A NEW EXTENSION OF THE (H.2) SUPERCONGRUENCE OF VAN HAMME FOR PRIMES $p \equiv 3 \pmod 4$

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Abstract. Using Andrews' multi-series generaliazation of Watson's ${}_8\phi_7$ transformation, we give a new extension of the (H.2) supercongruence of Van Hamme for primes $p \equiv 3 \pmod 4$, as well as its q-analogue. Meanwhile, applying the method of 'creative microscoping', recently introduced by the author and Zudilin, we establish some further q-supercongruences modulo $\Phi_n(q)^3$, where $\Phi_n(q)$ denotes the n-th cyclotomic polynomial in q.

1. Introduction

In 1997, Van Hamme [29, (H.2)] eastablished the following supercongruence: for any prime $p \equiv 3 \pmod 4$,

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}, \tag{1.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol. Since the p-adic order of $(1/2)_k/k!$ is 1 for $(p-1)/2 < k \leqslant p-1$, we may compute the sum in (1.1) for k up to $p-1$. For all kinds of generalizations of (1.1), we refer the reader to [14, 16, 17, 19, 20, 22, 26, 27].

The objective of this paper is to prove the following extension of (1.1).

Theorem 1.1. *Let $s \geqslant 0$ be an even integer. Let p be an odd prime with $p \geqslant 2s+3$ and $p \equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{(p-1)/2} (4k+1)^s \frac{(\frac{1}{2})_k^3}{k!^3} \equiv 0 \pmod{p^2}. \tag{1.2}$$

It is clear that (1.1) is the $s=0$ case of (1.5). We shall prove Theorem 1.1 by establishing its q-analogue.

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Theorem 1.2. *Let $s \geqslant 0$ be an even integer. Let n be a positive odd integer with $n \geqslant 2s+3$ and $n \equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{(n-1)/2} [4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{(1-2s)k} \equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.3}$$

Here we need to familiarize ourselves with the standard basic hypergeometric notation. The *q-integer* is defined by $[n]=[n]_q=(1-q^n)/(1-q)$. The *q-shifted factorial* is defined as $(a;q)_0=1$ and $(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n\geqslant 1$. Moreover, the n-th *cyclotomic polynomial* $\Phi_n(q)$ is given by

$$\Phi_n(q)=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(n,k)=1}}(q-\zeta^k),$$

where ζ is an n-th primitive root of unity. Note that $\Phi_n(q^2)=\Phi_n(q)\Phi_n(-q)$ for odd n.

Letting $n=p$ be a prime and taking the limits as $q\to 1$ in (1.3), we immediately obtain (1.5). The $s=0$ case of (1.3) is already known. It was first conjectured by the author and Zudilin [15, Conjecture 4.13] and was recently confirmed by themselves in [17]. For some other recent progress on q-congruences, the reader may consult [3–5, 7, 10, 12, 14, 16, 17, 21, 25, 30, 31, 33].

Recently, Mao and Pan [23] (see also Sun [27, Theorem 1.3]) showed that, if $p\equiv 1 \pmod 4$ is a prime, then

$$\sum_{k=0}^{(p+1)/2}\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv 0 \pmod{p^2}. \tag{1.4}$$

In this paper, we shall prove the following generalization of (1.4).

Theorem 1.3. *Let $s\geqslant 0$ be an even integer. Let p be an odd prime with $p\geqslant 2s-3$ and $p\equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{(p+1)/2}(4k-1)^s\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv 0 \pmod{p^2}. \tag{1.5}$$

For some recent supercongruences involving $(4k-1)^s$ with s odd, see [11]. As before, we have a q-analogue of Theorem 1.3 as follows.

Theorem 1.4. *Let $s\geqslant 0$ be an even integer. Let n be a positive odd integer with $n\geqslant \max\{3,2s-3\}$ and $n\equiv 1 \pmod 4$. Then*

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{(7-2s)k}\equiv 0 \pmod{\Phi_n(q)^2}. \tag{1.6}$$

Likewise, letting $n=p$ be a prime and taking the limits as $q\to 1$ in (1.6), we are led to (1.5). The $s=0$ case of (1.3) was first observed by the author and Schlosser [13, Conjecture 10.2] and was recently proved by the author and Zudilin [17].

We shall prove Theorems 1.2 and 1.4 in Sections 2 and 3 by making a careful use of Andrews' multi-series extension [1, Theorem 4] of Watson's ${}_8\phi_7$ transformation. It should be pointed out that Andrews' transformation plays an important part in combinatorics and number theory (see [8] and the introduction of [12] for such examples). We shall give generalizations of Theorems 1.2 and 1.4 modulo $\Phi_n(q)^3$ in Section 4 by using the 'creative microscoping' method, recently introduced by the author and Zudilin [15]. We end this paper by Section 5, where we propose two open problems on further generalizations of Theorems 1.1 and 1.3.

2. Proof of Theorem 1.2

We will use the following powerful transformation formula due to Andrews [1, Theorem 4]:

$$\begin{aligned}
&\sum_{k\geqslant 0}\frac{(a,q\sqrt{a},-q\sqrt{a},b_1,c_1,\dots,b_m,c_m,q^{-N};q)_k}{(q,\sqrt{a},-\sqrt{a},aq/b_1,aq/c_1,\dots,aq/b_m,aq/c_m,aq^{N+1};q)_k}\left(\frac{a^mq^{m+N}}{b_1c_1\cdots b_mc_m}\right)^k\\
&\quad=\frac{(aq,aq/b_mc_m;q)_N}{(aq/b_m,aq/c_m;q)_N}\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(aq/b_1c_1;q)_{j_1}\cdots(aq/b_{m-1}c_{m-1};q)_{j_{m-1}}}{(q;q)_{j_1}\cdots(q;q)_{j_{m-1}}}\\
&\qquad\times\frac{(b_2,c_2;q)_{j_1}\dots(b_m,c_m;q)_{j_1+\cdots+j_{m-1}}}{(aq/b_1,aq/c_1;q)_{j_1}\dots(aq/b_{m-1},aq/c_{m-1};q)_{j_1+\cdots+j_{m-1}}}\\
&\qquad\times\frac{(q^{-N};q)_{j_1+\cdots+j_{m-1}}}{(b_mc_mq^{-N}/a;q)_{j_1+\cdots+j_{m-1}}}\frac{(aq)^{j_{m-2}+\cdots+(m-2)j_1}q^{j_1+\cdots+j_{m-1}}}{(b_2c_2)^{j_1}\cdots(b_{m-1}c_{m-1})^{j_1+\cdots+j_{m-2}}}.
\end{aligned}\tag{2.1}$$

This transformation may be deemed a multi-series generalization of Watson's ${}_8\phi_7$ transformation formula (see [2, Appendix (III.18)]) which in standard notation for basic hypergeometric series [2, Section 1] may be written as:

$$\begin{aligned}
&{}_8\phi_7\left[\begin{matrix}a, & qa^{\frac12}, & -qa^{\frac12}, & b, & c, & d, & e, & q^{-n}\\ & a^{\frac12}, & -a^{\frac12}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1}\end{matrix};q,\ \frac{a^2q^{n+2}}{bcde}\right]\\
&\quad=\frac{(aq,aq/de;q)_n}{(aq/d,aq/e;q)_n}\,{}_4\phi_3\left[\begin{matrix}aq/bc,\ d,\ e,\ q^{-n}\\ aq/b,\ aq/c,\ deq^{-n}/a\end{matrix};q,\ q\right].
\end{aligned}$$

We assume that $s\geqslant 2$ (as mentioned before, the $s=0$ case has already been proved by the author and Zudilin [17]). It is easy to see that

$$\frac{(abq^2,q^2;q^4)_k}{(abq^4,q^4;q^4)_k}\equiv\frac{(aq^2,bq^2;q^4)_k}{(aq^4,bq^4;q^4)_k}\pmod{(1-a)(1-b)}.$$

Letting $a=q^{2n}$ and $b=q^{-2n}$ and noticing that $1-q^{2n}$ has the factor $\Phi_n(q)$, we obtain

$$\frac{(q^2;q^4)_k^2}{(q^4;q^4)_k^2}\equiv\frac{(q^{2+2n},q^{2-2n};q^4)_k}{(q^{4+2n},q^{4-2n};q^4)_k}\pmod{\Phi_n(q)^2}$$

for $0 \leqslant k \leqslant n-1$. Thus, modulo $\Phi_n(q)^2$, the left-hand side of (1.3) is congruent to

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2,q^{2+2n},q^{2-2n};q^4)_k}{(q^4,q^{4+2n},q^{4-2n};q^4)_k}q^{(1-2s)k}$$

$$=\sum_{k=0}^{(n-1)/2}\frac{(q^2,q^5,-q^5,\overbrace{q^5,\dots,q^5}^{(s-1)\text{'s } q^5},q^{2+2n},q^{2-2n};q^4)_k}{(q^4,q,-q,q,\dots,q,q^{4-2n},q^{4+2n};q^4)_k}q^{(1-2s)k},$$

which, by Andrews' transformation (2.1) with the parameter substitutions $m=s/2$, $q\mapsto q^4$, $a=q^2$, $b_1=c_1=\cdots=b_{m-1}=c_{m-1}=b_m=q^5$, $c_m=q^{2+2n}$, and $N=(n-1)/2$, is equal to

$$\frac{(q^6,q^{-2n-1};q^4)_{(n-1)/2}}{(q,q^{4-2n};q^4)_{(n-1)/2}}\sum_{j_1,\dots,j_{m-1}\geqslant 0}\frac{(q^{-4};q^4)_{j_1}\cdots(q^{-4};q^4)_{j_{m-1}}}{(q^4;q^4)_{j_1}\cdots(q^4;q^4)_{j_{m-1}}}$$
$$\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(q^5,q^{2+2n},q^{2-2n};q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^7;q^4)_{j_1+\cdots+j_{m-1}}}$$
$$\times q^{4(j_1+\cdots+j_{m-1})-4(j_{m-2}+\cdots+(m-2)j_1)}.$$

Since

$$\frac{(q^{-4};q^4)_k}{(q^4;q^4)_k}=\begin{cases}(-1)^kq^{-4k}, & \text{if } k=0,1,\\ 0, & \text{otherwise,}\end{cases}$$

we obtain

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{(1-2s)k}$$
$$\equiv\frac{(q^6,q^{-2n-1};q^4)_{(n-1)/2}}{(q,q^{4-2n};q^4)_{(n-1)/2}}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$
$$\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(q^5,q^{2+2n},q^{2-2n};q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^7;q^4)_{j_1+\cdots+j_{m-1}}}\pmod{\Phi_n(q)^2},\tag{2.2}$$

where $m=s/2$.

It is easy to see that $(q^6;q^4)_{(n-1)/2}$ contains the factor $1-q^{2n}$, and $(q^{-2n-1};q^4)_{(n-1)/2}$ contains the factor $1-q^{-n}$ for $n\equiv 3\pmod 4$ and $n\geqslant 7$. At the same time, the polynomial $(q,q^{4-2n};q^4)_{(n-1)/2}$ is relatively prime to $\Phi_n(q)$ for $n\equiv 3\pmod 4$, and so is $(q;q^4)_j$ for such n and $j\leqslant(n-1)/2$. Moreover, by the condition $n\geqslant 2s+3$ in the theorem, we have $j_1+\cdots+j_{m-1}\leqslant m-1=(s-2)/2\leqslant(n-7)/4$ and so $(q^7;q^4)_{j_1+\cdots+j_{m-1}}$ is also relatively prime to $\Phi_n(q)$ (for $n=7$ this q-shifted factorial is understood to be 1 since $m=1$ in this case). This proves that the right-hand side of (2.2) is congruent to 0 modulo $\Phi_n(q)^2$, thus establishing the theorem.

3. Proof of Theorem 1.4

Similarly to the proof of Theorem 1.2, modulo $\Phi_n(q)^2$, the left-hand side of (1.6) is congruent to

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2},q^{-2+2n},q^{-2-2n};q^4)_k}{(q^4,q^{4+2n},q^{4-2n};q^4)_k}q^{(7-2s)k}$$

$$=q^{-s-1}\sum_{k=0}^{(n-1)/2}\frac{(q^{-2},q^3,-q^3,\overbrace{q^3,\ldots,q^3}^{(s-1)\text{'s } q^3},q^{-2+2n},q^{-2-2n};q^4)_k}{(q^4,q^{-1},-q^{-1},q^{-1},\ldots,q^{-1},q^{4-2n},q^{4+2n};q^4)_k}q^{(7-2s)k}. \tag{3.1}$$

By Andrews' transformation (2.1) with the parameter replacements $m=s/2$, $q\mapsto q^4$, $a=q^{-2}$, $b_1=c_1=\cdots=b_{m-1}=c_{m-1}=b_m=q^3$, $c_m=q^{-2+2n}$, and $N=(n+1)/2$, the right-hand side of (3.1) can be written as

$$\begin{aligned}&q^{-s-1}\frac{(q^2,q^{1-2n};q^4)_{(n+1)/2}}{(q^{-1},q^{4-2n};q^4)_{(n+1)/2}}\sum_{j_1,\ldots,j_{m-1}=0}^{1}\frac{(q^{-4};q^4)_{j_1}\cdots(q^{-4};q^4)_{j_{m-1}}}{(q^4;q^4)_{j_1}\cdots(q^4;q^4)_{j_{m-1}}}\\&\quad\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(q^3,q^{-2+2n},q^{-2-2n};q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q;q^4)_{j_1+\cdots+j_{m-1}}}\\&\quad\times q^{4(j_1+\cdots+j_{m-1})-4(j_{m-2}+\cdots+(m-2)j_1)},\end{aligned} \tag{3.2}$$

where $m=s/2$.

Note that $(q^2;q^4)_{(n+1)/2}$ contains the factor $1-q^{2n}$, and $(q^{1-2n};q^4)_{(n+1)/2}$ contains the factor $1-q^{-n}$ for $n\equiv 1 \pmod 4$. Meanwhile, the polynomial $(q^{-1},q^{4-2n};q^4)_{(n+1)/2}$ is relatively prime to $\Phi_n(q)$ for $n\equiv 1 \pmod 4$, and so is $(q^{-1};q^4)_j$ for such n and $j\leqslant (n+1)/2$. Furthermore, the condition $n\geqslant \max\{3,2s-3\}$ in the theorem implies that $j_1+\cdots+j_{m-1}\leqslant m-1=(s-2)/2\leqslant (n-1)/4$ and so $(q;q^4)_{j_1+\cdots+j_{m-1}}$ is also relatively prime to $\Phi_n(q)$. This proves that the expression (3.2) is congruent to 0 modulo $\Phi_n(q)^2$, as desired.

4. Further q-congruences modulo $\Phi_n(q)^3$

Like the paper [17], we may give a generalization of Theorem 1.2 modulo $\Phi_n(q)^3$.

Theorem 4.1. *Let $n>1$ be an odd integer and let $s>0$ be an even integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-3k} \\
&\quad\equiv [n]_{q^2}\frac{(q^7;q^4)_{(n-1)/2}}{(q;q^4)_{(n-1)/2}}q^{5(1-n)/2}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)} \\
&\qquad\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(q^2,q^2,q^5;q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^7;q^4)_{j_1+\cdots+j_{m-1}}} \\
&\qquad\begin{cases}\pmod{\Phi_n(q)^2\Phi_n(-q)^3} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^3\Phi_n(-q)^3} & \text{if } n\equiv 3 \pmod 4.\end{cases}
\end{aligned}\tag{4.1}$$

where $m=s/2$.

For example, for odd $n>1$, modulo

$$\begin{cases}\Phi_n(q)^2\Phi_n(-q)^3 & \text{if } n\equiv 1 \pmod 4,\\ \Phi_n(q)^3\Phi_n(-q)^3 & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

we have

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-3k}\equiv[n]_{q^2}\frac{(q^7;q^4)_{(n-1)/2}}{(q;q^4)_{(n-1)/2}}q^{5(1-n)/2},$$

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^3\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-7k}\equiv[n]_{q^2}\frac{(q^7;q^4)_{(n-1)/2}([7]-[2]^2[5])}{(q;q^4)_{(n-1)/2}[7]}q^{5(1-n)/2},$$

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^5\frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3}q^{-11k}\equiv[n]_{q^2}\frac{(q^7;q^4)_{(n-1)/2}}{(q;q^4)_{(n-1)/2}}q^{5(1-n)/2}R(q),$$

where $R(q)=\frac{(5q^{12}+18q^{11}+39q^{10}+68q^9+100q^8+123q^7+131q^6+123q^5+100q^4+68q^3+39q^2+18q+5)q^2}{(q^{10}+q^9+q^8+q^7+q^6+q^5+q^4+q^3+q^2+q+1)(q^6+q^5+q^4+q^3+q^2+q+1)}$.

When $n=p$ is an odd prime and $q\to -1$, then (4.1) reduces to

$$\sum_{k=0}^{(p-1)/2}(-1)^k(4k+1)\frac{(\frac12)_k^3}{k!^3}\equiv p(-1)^{(p-1)/2}\pmod{p^3}\tag{4.2}$$

(tagged (B.2) on Van Hamme's list [29]). The supercongruence (4.2) was first confirmed by Mortenson [24] using a ${}_6F_5$ transformation, and later received another proof by Zudilin [32] via the Wilf–Zeilberger method. We point out that some different q-analogues of (4.2) were given in [6, 7, 17].

Proof of Theorem 4.1. We first establish the following parametric generalization of (4.1):

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(aq^2,q^2/a,q^2;q^4)_k}{(aq^4,q^4/a,q^4;q^4)_k}q^{-3k}$$

$$\equiv [n]_{q^2}\frac{(q^7;q^4)_{(n-1)/2}}{(q;q^4)_{(n-1)/2}}q^{5(1-n)/2}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}$$

$$\times\frac{(q^5;q^4)_{j_1}^2\cdots(q^5;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^2,q^2/a,q^5;q^4)_{j_1+\cdots+j_{m-1}}}{(q;q^4)_{j_1}^2\cdots(q;q^4)_{j_1+\cdots+j_{m-1}}^2(q^7;q^4)_{j_1+\cdots+j_{m-1}}}$$

$$\begin{cases}\pmod{\Phi_n(-q)(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^2(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 3 \pmod 4.\end{cases} \tag{4.3}$$

where $m=s/2$.

Note that

$$\frac{(q^6,q^{-2n-1};q^4)_{(n-1)/2}}{(q,q^{4-2n};q^4)_{(n-1)/2}}=[n]_{q^2}\frac{(q^7;q^4)_{(n-1)/2}}{(q;q^4)_{(n-1)/2}}q^{5(1-n)/2}.$$

The proof of Theorem 1.2 implies that, when $a=q^{-2n}$ or $a=q^{2n}$, both sides of (4.3) are equal. Namely, the q-congruence holds modulo $1-aq^{2n}$ and $a-q^{2n}$.

On the other hand, by [13, Lemma 3.1], for $0\leqslant k\leqslant (n-1)/2$, we have

$$\frac{(aq;q^2)_{(n-1)/2-k}}{(q^2/a;q^2)_{(n-1)/2-k}}\equiv(-a)^{(n-1)/2-2k}\frac{(aq;q^2)_k}{(q^2/a;q^2)_k}q^{(n-1)^2/4+k}\pmod{\Phi_n(q)}.$$

Applying the above q-congruence one can easily verify that, for odd $n>1$ and $0\leqslant k\leqslant (n-1)/2$, the k-th and $((n-1)/2-k)$-th summands on the left-hand side of (4.3) cancel each other modulo $\Phi_n(-q)$ (or modulo $\Phi_n(q^2)$ if $n\equiv 3 \pmod 4$). This proves that

$$\sum_{k=0}^{(n-1)/2}[4k+1]_{q^2}[4k+1]^{s-1}\frac{(aq^2,q^2/a,q^2;q^4)_k}{(aq^4,q^4/a,q^4;q^4)_k}q^{-3k}$$

$$\equiv 0\begin{cases}\pmod{\Phi_n(-q)} & \text{if } n\equiv 1 \pmod 4,\\ \pmod{\Phi_n(q)^2} & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

It is not difficult see that the right-hand side of (4.3) is congruent to 0 modulo $\Phi_n(-q)$ if $n\equiv 1 \pmod 4$ and modulo $\Phi_n(q^2)$ if $n\equiv 3 \pmod 4$. Hence, the q-congruence (4.3) is true modulo $\Phi_n(-q)$ if $n\equiv 1 \pmod 4$ and modulo $\Phi_n(q^2)$ if $n\equiv 3 \pmod 4$. Since $1-aq^{2n}$, $a-q^{2n}$ and $\Phi_n(-q)$ (or $\Phi_n(q^2)$) are pairwise relatively prime polynomials, we finish the proof of (4.3).

It is easy to see that the denominator of the reduced form of $[n]_{q^2}(q^5;q^4)_k/(q^7;q^4)_k$ is relatively prime to $\Phi_n(q^2)$ for any $k\geqslant 0$. Letting $a=1$ in (4.3) and noticing that $1-q^{2n}$ has the factor $\Phi_n(q^2)$, we immediately obtain (4.1). □

We also have the following generalization of Theorem 1.4 modulo $\Phi_n(q)^3$.

Theorem 4.2. *Let $n>1$ be an odd integer and let $s>0$ be an even integer. Then*

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{(7-2s)k}\\
&\quad\equiv [n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}q^{3(1-n)/2-s-1}\sum_{j_1,\ldots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}\\
&\qquad\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(q^{-2},q^{-2},q^3;q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q;q^4)_{j_1+\cdots+j_{m-1}}}\\
&\qquad\begin{cases}(\mathrm{mod}\ \Phi_n(q)^3\Phi_n(-q)^3) & \text{if } n\equiv 1 \pmod 4,\\ (\mathrm{mod}\ \Phi_n(q)^2\Phi_n(-q)^3) & \text{if } n\equiv 3 \pmod 4,\end{cases}
\end{aligned}\tag{4.4}$$

where $m=s/2$.

For example, for odd $n>1$, modulo

$$\begin{cases}\Phi_n(q)^3\Phi_n(-q)^3 & \text{if } n\equiv 1 \pmod 4,\\ \Phi_n(q)^2\Phi_n(-q)^3 & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

we have

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{3k}\equiv [n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}q^{3(1-n)/2-3},$$

$$\begin{aligned}\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^3\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{-k}&\equiv -[n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}q^{3(1-n)/2-7}\\
&\quad\times(q^4+3q^3+3q^2+3q+1),\end{aligned}$$

$$\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^5\frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3}q^{-5k}\equiv -[n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}q^{3(1-n)/2-9}S(q),$$

where

$$S(q)=\frac{3q^8+14q^7+32q^6+51q^5+59q^4+51q^3+32q^2+14q+3}{q^4+q^3+q^2+q+1}.$$

Letting $n=p$ be an odd prime and $q\to -1$ in Theorem 4.2, we are led to the following result:

$$\sum_{k=0}^{(p+1)/2}(-1)^k(4k-1)\frac{(-1/2)_k^3}{k!^3}\equiv p(-1)^{(p+1)/2} \pmod{p^3}. \tag{4.5}$$

Note that different q-analogues of (4.5) can be found in [17] and [15, Theorem 4.9] with $r=-1$, $d=2$ and $a=1$ (see also [13, Section 5]).

Proof of Theorem 4.2. The proof is similar to that of Theorem 4.1. Noticing that

$$\frac{(q^2,q^{1-2n};q^4)_{(n+1)/2}}{(q^{-1},q^{4-2n};q^4)_{(n+1)/2}}=\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}q^{3(1-n)/2},$$

and, for $0\leqslant k\leqslant (n+1)/2$,

$$\frac{(aq^{-1};q^2)_{(n+1)/2-k}}{(q^2/a;q^2)_{(n+1)/2-k}}\equiv (-a)^{(n+1)/2-2k}\frac{(aq^{-1};q^2)_k}{(q^2/a;q^2)_k}\,q^{(n-1)^2/4+3k-1} \pmod{\Phi_n(q)}$$

(see [13, (5.4)]), we can establish the following parametric generalization of (4.4):

$$\begin{aligned}
&\sum_{k=0}^{(n+1)/2}[4k-1]_{q^2}[4k-1]^{s-1}\frac{(aq^{-2},q^{-2}/a,q^{-2};q^4)_k}{(aq^4,q^4/a,q^4;q^4)_k}q^{(7-2s)k}\\
&\quad\equiv [n]_{q^2}\frac{(q^5;q^4)_{(n-1)/2}}{(q^3;q^4)_{(n-1)/2}}q^{3(1-n)/2-s-1}\sum_{j_1,\dots,j_{m-1}=0}^{1}(-1)^{j_1+\cdots+j_{m-1}}q^{-4(j_{m-2}+\cdots+(m-2)j_1)}\\
&\qquad\times\frac{(q^3;q^4)_{j_1}^2\cdots(q^3;q^4)_{j_1+\cdots+j_{m-2}}^2(aq^{-2},q^{-2}/a,q^3;q^4)_{j_1+\cdots+j_{m-1}}}{(q^{-1};q^4)_{j_1}^2\cdots(q^{-1};q^4)_{j_1+\cdots+j_{m-1}}^2(q;q^4)_{j_1+\cdots+j_{m-1}}}\\
&\quad\begin{cases}\pmod{\Phi_n(q^2)(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 1 \pmod 4,\\[4pt] \pmod{\Phi_n(-q)(1-aq^{2n})(a-q^{2n})} & \text{if } n\equiv 3 \pmod 4.\end{cases}
\end{aligned}\tag{4.6}$$

where $m=s/2$.

Observe that the denominator of the reduced form of $[n]_{q^2}(q^5;q^4)_k/(q^3;q^4)_k$ is relatively prime to $\Phi_n(q^2)$ for any $k\geqslant 0$. Letting $a=1$ in (4.6) and noticing that $1-q^{2n}$ contains the factor $\Phi_n(q^2)$, we arrive at (4.4). $\square$

5. Discussion

In 2015, Swisher [28] proposed 12 amazing conjectures on Dwork-type supercogn-ruences (see [9, 18] for a recent progress on such supercongruences). For example, she [28, (H.3)] conjectured that, for any integer $r\geqslant 2$ and prime $p>3$ with $p\equiv 3 \pmod 4$,

$$\sum_{k=0}^{(p^r-1)/2}\frac{(\frac12)_k^3}{k!^3}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/2}\frac{(\frac12)_k^3}{k!^3} \pmod{p^{3r-2}}, \tag{5.1}$$

It seems that there are no common generalizations of (1.5) and (5.1). Nevertheless, we find the following conjecture on Dwork-type supercognruences.

Conjecture 5.1. *Let p be an odd prime with $p\equiv 3 \pmod 4$. Let $s\geqslant 0$ be an even integer. Then, for $r\geqslant 2$, we have*

$$\sum_{k=0}^{(p^r-1)/2}(4k+1)^s\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p^2\sum_{k=0}^{(p^{r-2}-1)/2}(4k+1)^s\frac{(\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r-5}},$$

$$\sum_{k=0}^{p^r-1}(4k+1)^s\frac{(\frac{1}{2})_k^3}{k!^3}\equiv p^2\sum_{k=0}^{p^{r-2}-1}(4k+1)^s\frac{(\frac{1}{2})_k^3}{k!^3} \pmod{p^{3r-5}}.$$

We also have the following conjecture related to Theorem 1.3.

Conjecture 5.2. *Let p be a prime with $p\equiv 1 \pmod 4$. Let $s\geqslant 0$ be an even integer. Then, for $r\geqslant 1$, we have*

$$\sum_{k=0}^{(p^r+1)/2}(4k-1)^s\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv\sum_{k=0}^{p^r-1}(4k-1)^s\frac{(-\frac{1}{2})_k^3}{k!^3}\equiv 0 \pmod{p^{2r-1}}. \tag{5.2}$$

Unlike Theorems 1.1 and 1.3, we do not require additional conditions for p (such as $p\geqslant 2s+3$) in Conjecture 5.1 and 5.2.

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