The conformable space-time fractional Fokas-Lenells equation and its optical soliton solutions based on three analytical schemes

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Abstract

This paper is about the study of space-time fractional Fokas-Lenells equation that describes nonlinear wave propagation in optical fibers. Three prominent schemes are employed for extracting different types of exact soliton solutions. In particular, the exp_a function method, the hyperbolic function method and the simplest Riccati equation scheme are investigated for the said model. As a sequela, a series of soliton solutions are obtained and verified through MATHEMATICA. The obtained solutions are significant additions in some specific fields of physics and engineering. Furthermore, the $3D$ graphical descriptions are left to analyze the pulse propagation for the reader.

Keywords: Conformable space-time fractional Fokas-Lenells equation; Soliton solutions; three integration schemes.

1 Introduction

Many complex nonlinear expressions arising in different fields related to science and engineering such as plasma physics, biophysics, optical fibers, biology and nonlinear optics are best described by the nonlinear fractional differential equations (NFPDEs). A series of powerful schemes have been composed and executed to find the exact and other solutions of NFPDEs. For this purpose, different wave transformations are applied to change the non-linear partial differential equation into a non-linear ordinary differential equation, which leads for further procedure to reach different types of solutions. The word soliton is used to describe the particle characteristics of waves propagating in many complex phenomena in nonlinear sciences, like non-linear optics, fluid dynamics, the linear fractional materialistic progression of the waves in non-linear fibre optics [1].

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Many powerful approaches have established to search exact soliton solutions in different research articles including the fractional exp-function method [2], the ansatz [3, 4], fractional (G'/G)-expansion scheme [5], modified simple equation [6], the extended trial equation [7], the fractional functional variable scheme [8], the unified method [9], the first integral scheme [10, 11], sine-cosine approach [12] and Lie symmetry analysis [13]. There are many other non-linear approaches that are taken into applications, Kerr law, power law, parabolic law and dual-power law [14, 15]. The chiral nonlinear Schrödinger equation consists of a perturbation term and a coefficient of Bohm potential. The equation admits a rich variety of families of optical solitons for a range of five parameters [16]. By using the F-expansion scheme the explicit Jacobian elliptic solitons in birefringent fibers with Spatio-temporal dispersion are obtained [17]. Optical solitons of the time-fractional wu-Zhang system are obtained by using the first integral method [18]. To obtain the soliton solutions of the space-time non-linear conformable fractional differential equations, Bogoyavlenskii equations, the Schrödinger-Hirota equation and the modified KDV-Zakharov-Kuznetsov equation, two different schemes naming the first integral scheme [19] and the functional variable scheme are applied [20]. The Sine-Gordon expansion scheme is used to obtain the solitons of Lakshmanan-Porsezian-Daniel model [21, 22]. The solutions of fractional Zakharov-Kuznetsov equation along dual-power law non-linearity in the sight of conformable derivative, the Riccti sub equation scheme is used [23].
Khalil's conformable fractional derivative as well as Liu's extended method are used to obtain the exact solutions along quadratic-cubic-septic non-linearity that are consists on some perturbation terms. Results obtained are very useful in telecommunication industry [24]. To describe the traveling waves of solutions in magneto-electro-elastic circular rod, the non-linear longitudinal wave equation is investigated analytically [25–27]. In non-linear optics, dark solutions show more diligence because of their balanced manual characteristics. By adjusting corresponding parameters, one can control the periodicity and dimension of the dark optical solutions of vibration [28, 29].

The exact solutions of dynamics in optics with conformable space-time fractional Fokas–Lenells equation [30] is described in this paper. The aforesaid equation is read as below:

$$\begin{aligned} &\iota D_t^{\alpha}\phi + a_1 D_t^{2\beta}\phi + a_2 D_t^{\alpha} D_x^{\beta}\phi + |\phi|^2(b\phi + \iota\sigma D_x^{\beta}\phi) \\ &\qquad - \iota\delta D_x^{\beta} - \iota\rho D_x^{\beta}(|\phi|^{2n}\phi) - \iota\gamma\phi D_x^{\beta}(|\phi|^{2n}) = 0,\ 0 < \alpha,\ \beta \leq 1, \end{aligned} \tag{1}$$

while $\iota = \sqrt{-1}$, $\phi(x,t)$ represents solitons molecules and other forms of nonlinear waves where x and t are independent spatial and temporal variables respectively. The first term in Eq. (1) represents the linear fractional physical transformation of the pulses in the non-linear optical fibres. The coefficients a_1, a_2, δ, ρ and γ are the spatio-temporal dispersion (STD), group velocity dispersion (GVD), inter-model dispersion (IMD), self-steepening perturbation term and non-linear dispersion (ND) coefficient, respectively. The parameter n represents the full nonlinearity of the Eq. (1). If we put $\alpha = \beta = 1$, then Eq. (1) gets

the form of the original Fokas-Lenells equation shown in the [37–39].
Three different types of approaches, the exp_a function method [31–33], the hyperbolic function method [34–36] and the simplest equation method [43], are applied to obtain optical solitons along with some forms of combo-solitons. The scheme of the paper is represented as: Section (2) is about the limited explanation of the Exp_a function method and the hyperbolic function method. Section (3) explain how to apply these methods for finding new explicit solitons of conformable space-time fractional Fokas-Lenells equation. At the end, the results are shown in the form of graphs.

Conformable Derivative: Some characteristics of conformable fractional derivative are [40].

Definition 1: Consider $p : (0, \infty) \to \mathbb{R}$ be a function. Then, for all $t > 0$,

$$D_t^\gamma(p(t)) = \lim_{\varepsilon \to 0} \frac{p(t + \varepsilon t^{1-\gamma}) - p(t)}{\varepsilon}$$

is known as γ, $0 < \gamma \leq 1$ order conformable fractional derivative of p. The following are some useful properties:
$D_t^\gamma(a\ p + b\ q) = aD_t^\gamma(p) + bD_t^\gamma(q)$, for all $a,\ b \in \mathbb{R}$
$D_t^\gamma(p\ q) = p\ D_t^\gamma(q) + q\ D_t^\gamma(p)$
Let $p : \mathbb{R}_{>0} \to \mathbb{R}$ be an γ-differentiable function, where q is a function that is differentiable over the range defined for p.

$$D_t^\gamma(p \circ q(t)) = t^{1-\gamma}\ q'(t)\ p'(q(t)).$$

On the above of that, the following rules are valid.
$D_t^\gamma(t^h) = h\ t^{h-\gamma}$, for all $h \in \mathbb{R}$
$D_t^\gamma(\delta) = 0$, where δ is constant.
$D_t^\gamma(p/q) = \frac{qD_t^\gamma(p) - pD_t^\gamma(q)}{q^2}$.
likewise, if p is differentiable, then $D_t^\gamma(p(t)) = t^{1-\gamma}\frac{dp(t)}{dt}$.

2 Optical soliton solutions of Eq. (1) via three integration schemes

Let us start with the complex-valued function $\phi(x, t) = \psi(\eta)\exp^{\iota\Theta}$ and the fractional traveling wave transformation

$$\eta = \nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha}) \quad , \Theta = -\kappa\frac{x^\beta}{\beta} + \omega\frac{t^\alpha}{\alpha} + \theta \tag{2}$$

$$D_t^\alpha\phi = -c\nu\psi' \exp^{\iota\Theta} + \omega\psi\exp^{\iota\Theta}\iota \tag{3}$$

$$D_x^\beta\phi = \nu\psi' \exp^{\iota\Theta} - \kappa\psi\exp^{\iota\Theta}\iota \tag{4}$$

$$D_x^{2\beta}\phi = \nu^2\psi^{''}\exp^{\iota\Theta} - 2\nu\kappa\psi^{'}\exp^{\iota\Theta}\iota - \kappa^2\psi\exp^{\iota\Theta} \tag{5}$$

$$D_t^{\alpha}D_x^{\beta}\phi = -c\nu^2\psi^{''}\exp^{\iota\Theta} + \nu\omega\psi^{'}\exp^{\iota\Theta}\iota + c\nu\kappa\psi^{'}\exp^{\iota\Theta}\iota + \kappa\omega\psi\exp^{\iota\Theta} \tag{6}$$

$$D_x^{\beta}(|\phi|^{2n}) = 2n\nu\psi^{'}\psi^{2n+1} \tag{7}$$

$$D_x^{\beta}(|\phi|^{2n}\phi) = (2n+1)\nu\psi^{2n}\psi^{'}\exp^{\iota\Theta} - \kappa\psi^{2n+1}\exp^{\iota\Theta}\iota \tag{8}$$

By using the Eqs. (3)-(7) and (8) in Eq. (1), we get
Equating the real part:

$$\nu^2(a_1 - a_2c)\psi^{''} + (a_2\kappa\omega - \omega - a_1\kappa^2 - \delta\kappa)\psi + (b + \kappa\sigma)\psi^3 - \kappa\rho\psi^{2n+1} = 0. \tag{9}$$

Equating the imaginary part:

$$(c + \delta + 2a_1\kappa - a_2(c\kappa + \omega) - \sigma\psi^2 + (\rho + 2n\rho + 2n\gamma)\psi^{2n})\psi^{'} = 0. \tag{10}$$

Considering $n = 1$, Eqs. (1)-(9) and (10) become:

$$\begin{aligned}\iota D_t^{\alpha}\phi + a_1D_t^{2\beta}\phi + a_2D_t^{\alpha}D_x^{\beta}\phi + |\phi|^2(b\phi + \iota\sigma D_x^{\beta}\phi)\\ - \iota\delta D_x^{\beta} - \iota\rho D_x^{\beta}(|\phi|^2\phi) - \iota\gamma\phi D_x^{\beta}(|\phi|^2) = 0,\ 0 < \alpha,\ \beta \leq 1.\end{aligned} \tag{11}$$

$$\nu^2(a_1 - a_2c)\psi^{''} + (a_2\kappa\omega - \omega - a_1\kappa^2 - \delta\kappa)\psi + (b + \kappa\sigma)\psi^3 - \kappa\rho\psi^3 = 0. \tag{12}$$

and

$$(c + \delta + 2a_1\kappa - a_2(c\kappa + \omega) - \sigma\psi^2 + (3\rho + 2\gamma)\psi^2)\psi^{'} = 0. \tag{13}$$

respectively.
Putting $(3\rho + 2\gamma - \sigma) = 0$ in Eq. (13), we obtained the following form:

$$\sigma = 3\rho + 2\gamma \tag{14}$$

and

$$c = \frac{\delta + 2a_1\kappa - a_2\omega}{a_2\kappa - 1}, \quad a_2\kappa \neq 1. \tag{15}$$

2.1 The exp_a function scheme

Balancing $\psi^{''}$ and ψ^3 in Eq. (12), we attain $N = 1$. Consequently, the solution of Eq. (12) can be represented in symmetrical extended form as [31–33]:

$$\psi(\eta) = \frac{\alpha_0 + \alpha_1a^{\eta}}{\beta_0 + \beta_1a^{\eta}} \tag{16}$$

By putting Eq. (16) into Eq. (12) and using Eq. (2), we obtain a series of polynomials in the powers of a^{η}. By assimilating all the coefficients of these polynomials to zero, we

attain a system of non-linear equations. By Mathematical tool, the below different types of soliton sets are obtained:
Set 1:

$$\begin{aligned} \alpha_0 &= -\frac{\iota\beta_0\sqrt{\nu_0}}{\sqrt{-b+\kappa\rho-\kappa\sigma}}, \quad \alpha_1 = \frac{\iota\beta_1\sqrt{\nu_0}}{\sqrt{-b+\kappa\rho-\kappa\sigma}}, \\ \nu &= \mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2 c\log^2(a) - a_1\log^2(a)}}, \quad \nu_0 = a_1\kappa^2 - a_2\kappa\omega + \delta\kappa + \omega. \end{aligned} \tag{17}$$

$$\phi_1(x,t) = -\frac{\iota\sqrt{\nu_0}(\beta_0 - \beta_1 a^{\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})})}{\sqrt{\kappa(\rho-\sigma)-b}(\beta_1 a^{\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})}+\beta_0)} \times \exp(\iota(\theta + \frac{\omega t^\alpha}{\alpha} + \frac{(-\kappa)x^\beta}{\beta})) \tag{18}$$

where ν and ν_0 given in Eq. (17).
Set 2:

$$\begin{aligned} \alpha_0 &= \frac{\iota\beta_0\sqrt{\nu_0}}{\sqrt{-b+\kappa\rho-\kappa\sigma}}, \quad \alpha_1 = -\frac{\iota\beta_1\sqrt{\nu_0}}{\sqrt{-b+\kappa\rho-\kappa\sigma}}, \\ \nu &= \mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2 c\log^2(a) - a_1\log^2(a)}}, \quad \nu_0 = a_1\kappa^2 - a_2\kappa\omega + \delta\kappa + \omega. \end{aligned} \tag{19}$$

$$\phi_2(x,t) = \frac{\iota\sqrt{\nu_0}(\beta_0 - \beta_1 a^{\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})})}{\sqrt{\kappa(\rho-\sigma)-b}(\beta_1 a^{\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})}+\beta_0)} \times \exp(\iota(\theta + \frac{\omega t^\alpha}{\alpha} + \frac{(-\kappa)x^\beta}{\beta})) \tag{20}$$

where ν and ν_0 are given in Eq. (19).
Here the three types of $3D$ graphics are displayed in figures 1-2 for some of solutions.

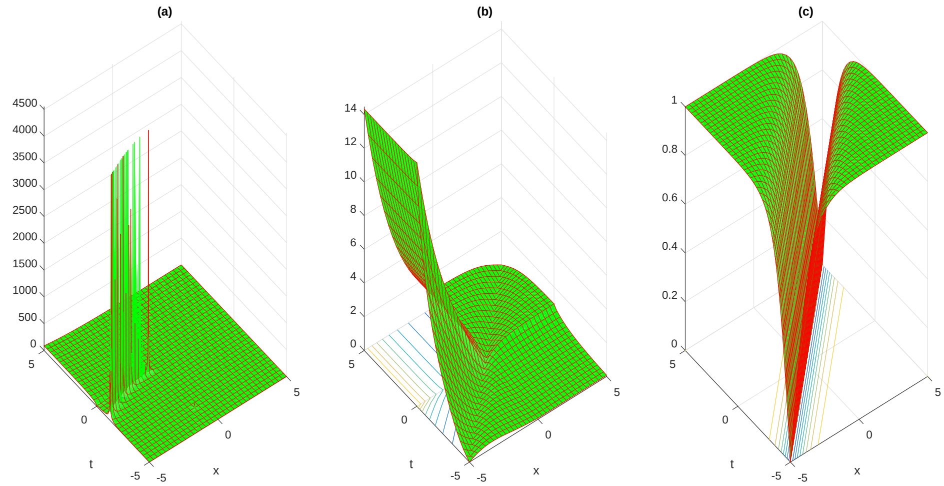

Figure 1: $3D$ wave profiles of $|\phi(x,t)_1|$ for different $\alpha = 0.5, 0.8, 1.$ values, appears in Eq. (18), are displayed corresponding to $b = -3$, $a_2 = 2$, $\omega = \kappa = \delta = 1$, and $a_1 = 1$.

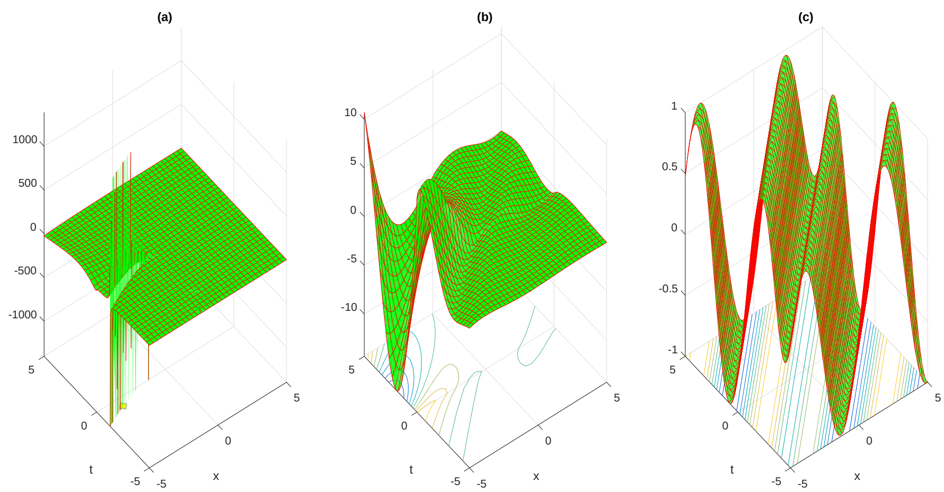

Figure 2: $3D$ wave profiles of $Real(\phi(x,t)_1)$ for different $\alpha = 0.5, 0.8, 1.$ values, appears in Eq. (18), are displayed corresponding to $b = -3$, $a_2 = 2$, $\omega = \kappa = \delta = 1$, and $a_1 = 1$.

2.2 The hyperbolic function scheme

Now we apply the hyperbolic functions scheme [34,35,41,42] to the conformable space-time fractional Fokas-Lnells Eq. (11).
Case 1:

$$\frac{d\rho}{d\eta} = \sinh(\rho) \tag{21}$$

Balancing $\psi^{''}$ and ψ^3 in Eq. (12), we attain $N = 1$.

$$\psi(\xi) = A_1 \cosh(w(\xi)) + A_0 + B_1 \sinh(w(\xi)). \tag{22}$$

where A_1 and B_1 can not be both zero at a time.
By putting the Eq. (22) into the Eq. (12), and comparing coefficients, we will obtain a sets of non-linear algebraic equations and finally by solving them we get the following different solutions:
Set 1:

$$A_0 = 0, A_1 = 0, B_1 = -\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b - \kappa\rho + \kappa\sigma}}, \nu = \mp\frac{\sqrt{\nu_0}}{\sqrt{a_2 c - a_1}},\ \nu_0 = -(a_1\kappa^2 - a_2\kappa\omega + \delta\kappa + \omega). \tag{23}$$

$$\phi_1(x,t) = \frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b + \kappa(\sigma - \rho)}} \text{csch}(\nu(\frac{x^\beta}{\beta} - \frac{ct^\alpha}{\alpha})) \times \exp(\iota(\theta + \frac{\omega t^\alpha}{\alpha} + \frac{(-\kappa)x^\beta}{\beta})) \tag{24}$$

Set 2:

$$A_0 = 0, A_1 = 0, B_1 = \frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b - \kappa\rho + \kappa\sigma}}, \nu = \mp\frac{\sqrt{\nu_0}}{\sqrt{a_2 c - a_1}},\ \nu_0 = -(a_1\kappa^2 - a_2\kappa\omega + \delta\kappa + \omega). \tag{25}$$

$$\phi_2(x,t) = -\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b + \kappa(\sigma - \rho)}} \text{csch}(\nu(\frac{x^\beta}{\beta} - \frac{ct^\alpha}{\alpha})) \times \exp(\iota(\theta + \frac{\omega t^\alpha}{\alpha} + \frac{(-\kappa)x^\beta}{\beta})) \tag{26}$$

Set 3:

$$A_0 = 0, A_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b - \kappa\rho + \kappa\sigma}}, B_1 = 0, \nu = \mp\frac{\sqrt{\nu_0}}{\sqrt{2}\sqrt{a_2 c - a_1}}, \nu_0 = a_1\kappa^2 - a_2\kappa\omega + \delta\kappa + \omega. \tag{27}$$

$$\phi_3(x,t) = \frac{\sqrt{\nu_0}}{\sqrt{b + \kappa(\sigma - \rho)}} \coth(\nu(\frac{x^\beta}{\beta} - \frac{ct^\alpha}{\alpha})) \times \exp(\iota(\theta + \frac{\omega t^\alpha}{\alpha} + \frac{(-\kappa)x^\beta}{\beta})) \tag{28}$$

Set 4:

$$\begin{aligned} &A_0 = 0, \quad A_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b - \kappa\rho + \kappa\sigma}}, \quad B_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b - \kappa\rho + \kappa\sigma}}, \\ &\nu = \mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2 c - a_1}}, \quad \nu_0 = a_1\kappa^2 - a_2\kappa\omega + \delta\kappa + \omega. \end{aligned} \tag{29}$$

$$\phi_4(x,t)=\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\coth(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))+\operatorname{csch}(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \tag{30}$$

Set 5:

$$\begin{aligned} &A_0=0,\quad A_1=-\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\quad B_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\\ &\nu=\mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\quad \nu_0=a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega. \end{aligned} \tag{31}$$

$$\phi_5(x,t)=\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\operatorname{csch}(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))-\coth(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \tag{32}$$

Set 6:

$$A_0=0,\ A_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\ B_1=0,\ \nu=\mp\frac{\sqrt{\nu_0}}{\sqrt{2}\sqrt{a_2c-a_1}},\ \nu_0=a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega. \tag{33}$$

$$\phi_6(x,t)=-\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}}\coth\left(\nu\left(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}\right)\right)\times\exp\iota\left(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta}\right) \tag{34}$$

Set 7:

$$\begin{aligned} &A_0=0,\ A_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\ B_1=-\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\\ &\nu=\mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\quad \nu_0=a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega. \end{aligned} \tag{35}$$

$$\phi_7(x,t)=\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\coth(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))+\operatorname{csch}(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \tag{36}$$

Set 8:

$$\begin{aligned} &A_0=0,A_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\quad B_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\\ &\nu=\mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\quad \nu_0=a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega. \end{aligned} \tag{37}$$

$$\phi_8(x,t)=-\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\coth(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))+\operatorname{csch}(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \tag{38}$$

Here the three types of $3D$ graphics are displayed in figures 3-4 for some of solutions.

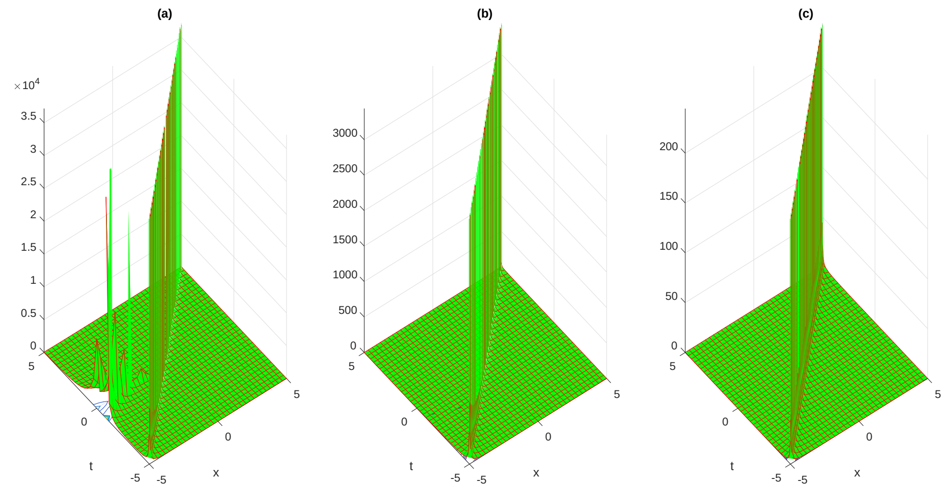

Figure 3: $3D$ wave profiles of $|\phi(x,t)_1|$ for different $\alpha = 0.5, 0.8, 1.$ values, appears in Eq. (24), are displayed corresponding to $b = -3$, $a_2 = 2$, $\omega = \kappa = \delta = 1$, and $a_1 = 1$.

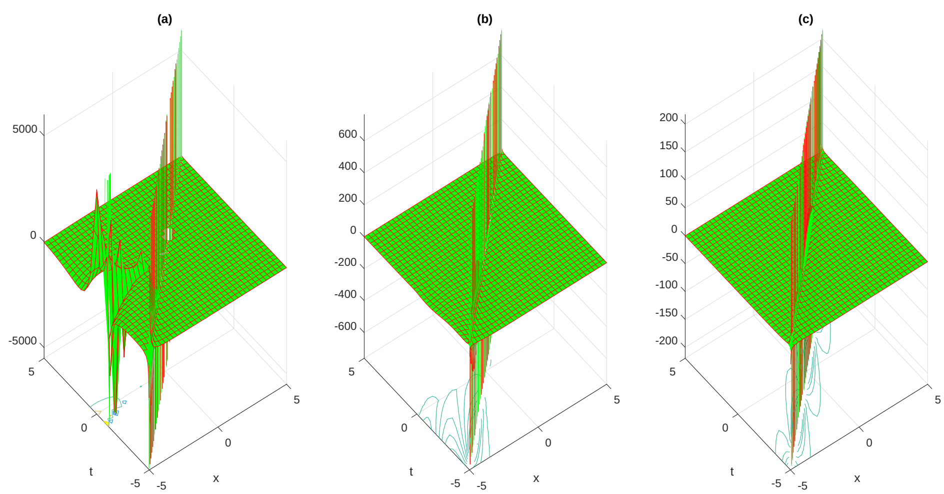

Figure 4: $3D$ wave profiles of $Real(\phi(x,t)_1)$ for different $\alpha = 0.5, 0.8, 1.$ values, appears in Eq. (24), are displayed corresponding to $b = -2$, $\kappa = 2$, $a_2 = \omega = \delta = 1$, and $a_1 = -1$.

Case 2:

$$\frac{d\rho}{d\eta} = \cosh(\rho) \tag{39}$$

In the same way describe above, we can get a system of non-linear algebraic equations and solving them we get the below solutions:
Set 1:

$$A_0 = 0,\ A_1 = -\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\ B_1 = 0,\ \nu = \mp\frac{\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\ \nu_0 = a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega. \quad (40)$$

$$\phi_1(x,t) = -\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}\csc(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \quad (41)$$

Set 2:

$$A_0 = 0,\ A_1 = \frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\ B_1 = 0,\ \nu = \mp\frac{\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\ \nu_0 = a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega. \quad (42)$$

$$\phi_2(x,t) = \frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}}\csc(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \quad (43)$$

Set 3:

$$A_0 = 0,\ \ A_1 = 0,\ \ B_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\nu = \mp\frac{\sqrt{\nu_0}}{\sqrt{2}\sqrt{a_2c-a_1}},\ \nu_0 = -(a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega). \quad (44)$$

$$\phi_3(x,t) = \frac{\sqrt{-a_1\kappa^2+a_2\kappa\omega-\delta\kappa-\omega}}{\sqrt{b+\kappa(\sigma-\rho)}}\cot(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))\times\ \exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \quad (45)$$

Set 4:

$$\begin{aligned} &A_0 = 0,\ A_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\ B_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}}, \\ &\nu = \mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\ \nu_0 = -(a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega). \end{aligned} \quad (46)$$

$$\phi_4(x,t) = -\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\csc(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))-\cot(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \quad (47)$$

Set 5:

$$\begin{aligned} &A_0 = 0,\ A_1 = \frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\ B_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}}, \\ &\nu = \mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\ \nu_0 = -(a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega). \end{aligned} \quad (48)$$

$$\phi_5(x,t) = \frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\cot(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))+\csc(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \quad (49)$$

Set 6:

$$A_0=0,\ A_1=0,\ B_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\nu=\mp\frac{\sqrt{\nu_0}}{\sqrt{2}\sqrt{a_2c-a_1}},\ \nu_0=-(a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega). \tag{50}$$

$$\phi_6(x,t)=-\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}\cot(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \tag{51}$$

Set 7:

$$\begin{aligned}A_0=0,\ A_1=-\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\ B_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\\ \nu=\mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\ \nu_0=-(a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega).\end{aligned} \tag{52}$$

$$\phi_7(x,t)=-\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\cot(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))+\csc(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta})) \tag{53}$$

Set 8:

$$\begin{aligned}A_0=0,\ A_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\quad B_1=\frac{\sqrt{\nu_0}}{\sqrt{b-\kappa\rho+\kappa\sigma}},\\ \nu=\mp\frac{\sqrt{2}\sqrt{\nu_0}}{\sqrt{a_2c-a_1}},\quad \nu_0=-(a_1\kappa^2-a_2\kappa\omega+\delta\kappa+\omega).\end{aligned} \tag{54}$$

$$\begin{aligned}\phi_8(x,t)=\frac{\sqrt{\nu_0}}{\sqrt{b+\kappa(\sigma-\rho)}}(\csc(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha}))-\cot(\nu(\frac{x^\beta}{\beta}-\frac{ct^\alpha}{\alpha})))\\ \times\exp(\iota(\theta+\frac{\omega t^\alpha}{\alpha}+\frac{(-\kappa)x^\beta}{\beta}))\end{aligned} \tag{55}$$

2.3 The simplest riccati equation scheme

By putting the solution in a finite series form given in [43] as:

$$U(\eta)=\sum_{i=1}^{m}b_i\varphi^i(\eta) \tag{56}$$

where $b_i(i=1,2,...,m)$ are non-variables to be find out and factor $b_m\neq 0$. The function $\varphi(\eta)$ fulfills the some ODEs. In this research, we use the Riccati equations as the simplest equation

$$\varphi'(\eta)=\varphi^2(\eta)+\Omega \tag{57}$$

where Ω is a non-variable and the prime represent derivative w.r.t η. After that we get family of solutions to Eq. (57) due to variations of Ω, see [43]. Using Eq. (56) in Eq. (12) with along Eq. (57), one may establish a polynomial in φ. After comparing the coefficients of said polynomial equal to zero, one can obtain a set of non-linear algebraic equations. With the help of symbolic software mathematica, we solve the obtained set of equations for the values of b_0, b_1, ν and are approached to the following results:

Case 1:

$$b_0 = 0, b_1 = -\frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}, \nu = \pm\frac{\sqrt{\nu}}{\sqrt{2}\sqrt{a_2 c\Omega - a_1\Omega}},\ \nu = a_1k^2 - a_2k\omega + \delta k + \omega.$$

When $\Omega < 0$,
Hence, we gain the solitary wave solution

$$u(x,t) = -\frac{\sqrt{\nu}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(-\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)} \tag{58}$$

or

$$u(x,t) = -\frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(-\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)}. \tag{59}$$

When $\Omega > 0$,
Hence, we gain the periodic function solution

$$u(x,t) = -\frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(\sqrt{\Omega}\tan\left(\sqrt{\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)} \tag{60}$$

or

$$u(x,t) = -\frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(-\sqrt{\Omega}\cot\left(\sqrt{\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)}. \tag{61}$$

Case 2:

$$b_0 = 0, \quad b_1 = \frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}, \nu = \pm\frac{\sqrt{\nu_0}}{\sqrt{2}\sqrt{a_2 c\Omega - a_1\Omega}},\ \nu_0 = a_1k^2 - a_2k\omega + \delta k + \omega.$$

When $\Omega < 0$,
Hence, we gain the solitary wave solution

$$u(x,t) = \frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(-\sqrt{-\Omega}\tanh\left(\sqrt{-\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)} \tag{62}$$

or

$$u(x,t) = \frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(-\sqrt{-\Omega}\coth\left(\sqrt{-\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)}. \tag{63}$$

When $\Omega > 0$,
Hence, we gain the periodic function solution

$$u(x,t) = \frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(\sqrt{\Omega}\tan\left(\sqrt{\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)} \tag{64}$$

or

$$u(x,t) = \frac{\sqrt{\nu_0}}{\sqrt{b\Omega - k\rho\Omega + k\sigma\Omega}}(-\sqrt{\Omega}\cot\left(\sqrt{\Omega}\left(\nu(\frac{x^\beta}{\beta} - c\frac{t^\alpha}{\alpha})\right)\right)) \times e^{i(-k\frac{x^\beta}{\beta}+\omega\frac{t^\alpha}{\alpha}+\theta)}. \tag{65}$$

Here the three types of $3D$ graphics are displayed in figures 5-6 for some of solutions.

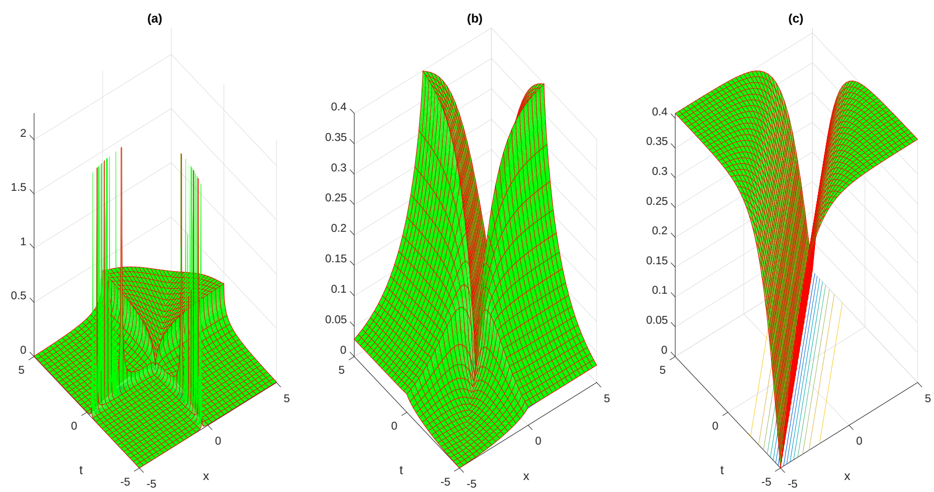

Figure 5: $3D$ wave profiles of $|\phi(x,t)_1|$ for different $\alpha = 0.5, 0.8, 1.$ values, appears in Eq. (58), are displayed corresponding to $b = -2$, $a_2 = \omega = \delta = 1$, and $a_1 = \kappa = -1$.

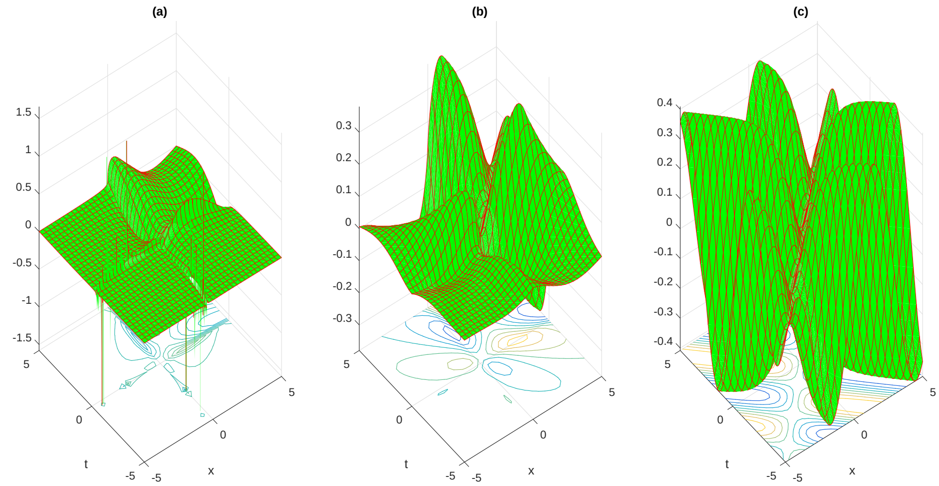

Figure 6: $3D$ wave profiles of $Real(\phi(x,t)_1)$ for different $\alpha = 0.5, 0.8, 1.$ values, appears in Eq. (58), are displayed corresponding to $b = -2$, $a_2 = \omega = \delta = 1$, and $a_1 = \kappa = -1$.

3 Conclusion

The dark, singular and combined exact soliton solutions of the space-time fractional Fokas-Lenells equation that modelled the nonlinear propagation in optical fibers have been established. A complex travelling wave transformation and the conformable differentiation are used to transform the fractional-order derivative into an ordinary one. In particular, three important integration schemes, the Exp_a function, the hyperbolic function and the simplest equation schemes, are employed. The obtained solutions, with certain conditions between constants, are also described by their $3D$-graphs via numerical simulation.

References

[1] A Biswas et al., *Optik* **165**, 29 (2018)

[2] A Bekir, Ö Güner and A Cevikel, **2013**, (2013)

[3] Q Zhou et al., *Jornal of Modern Optics.* **63**(10), 427 (2016)

[4] K Hosseini, P Mayeli and R Ansari, *Waves in Random and Complex Media* **28**(3), 426 (2018)

[5] S. Sahoo and S. Saha Ray, *Physica A* **448**, 265 (2016)

[6] M D Petković Jawad, A J Mohamad and A Biswas, *Applied Mathematics and Computation* **2017**(2), 869 (2010)

[7] Khaled A Gepreel, Taher A Nofal and Nehal S Al-Sayali, *Engineering Letters* **24**(3), 274 (2016)

[8] A. Biswas et al, *Frequenz* **68**(11-12), 525 (2014)

[9] M S Osman, *Pramana – J. Phys.* **93**, 26 (2019)

[10] B. Lu, *J. Math. Anal. Appl.* **395**, 684 (2012)

[11] M. Eslami, *Nonlinear Dynam* **85**(2), 813(2016)

[12] Bekir A, *Physica Scripta* **77**(4), 1 (2008)

[13] Zhang, Y., Gao, B. *Pramana - J Phys* **93**, 100 (2019)

[14] Q. Zhou, M. Mirzazadeh and M. Eslami, *J. Nonlinear Opt. Phys. Mater.* **24**(2), 1550017 (2015)

[15] D Lu et al., *Pramana*, **93**(3), 44 (2019)

[16] S. A. Mahmood et al., *Opt. Quantum Electron* **48**, 542 (2016)

[17] S. Ahmad et al., *Optik* **142**, 327 (2017)

[18] H. Rezazadeh and M. Eslami, *Calcolo* **53**(3), 475 (2016)

[19] F. Nazari et al., *Opt. Quantum Electron* **49**(12), 391 (2017)

[20] M Rezazadeh et al., *Opt. Quantum Electron*, **49**(8), (2017)

[21] S M Mirhosseini-Alizamini et al., *Optik* **164**, 414 (2018)

[22] M. Mirzazadeh et al., *Optik*, **164**, 380 (2018)

[23] M Eslami et al., *Opt. Quantum Electron* **49**(11), 384 (2017)

[24] M Eslami et al, *Optik* **164**, 84 (2018)

[25] Q Zhou, *Nonlinear Dyn.* **83**(3), 1403 (2016)

[26] R Yilmazer, *Math. Commun.* **15**, 489 (2010)

[27] M Eslami, *Optik* **126**(23), 3987 (2016)

[28] M Mirzazadeh et al., *Optik* **165**, 341(2018)

[29] A Biswas et al., *Optik*, **142**, 73 (2017)

[30] H Bulut et al., *Optik***172**, 20 (2018)

[31] Ahmad T. Ali and Ezzat R. Hassan, *Applied Mathematics and Computation* **217**(2), 451 (2010)

[32] EME Zayed and AG Al-Nowehy, *J. Space Explor* **6**, 1 (2017)

[33] A Zafar, *Nonlinear Engineering* **8**(1), 728 (2019)

[34] F Xie, Z Yan, and H Zhang, *Physics Letters A* **285**(1-2), 76 (2001)

[35] Chenglin Bai, *Physics Letters A* **288**(3-4), 191 (2001)

[36] A Zafar, *Nonlinear Engineering* (2018)

[37] H Triki and AM Wazwaz, *Waves in Random and Complex Media*, **27**(4), 587(2017)

[38] H Triki and AM Wazwaz, *International Journal of Numerical Methods for Heat & Fluid Flow* **27**(7), 1596 (2017)

[39] A Biswas et al, *Optik* **165**, 288 (2018)

[40] R. Khalil et al., *J.Comput.Appl.Math.* **264**, 65 (2014)

[41] Aly R Seadawy et al., *Results in Physics* **9**, 1631 (2018)

[42] Asim Zafar and Aly R. Seadawy, *Journal of King Saud University - Science* **31**(4), 1478 (2019)

[43] Cheng Chen and Yao-Lin Jiang, *Computers & Mathematics with Applications* **75**(8), 2978 (2018)