \frac{1}{h^{\gamma(x)}} \sum_{k=0}^{[\frac{x}{h}]} \Omega_k^{(\gamma(x))} f(x-hk), \qquad x \geq 0, \tag{5}$$

where $[\frac{x}{h}]$ is the integer part of $\frac{x}{h}$, and $\Omega_k^{(\gamma(x))}$ are the normalized Grünwald weights which can be written as $\Omega_k^{(\gamma(x))} = (-1)^k \begin{pmatrix} \gamma(x) \\ k \end{pmatrix}$.

2. Derivation of the numerical scheme:

The WAFDM is applied to obtain the discretization finite difference scheme of the two-dimensional variable-order Cable equation. Assume that A, B, $C \in Z^+$. We use $\Delta x, \Delta y$ and Δt, to express the discretizations for the space-step lengths and time-step length, respectively. And let $x_a = a\Delta x$, $(a = 0, 1, ..., A)$, $y_b = b\Delta y$, $(b = 0, 1, ..., B)$ and $t_c = c\Delta t$, $(c = 0, 1, ..., C)$ be the coordinates of the mesh grade points of D and the exact values of the solution $u(x, y, t)$ on these grid points can be denoted by $u(x_a, y_b, t_c) \equiv u^c_{a,b} \simeq U^c_{a,b}$, where $\Delta x = \frac{l}{A}$, $\Delta y = \frac{h}{B}$ and $\Delta t = \frac{\tau}{C}$. The reader can read [16] for more information and a lot of details about discretization in fractional calculus.

Also, the variable-order differential operators are discretized as in the following [15]:

$$\frac{\partial u}{\partial t}|_{x_a,y_b,t_{c+\frac{1}{2}}} = \delta_t u_{a,b}^{c+\frac{1}{2}} + O(\Delta t) \equiv \frac{u_{a,b}^{c+1} - u_{a,b}^c}{\Delta t} + O(\Delta t), \tag{6}$$

$$\frac{\partial^2 u}{\partial x^2}|_{x_a,y_b,t_c} = \delta_{xx} u^c_{a,b} + O((\Delta x)^2) \equiv \frac{u^c_{a-1,b} - 2u^c_{a,b} + u^c_{a+1,b}}{(\Delta x)^2} + O((\Delta x)^2), \tag{7}$$

and

$$\frac{\partial^2 u}{\partial y^2}|_{x_a,y_b,t_c} = \delta_{yy} u^c_{a,b} + O((\Delta y)^2) \equiv \frac{u^c_{a,b-1} - 2u^c_{a,b} + u^c_{a,b-1}}{(\Delta y)^2} + O((\Delta y)^2). \tag{8}$$

The discretization of the Riemann-Liouville operator is given by:

$$D_t^{1-\gamma^c_{a,b}} u(x, y, t)|_{x_a,y_b,t_c} = \delta_t^{1-\gamma^c_{a,b}} u^c_{a,b} + O(h^r), \tag{9}$$

where

$$\delta_t^{1-\gamma^c_{a,b}} u^c_{a,b} \equiv \frac{1}{h^{1-\gamma^c_{a,b}}} \sum_{i=0}^{[\frac{t_c}{\Delta t}]} \Omega_i^{(1-\gamma^c_{a,b})} u(x_a, y_b, t_c - ih) = \frac{1}{\Delta t^{1-\gamma^c_{a,b}}} \sum_{i=0}^{c} \Omega_i^{(1-\gamma^c_{a,b})} u^{c-l}_{a,b}, \tag{10}$$

and $[\frac{t_c}{\Delta t}]$ usual Dirichlet function of $\frac{t_c}{\Delta t}$ and r is the order of the approximation which is varies according to the choice of $\Omega_i^{(1-\gamma^c_{a,b})}$. One can note that this above expression is not the only expression due to the many different expressions of the weights functions $\Omega_i^{(\gamma)}$ [15]. Let $\Omega_i^{(\gamma_{a,b}^c)}$ be the general function of the functions $\Omega(z,\gamma_{a,b}^c)$, so we have,

$$\Omega(z,\gamma_{a,b}^c)=\sum_{i=0}^{\infty}\Omega_i^{(\gamma_{a,b}^c)}z^i. \tag{11}$$

Lemma 0.1. *[17] Assume that $0<\gamma_{min}<\gamma(x,y,t)<\gamma_{max}<1$, for $a=1,2,\dots,A, b=1,2,\dots,B, c=1,2,\dots,C$, and $i=0,1,\dots,$ we have the following:*

1. $\Omega_0^{(\gamma_{a,b}^c)}=1;\ \Omega_i^{(\gamma_{a,b}^c)}=(\gamma_{a,b}^c)-1<0\ ;\ \Omega_i^{(\gamma_{a,b}^c)}<0,\ i=2,3,;$
2. $\sum_{i=0}^{\infty}\Omega_i^{(\gamma_{a,b}^c)}=0;$ *and* $-\sum_{l=0}^{h}\Omega_l^{(\gamma_{i,j}^k)}<1$, *for* $h=1,2,\dots.$

The required WAFD scheme of the two-dimensional variable-order reaction sub-diffusion equation (1) will be obtained by evaluating this big equation at the intermediate point of the grid $(x_a,y_b,t_c+\frac{\Delta t}{2})$

$$[u_t(x,y,t)-D_t^{1-\gamma(x_a,y_b,t_c+\frac{\Delta t}{2})}[u_{xx}(x,y,t)+u_{yy}(x,y,t)]]_{(x_a,y_b,t_c+\frac{\Delta t}{2})}+\varsigma D_t^{1-\gamma(x_a,y_b,t_c)}u(x_a,y_b,t_c)=0. \tag{12}$$

By using the equations (6), (7) and (8) at the times t_c and t_{c+1}, we get

$$\delta_t u_{a,b}^{c+\frac{1}{2}}-\{\theta\delta_t^{1-\gamma_{a,b}^c}[\delta_{xx}u_{a,b}^c+\delta_{yy}u_{a,b}^c]+(1-\theta)\delta_t^{1-\gamma_{a,b}^c}[\delta_{xx}u_{a,b}^{c+1}+\delta_{yy}u_{a,b}^{c+1}]\}+\varsigma\delta_t^{1-\gamma_{a,b}^c}u_{a,b}^c=E_{a,b}^{c+\frac{1}{2}}, \tag{13}$$

where θ is the weight factor and $E_{a,b}^{c+\frac{1}{2}}$ is the usual truncation error. The standard difference formula is given by:

$$\delta_t U_{a,b}^{c+\frac{1}{2}}-\{\theta\delta_t^{1-\gamma_{a,b}^c}[\delta_{xx}U_{a,b}^c+\delta_{yy}U_{a,b}^c]+(1-\theta)\delta_t^{1-\gamma_{a,b}^{c+1}}[\delta_{xx}U_{a,b}^{c+1}+\delta_{yy}U_{a,b}^{c+1}]\}+\varsigma\delta_t^{1-\gamma_{a,b}^c}U_{a,b}^c=0. \tag{14}$$

Now, by substitution using equations (6)–(8) and (10), with the assumption that $\epsilon_{\gamma_{a,b}^c}=\frac{(\Delta t)^{\gamma_{a,b}^c}}{(\Delta x)^2}$, $\varepsilon_{\gamma_{a,b}^c}=\frac{(\Delta t)^{\gamma_{a,b}^c}}{(\Delta y)^2}$, and $\eta_{\gamma_{a,b}^c}=(\Delta t)^{\gamma_{a,b}^c}$,then Put $\rho=(1-\theta)\epsilon_{\gamma_{a,b}^c}$ and $\chi=(1-\theta)\varepsilon_{\gamma_{b,c}^a}$. Under some calculations, we can obtain the following WAFD scheme:

$$\rho[U_{a-1,b}^{c+1}+U_{a+1,b}^{c+1}]+\chi[U_{a,b-1}^{c+1}+U_{a,b+1}^{c+1}]-[2(\chi+\rho)+1]U_{a,b}^{c+1}=\Gamma. \tag{15}$$

Where

$$\begin{aligned}\Gamma &= -\epsilon_{\gamma_{a,b}^c}\sum_{i=0}^{c}[\theta\Omega_i^{(1-\gamma_{a,b}^c)}+(1-\theta)\Omega_{i+1}^{(1-\gamma_{a,b}^c)}][U_{a-1,b}^{c-i}-2U_{a,b}^{c-i}+U_{a+1,b}^{c-i}]\\ &- \varepsilon_{\gamma_{a,b}^c}\sum_{i=0}^{c}[\theta\Omega_i^{(1-\gamma_{a,b}^c)}+(1-\theta)\Omega_{i+1}^{(1-\gamma_{a,b}^c)}][U_{a,b-1}^{c-i}-2U_{a,b}^{c-i}+U_{a,b+1}^{c-i}]\\ &- U_{a,b}^c+\varsigma\eta_{\gamma_{a,b}^c}\sum_{i=0}^{c}\Omega_i^{(1-\gamma_{a,b}^c)}U_{a,b}^{c-i}.\end{aligned} \tag{16}$$

The system of equations (15) is a tridiagonal system which easy can be solved by the help of the conjugate gradian famous method. When $\theta = 1$ and $\theta = \frac{1}{2}$, we will have the backward Euler fractional quadrature method and the Crank-Nicholson quadrature methods, respectively, but at $\theta = 0$ the resulting scheme will be fully implicit.

3. Stability analysis

The stability analysis for the resulting scheme for the two-dimensional variable-order reaction sub-diffusion equation (15) will be studied by using John von Neumann stability analysis technique and an accurate stability condition for the resulting numerical scheme (15) will be derived. In the stability analysis study we neglected the source term (i.e., $v(u, x, y, t) = 0$).

Proposition 1. *Assume that* $U^c_{a,b} = \xi e^{ap(a\Delta x + b\Delta y)}$, $\xi_{m+1} = \eta\xi_m$, $\varpi_1 = \epsilon_{\gamma^c_{a,b}} \sin^2(\frac{p\Delta x}{2})$ *and* $\varpi_2 = \varepsilon_{\gamma^c_{a,b}} \sin^2(\frac{p\Delta y}{2})$ *then the scheme will be stable as long as*

$$-1 \leq \frac{1 - \varsigma\zeta_{\gamma^c_{a,b}} \sum_{i=0}^{c} \Omega_i^{(1-\gamma^c_{a,b})} \eta^{-i} - 4\varpi_1 \sum_{i=0}^{c} (\theta\Omega_i^{(1-\gamma^c_{a,b})} + (1-\theta)\Omega_{i+1}^{(1-\gamma^c_{a,b})})\eta^{-i} - 4\varpi_2 \sum_{i=0}^{c} (\theta\Omega_i^{(1-\gamma^c_{a,b})} + (1-\theta)\Omega_{i+1}^{(1-\gamma^c_{a,b})})\eta^{-i}}{1 - 4(1-\theta)\varpi_1 - 4(1-\theta)\varpi_2} \leq 1. \tag{17}$$

Proposition 2. *Assuming that*

$$\frac{4(2-\theta)(1 - \sum_{i=0}^{c} \Omega_i^{(1-\gamma^c_{a,b})}(-1)^{-i}) + (-1)^m(1-\theta)\Omega_{m+1}^{1-\gamma^c_{a,b}}}{2 - \varsigma\zeta_{\gamma^c_{a,b}} \sum_{i=0}^{c} \Omega_i^{(1-\gamma^c_{a,b})}(-1)^{-i}} = \frac{1}{\vartheta_m}. \tag{18}$$

Then (15) is stable if the following identity holds:

$$\frac{1}{\varpi_1 + \varpi_2} \geq \frac{1}{\vartheta_m}.$$

Theorem 1. *The resulting WAFD numerical scheme in (15) is stable under the following stability condition:*

$$\frac{1}{\epsilon_{\gamma^c_{a,b}} + \varepsilon_{\gamma^c_{a,b}}} \geq \frac{4(2-\theta)(2 - 2^{1-\gamma^c_{a,b}})}{1 - \varsigma\zeta_{\gamma^c_{a,b}} 2^{-\gamma^c_{a,b}}}. \tag{19}$$

Proof. Since ϑ_m tends quickly towards its limit value due to the dependence of ϑ depends on m,

$$\vartheta = \lim_{m\to\infty} \vartheta_m. \tag{20}$$

According to this limit notation, the above stability condition will be

$$\varpi_1 + \varpi_2 \leq \frac{2 - \varsigma\zeta_{\gamma^c_{a,b}} \sum_{i=0}^{\infty} \Omega_i^{1-\gamma^c_{a,b}}(-1)^{-i}}{4\{(2-\theta)[1 - \sum_{i=1}^{\infty} \Omega_i^{1-\gamma^c_{a,b}}] + \lim_{m\to\infty}(-1)^m(1-\theta)\Omega_{m+1}^{1-\gamma^c_{a,b}}\}}, \tag{21}$$

by the help of Eq.(11) with $z=-1$, we get that $\sum_{i=0}^{\infty}(-1)^i\Omega_i^{1-\gamma}=2^{1-\gamma}$, then we have

$$\vartheta=\frac{2-\varsigma\zeta_{\gamma_{a,b}^c}2^{1-\gamma_{a,b}^c}}{4\{(2-\theta)[2-2^{1-\gamma_{a,b}^c}]+\lim_{m\to\infty}(-1)^m(1-\theta)\Omega_{m+1}^{1-\gamma_{a,b}^c}\}},\tag{22}$$

but since $\varpi_1=\epsilon_{\gamma_{a,b}^c}sin^2(\frac{p\Delta x}{2})$ and $\varpi_2=\varepsilon_{\gamma_{a,b}^c}sin^2(\frac{p\Delta y}{2})$, and by replacing $sin^2(\frac{p\Delta x}{2})$ and $sin^2(\frac{p\Delta y}{2})$ by their maximum values and since $\lim_{m\to\infty}(-1)^m(1-\theta)\Omega_{m+1}^{1-\gamma_{a,b}^c}=0$, i.e. $\epsilon_{\gamma_{a,b}^c}\le\varpi_1$ and $\varepsilon_{\gamma_{a,b}^c}\le\varpi_2$, therefore

$0\le\epsilon_{\gamma_{a,b}^c}+\varepsilon_{\gamma_{a,b}^c}\le\varpi_1+\varpi_2$, then $\frac{1}{\epsilon_{\gamma_{a,b}^c}+\varepsilon_{\gamma_{a,b}^c}}\ge\frac{1}{\varpi_1+\varpi_2}\ge\frac{1}{\vartheta_m}$.

The proof is ended because now we can write $\frac{1}{\epsilon_{\gamma_{a,b}^c}+\varepsilon_{\gamma_{a,b}^c}}\ge\frac{4(2-\theta)(2-2^{1-\gamma_{a,b}^c})}{1-\varsigma\zeta_{\gamma_{a,b}^c}2^{-\gamma}}$; $\quad 0\le\theta\le1$. □

4. Numerical results

Now, we are ready to present linear and non-linear numerical treatments to check the accuracy and the stability of the resulting scheme, by applying to solve numerically the two-dimensional variable-order reaction sub-diffusion equation.

Example 1.

Let us onsider the following two-dimensional variable-order linear reaction sub-diffusion equation:

$$\begin{cases}\frac{\partial u}{\partial t}=D_t^{1-\gamma}\Big(-u+\Delta u\Big)+v(x,y,t) & (x,y)\in D, 0<T<1,\\ u(x,y,t)=0 & (x,y)\in D, 0<T<1,\\ u(x,y,0)=0 & (x,y)\in D,\end{cases}$$

where $\Omega=[0,1]\times[0,1]$, $0<\gamma<1$, and $v(x,y,t)$ is given by :

$$v(x,y,t)=2\Big(t+\frac{1}{\Gamma(2+\gamma)}t^{1+\gamma}+\frac{2\pi^2}{\Gamma(2+\gamma)}t^{1+\gamma}\Big)\sin(\pi x)\sin(\pi y).$$

The exact solution of the equation is: $u(x,y,t)=t^2\sin(\pi x)\sin(\pi y)$.
In Figures 2-5 , we study many cases with various values of the parameters ($\theta,\gamma,\Delta x,\Delta y,\Delta t,\varsigma$, and the final time T_f) and the error is plotted in some cases to check the presented numerical scheme, where in Figure 6, we present a case for the behavior of the (unstable) numerical solution. Table 1 is created to show the dependency of maximum absolute error on Δx, Δy and Δt in two different cases under two different set of values for the parameters.

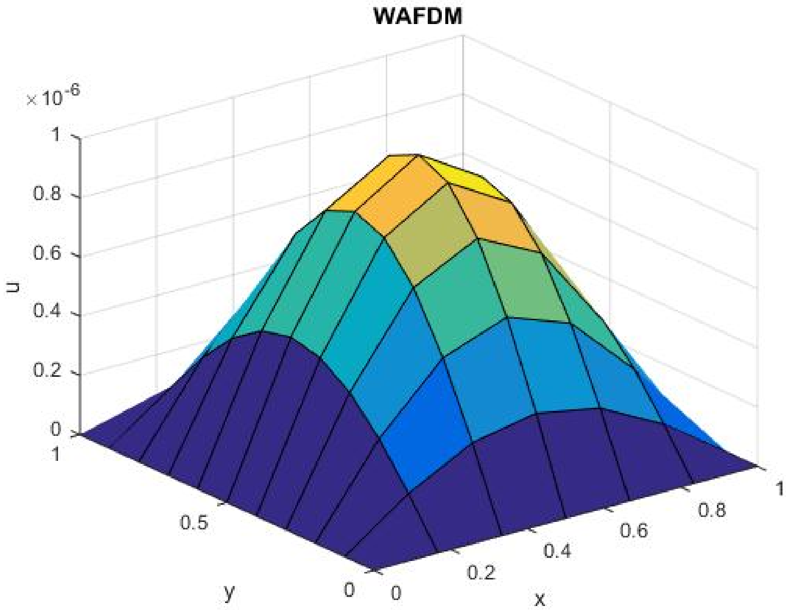

Figure 2: The behavior of the numerical solution with the following values of the parameters: $\theta = 1$ for $\gamma = \frac{3}{2} + \frac{1}{4}\cos(xy)\sin(t^2)$, $\Delta x = \frac{1}{6}$, $\Delta y = \frac{1}{10}$, $\Delta t = 0.00998$, $\varsigma = 0.9$, $T_f = 0.5$, and the maximum error is $Max_{error} = 1.2056 \times 10^{-4}$.

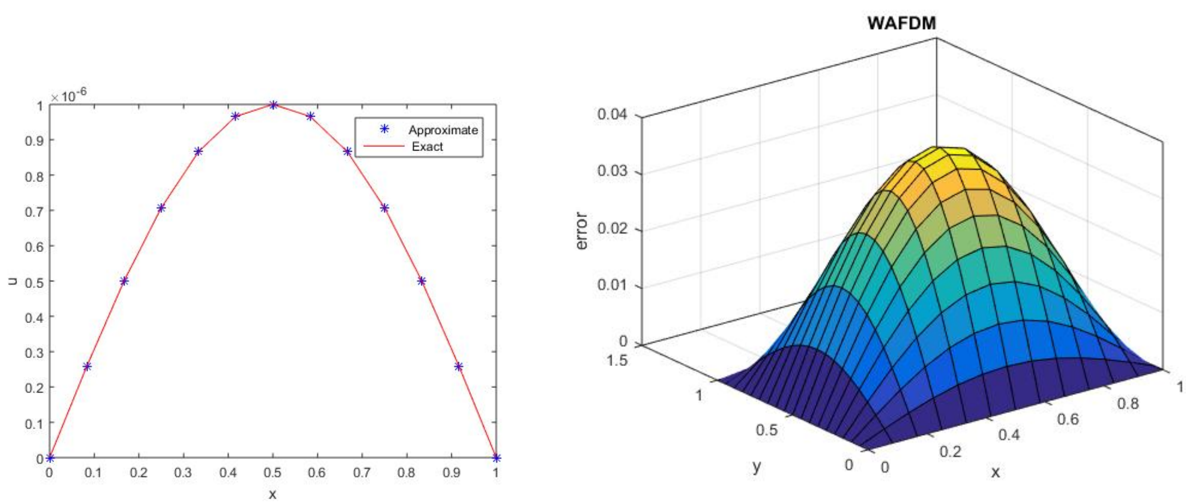

Figure 3: The behavior of the numerical and the exact solutions (at $y = 0.4$) together with the resulting error with the following values of the parameters: $\theta = 1$ for $\gamma = \big(2 - \cos x \cos(xy)\big)\sin^2(t)$, $\Delta x = \frac{1}{12}$, $\Delta y = \frac{1}{18}$, $\Delta t = 0.1999375$, $\varsigma = 0.25$, and $T_f = 0.2$.

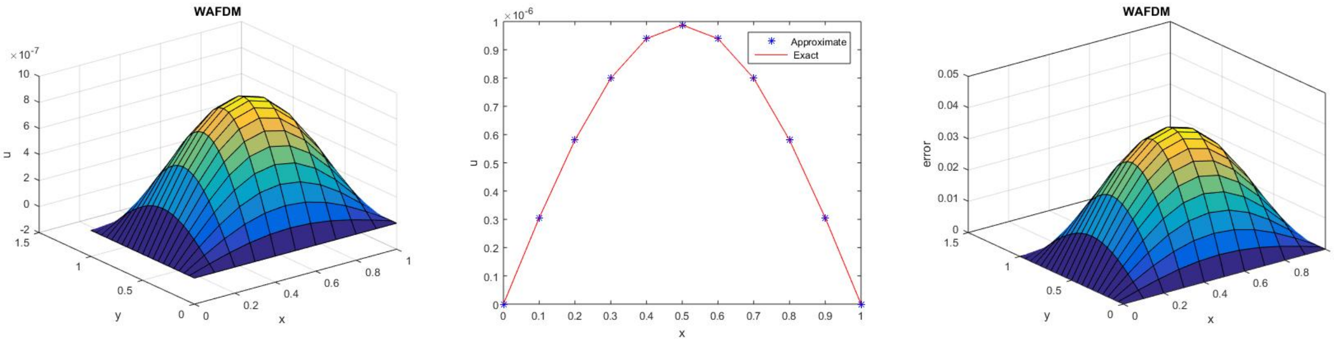

Figure 4: The behavior of the numerical and the exact solutions (at $y = 0.235$) together with the resulting error with the following values of the parameters: $\theta = 0$ for $\gamma = \frac{16-e^{xyt}}{17}$, $\Delta x = \frac{1}{10}$, $\Delta y = \frac{1}{20}$, $\Delta t = 0.0132666666$, $\varsigma = 0.6$, and $T_f = 0.2$.

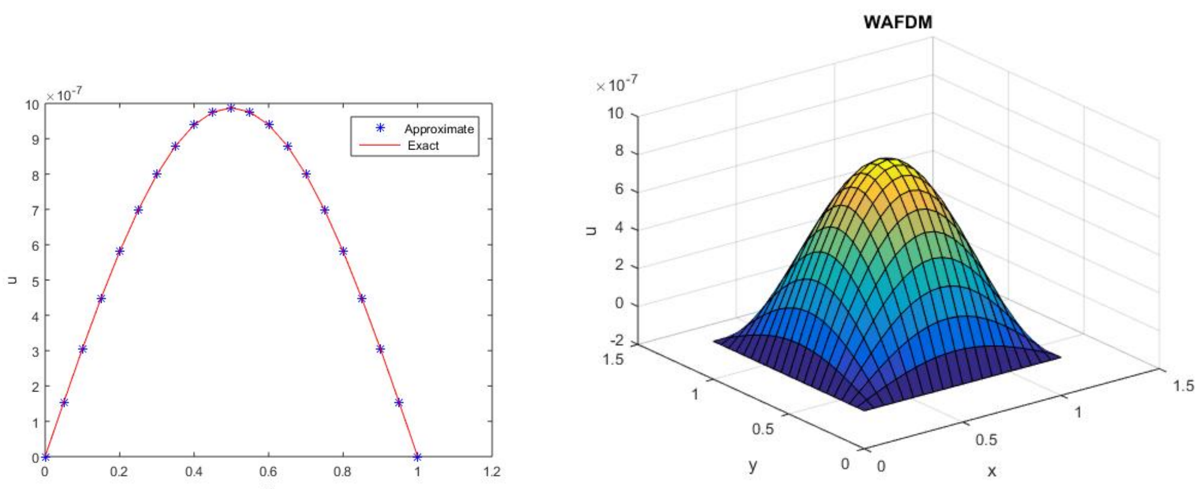

Figure 5: The behavior of the numerical and the exact solutions (at $y = 0.325$) with the following values of the parameters: $\theta = 0.5$ for $\gamma = \frac{20-(xyt)^4}{20}$, $\Delta x = \frac{1}{20}$, $\Delta y = \frac{1}{20}$, $\Delta t = 0.00495$, $\varsigma = 0.4$, $T_f = 0.1$, and the maximum error is $Max_{error} = 3.5403 \times 10^{-5}$.

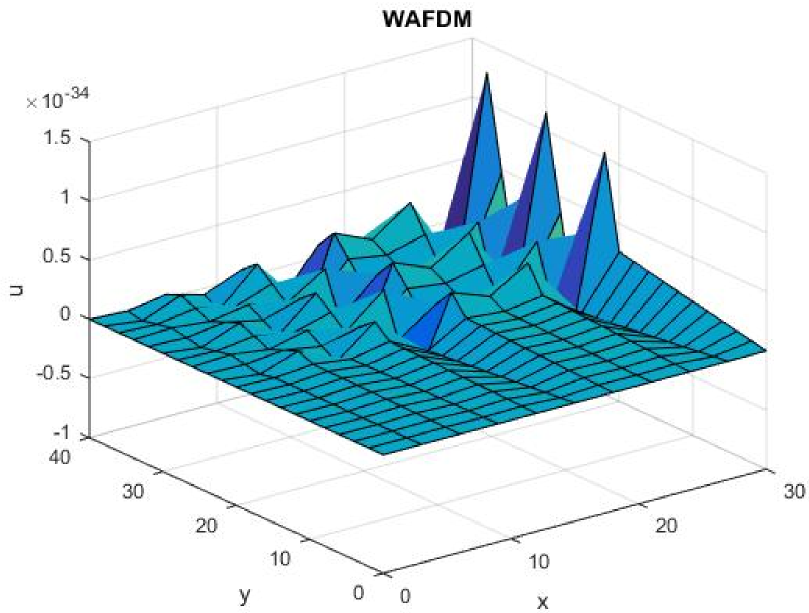

Figure 6: The behavior of the (unstable) numerical solution with the following values of the parameters: $\theta = 1$ for $\gamma = (2 - \cos^2 x \cos^2 y)\sin^2(ty)$, $\Delta x = 3$, $\Delta y = 2$, $\Delta t = 0.7999333333$, $\varsigma = 0.75$, $T_f = 0.8$, and the maximum error is $Max_{error} = 5.1622 \times 10^{-29}$.

The value of the variable order $\gamma \Rightarrow$			$\gamma = \frac{xyt}{241\pi^2} + \frac{3}{2}$	$\gamma = \frac{3+e^{1-(xyt)^2}}{2}$
The values of the discretizations			Maximum Error	
Δx	Δy	Δt	$\theta = 0.5, \varsigma = 0.9$, $T = 0.3$,	$\theta = 0, \varsigma = 0.6$, $T = 0.2$,
1/4	1/5	3/40	3.7×10^{-3}	6.5×10^{-3}
1/5	1/5	3/50	3.3×10^{-3}	5.9×10^{-3}
1/5	1/10	3/100	9.5401×10^{-4}	7.3261×10^{-4}
1/10	1/10	3/100	9.5401×10^{-4}	6.325×10^{-4}
1/15	1/10	3/100	9.4958×10^{-4}	4.1543×10^{-4}
1/15	1/15	3/100	4.358×10^{-4}	4.3681×10^{-4}

Table 1: This table is to show the dependency of maximum absolute error on Δx, Δy and Δt in two different cases under two different set of values for the parameters [The first case when $\theta = 0.5, =0.9$, $T = 0.3$, and $\gamma = \frac{xyt}{241\pi^2} + \frac{3}{2}$, where the second case when $\theta = 0, \mathrm{s} = 0.6$, $T = 0.2$, and $\gamma = \frac{3+e^{1-(xyt)^2}}{2}$.

Example 2.

Consider the two-dimensional variable-order non-linear reaction sub-diffusion equation

$$\frac{\partial u(x,y,t)}{\partial t} = D_t^{1-\gamma(x,y,t)}\Big(\Delta u(x,y,t) - \varsigma u(x,y,t)\Big) + v(x,y,t), \quad t \in (0,1), \tag{23}$$

above the square $D = [0,1] \times [0,1]$, with initial condition

$$u(x,y,0) = 0, \quad (x,y) \in D, \tag{24}$$

and the Dirichlet boundary conditions

$$u(0,y,t) = 0,\ u(1,y,t) = \sin(1)e^y t^2,\ u(x,0,t) = \sin(x)t^2,\ \text{and}\ u(x,1,t) = \sin(x)et^2, \tag{25}$$

where

$$v(x,y,t) = e^y t\sin(x)\Big(2 + \frac{2+t^\gamma}{\Gamma(2+\gamma)} + e^y t^3 \sin(x)\Big) - u^2(x,y,t). \tag{26}$$

The exact solution of the equation is: $u(x,y,t) = \sin(x)e^y t^2$.

In Figures 7, 8, 10, 11, we study many cases with various values of the parameters ($\theta, \gamma, \Delta x, \Delta y, \Delta t, \varsigma$, and the final time T_f) and the error is plotted in some cases to check the presented numerical scheme, where in Figure 9, we present a case for the behavior of the (unstable) numerical solution.

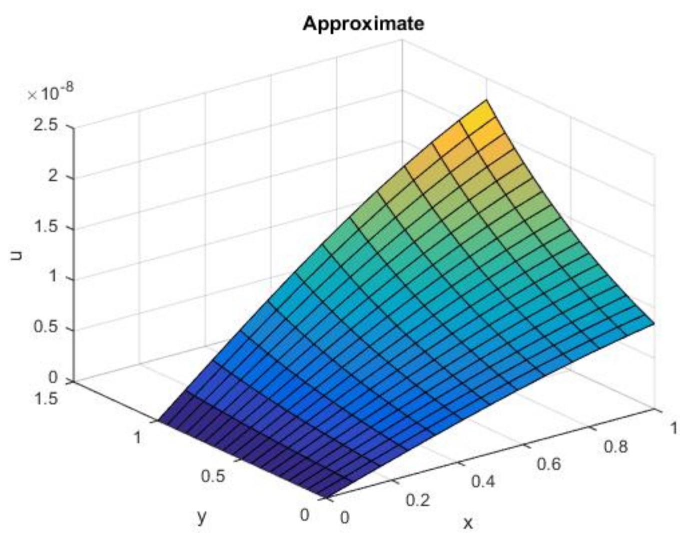

Figure 7: The behavior of the numerical solution with the following values of the parameters: $\theta = 0.5$ for $\gamma = 2 - \cos^2(xy)\sin^2(t)$, $\Delta x = \frac{1}{12}$, $\Delta y = \frac{1}{18}$, $\Delta t = 0.0070714$, $\varsigma = 0.1$, and $T_f = 0.08$.

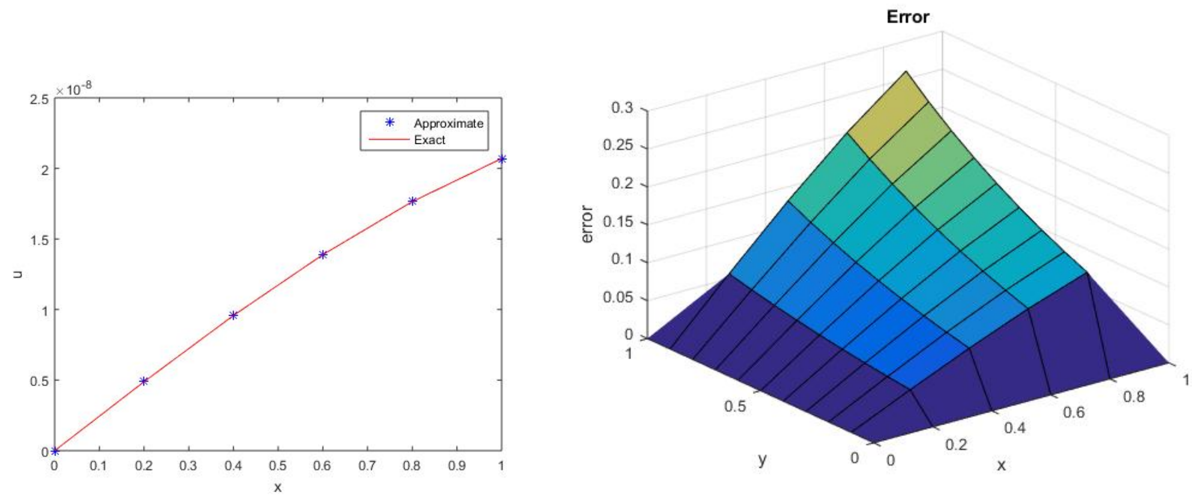

Figure 8: The behavior of the numerical and the exact solutions (at $y = 0.4$) together with the resulting error with the following values of the parameters: $\theta = 0$ for $\gamma = \frac{3}{2} + \frac{1}{4}\cos(xy)$, $\Delta x = \frac{1}{5}$, $\Delta y = \frac{1}{10}$, $\Delta t = 0.0399$, $\varsigma = 0.4$, and $T_f = 0.4$.

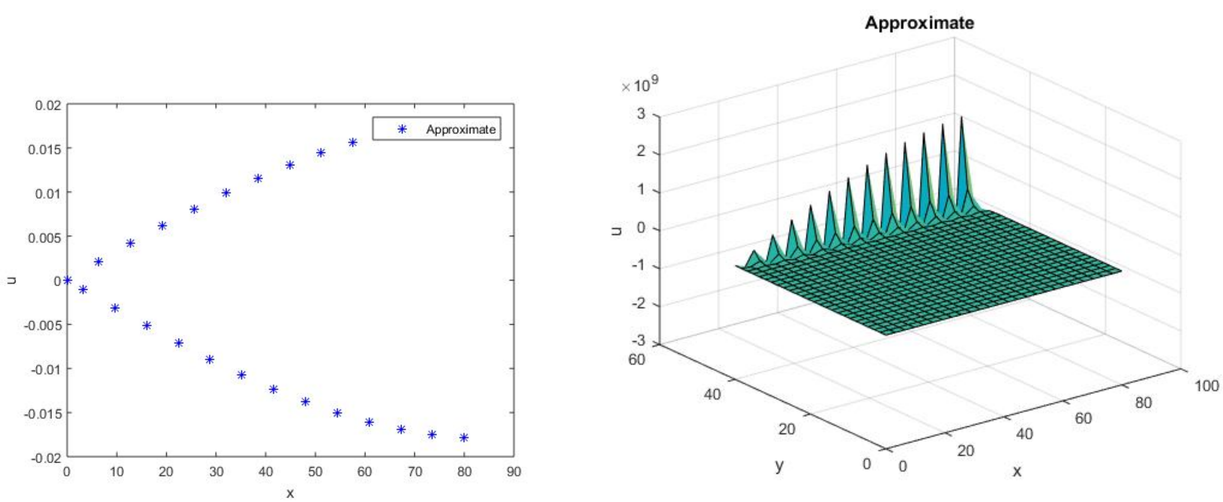

Figure 9: The behavior of the (unstable) numerical solution with the following values of the parameters: $\theta = 1$ for $\gamma = \frac{15-(xyt)^3}{15}$, $\Delta x = 3.2$, $\Delta y = 1.6$, $\Delta t = 0.0039$, $\varsigma = 0.75$, and the final time $T_f = 0.04$.

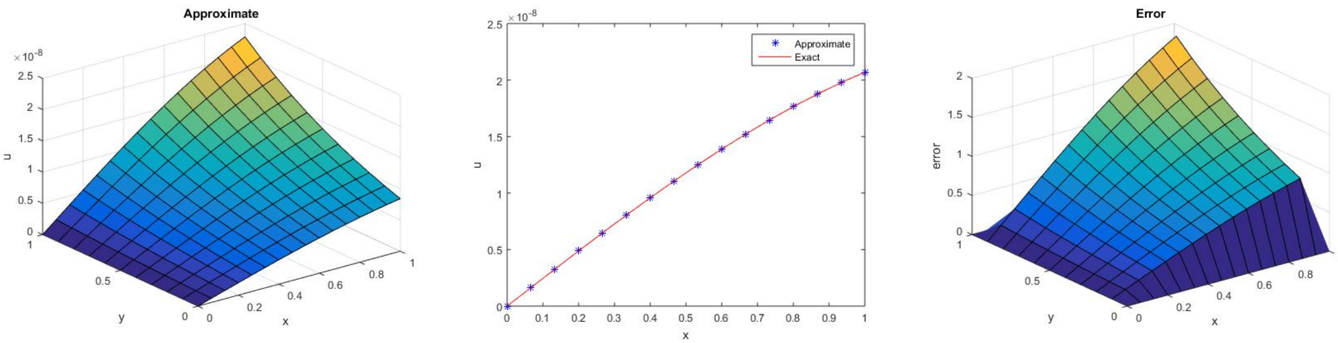

Figure 10: The behavior of the numerical and the exact solutions (at $y = 0.235$) together with the resulting error with the following values of the parameters: $\theta = 0$ for $\gamma = \frac{3}{2} + \frac{1}{2}e^{-xyt^2-1}$, $\Delta x = \frac{1}{15}$, $\Delta y = \frac{1}{10}$, $\Delta t = 0.0399$, $\varsigma = 0.4$, and $T_f = 1$.

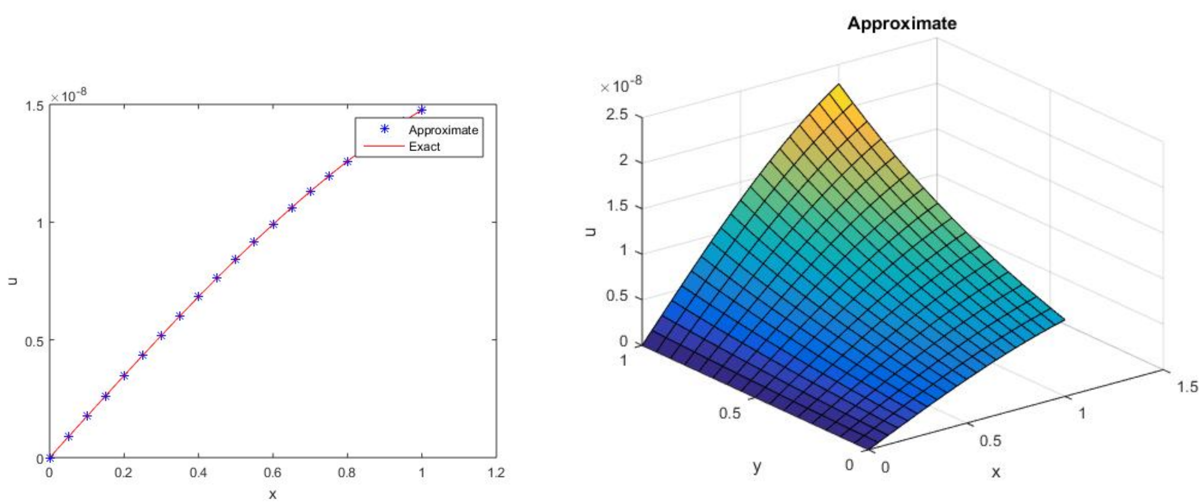

Figure 11: The behavior of the numerical and the exact solutions (at $y = 0.325$) with the following values of the parameters: $\theta = 1$ for $\gamma = \frac{15-\sin((xyt)^4)}{10}$, $\Delta x = \frac{1}{20}$, $\Delta y = \frac{1}{16}$, $\Delta t = 0.0249166$, $\varsigma = 0.8$, and $T_f = 0.3$.

5. Conclusion and discussion

In this manuscript, we have solved the two-dimensional variable-order reaction sub-diffusion equation by the help of WAFDM. The stability analysis of the resulting WAFDM scheme is presented using an extension of the John von Neumann stability technique, and we now can write that that this procedure is suitable and leads to very good predictions for the stability bounds. Two numerical treatments with their exact solutions are presented, and the derived stability condition is checked. From this study. The computations in this work were done using Matlab programming.

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