A SOLUTION FORMULA AND THE $\mathscr{R}$-BOUNDEDNESS FOR THE GENERALIZED STOKES RESOLVENT PROBLEM IN AN INFINITE LAYER WITH NEUMANN BOUNDARY CONDITION

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Abstract. We consider the generalized Stokes resolvent problem in an infinite layer with Neumann boundary conditions. This problem arises from a free boundary problem describing the motion of incompressible viscous one-phase fluid flow without surface tension in an infinite layer bounded both from above and from below by free surfaces. We derive a new exact solution formula to the generalized Stokes resolvent problem and prove the $\mathscr{R}$-boundedness of the solution operator families with resolvent parameter λ varying in a sector $\Sigma_{\varepsilon,\gamma_0}$ for any $\gamma_0 > 0$ and $0 < \varepsilon < \pi/2$, where $\Sigma_{\varepsilon,\gamma_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \leq \pi - \varepsilon,\ |\lambda| > \gamma_0\}$. As applications, we obtain the maximal L_p-L_q regularity for the nonstationary Stokes problem and then establish the well-posedness locally in time of the nonlinear free boundary problem mentioned above in L_p-L_q setting. We make full use of the solution formula to take $\gamma_0 > 0$ arbitrarily, while in general domains we only know the $\mathscr{R}$-boundedness for $\gamma_0 \gg 1$ from the result by Shibata. As compared with the case of Neumann-Dirichlet boundary condition studied by Saito, analysis is even harder on account of higher singularity of the symbols in the solution formula.

1. Introduction

We consider the generalized Stokes resolvent problem and the nonstationary Stokes problem in an infinite layer Ω with Neumann boundary conditions:

$$
\begin{cases} \lambda\mathbf{u} - \operatorname{Div}\mathbf{S}(\mathbf{u},\theta) = \mathbf{f}, \quad \operatorname{div}\mathbf{u} = g & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{h} & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

$$
\begin{cases} \partial_t\mathbf{U} - \operatorname{Div}\mathbf{S}(\mathbf{U},\Theta) = \mathbf{F}, \quad \operatorname{div}\mathbf{U} = G & \text{in } \Omega \times (0,\infty), \\ \mathbf{S}(\mathbf{U},\Theta)\nu = \mathbf{H} & \text{on } \partial\Omega \times (0,\infty), \\ \mathbf{U}|_{t=0} = 0 & \text{in } \Omega, \end{cases} \tag{1.2}
$$

where the domain Ω is given by

$$
\Omega = \{x = (x', x_N)^\mathsf{T} \in \mathbb{R}^N \mid x' = (x_1, \cdots, x_{N-1})^\mathsf{T} \in \mathbb{R}^{N-1},\ 0 < x_N < \delta\} \quad (\delta > 0)
$$

and $N \geq 2$. Here, by $\mathbf{u} = (u_1(x), \cdots, u_N(x))^\mathsf{T}$ and $\theta = \theta(x)$, we denote respectively unknown N-component velocity vector and scalar pressure, while vector fields $\mathbf{f} = (f_1(x), \cdots, f_N(x))^\mathsf{T}$, $\mathbf{h} = (h_1(x), \cdots, h_N(x))^\mathsf{T}$ and scalar function $g = g(x)$ are prescribed. Concerning $\mathbf{U} = \mathbf{U}(x,t)$, $\Theta = \Theta(x,t)$, $\mathbf{F} = \mathbf{F}(x,t)$, $G = G(x,t)$ and $\mathbf{H} = \mathbf{H}(x,t)$, it would be obvious what they are. The stress tensor $\mathbf{S}(\mathbf{u},\theta)$ is given by $\mu\mathbf{D}(\mathbf{u}) - \theta\mathbf{I}$, where μ is a positive constant which denotes the viscosity coefficient, $\mathbf{I}$ is the $N \times N$ identity matrix, and $\mathbf{D}(\mathbf{u})$ is the doubled deformation

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tensor whose (j,k) component is $\mathbf{D}_{jk}(\mathbf{u}) = \partial_k u_j + \partial_j u_k$ with $\partial_j = \partial/\partial x_j$. We denote by $\nu = (\nu_1(x), \cdots, \nu_N(x))^\mathsf{T}$ the unit outer normal vector to $\partial\Omega$. Finally, we set $\operatorname{div}\mathbf{u} = \sum_{j=1}^N \partial_j u_j$ and, given matrix field $\mathbf{M}$ with (j,k) component M_{jk}, $\operatorname{Div}\mathbf{M}$ is defined by the N-component vector whose j-th component is $\sum_{k=1}^N \partial_k M_{jk}$. In this paper, we derive a new exact solution formula to the generalized Stokes resolvent problem and prove the $\mathscr{R}$-boundedness of the solution operator families. As applications, we obtain the maximal L_p-L_q regularity for (1.2) and then establish the well-posedness locally in time for the free boundary problem (1.5) below for the Navier-Stokes equations.

Problems (1.1) and (1.2) in the layer have been studied in the case of other boundary conditions. For the problem in which the boundary condition on the lower boundary is replaced by the Dirichlet one

$$\mathbf{u} = 0 \text{ on } \Gamma_0 = \{x = (x', x_N) \in \mathbb{R}^N \mid x_N = 0\},$$

Abe [1] provided a solution formula of (1.1) and proved the resolvent estimate for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ with any $\gamma_0 > 0$ and $0 < \varepsilon < \pi/2$, where

$$\Sigma_{\varepsilon,\gamma_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\lambda| \le \pi - \varepsilon,\ |\lambda| > \gamma_0\}. \tag{1.3}$$

Abels showed the resolvent estimate for $\lambda \in \Sigma_{\varepsilon,0} \cup \{0\}$ with any $0 < \varepsilon < \pi/2$ in [9] and obtained the same result for asymptotically flat layers in [7]. Moreover, he showed that the Stokes operator admits a bounded H_∞-calculus in [8] and proved the maximal regularity for $\mathbf{f} \in L_q(0,\infty; L_q(\Omega))$ with $3/2 < q < \infty$ in [6]. In [25], Saito provided a new solution formula to (1.1) subject to Neumann-Dirichlet boundary condition mentioned above and established the $\mathscr{R}$-boundedness of the solution operator families with resolvent parameter $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ for any $\gamma_0 > 0$ and $0 < \varepsilon < \pi/2$. He obtained the maximal L_p-L_q regularity for $1 < p, q < \infty$ as a corollary. In the case of Dirichlet boundary conditions on both upper and lower boundaries, Abe and Shibata [2, 3], Abels and Wiegner [10], Abe and Yamazaki [4] and von Below and Bolkart [42] derived solution formulas and obtained the resolvent estimates. In addition, in [5], Abels proved the existence of bounded imaginary powers of the Stokes operator and, as a consequence, the maximal L_p-L_q regularity. However, analysis of (1.1) and (1.2) with Neumann boundary conditions on both sides of the boundary $\partial\Omega$ is less developed. We do not know any solution formula of the generalized Stokes resolvent problem (1.1).

On the other hand, in general domains Ω subject to Dirichlet boundary condition on $\Gamma_b \subset \partial\Omega$ and Neumann boundary condition on $\Gamma = \partial\Omega \setminus \Gamma_b$, Shibata [27] showed the resolvent estimate under the assumption: the unique existence of solution $\theta \in \mathcal{W}_q^1(\Omega)$ to the weak Dirichlet-Neumann problem

$$(\nabla\theta, \nabla\varphi)_\Omega = (\mathbf{f}, \nabla\varphi)_\Omega \text{ for any } \varphi \in \mathcal{W}_q^1(\Omega) \tag{1.4}$$

for any $\mathbf{f} \in L_q(\Omega)^N$. Here, $\mathcal{W}_q^1(\Omega)$ is any closed subspace of $\widehat{W}_{q,\Gamma}^1(\Omega)$ containing $W_{q,\Gamma}^1(\Omega)$, where $\widehat{W}_{q,\Gamma}^1(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\overline{\Omega}) \mid \nabla\theta \in L_q(\Omega)^N,\ \theta|_\Gamma = 0\}$ and $W_{q,\Gamma}^1(\Omega) = \{\theta \in W_q^1(\Omega) \mid \theta|_\Gamma = 0\}$. In [28], this result was further developed by showing the $\mathscr{R}$-boundedness and the maximal L_p-L_q regularity. As for the case of Neumann boundary conditions on both sides of the boundary $\partial\Omega$ of the layer Ω, the unique solvability of (1.4) with $\mathcal{W}_q^1(\Omega) = W_{q,0}^1(\Omega)$ was proved by Simader and Ziegler [36]. As a consequence, the $\mathscr{R}$-boundedness of solution operator families with $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ of (1.1) is available provided $\gamma_0 > 0$ is large enough, however, it is better to develop the theory with $\gamma_0 > 0$ arbitrarily small for the layer without relying on the framework of [27, 28]. We note that, as a corollary of a main result of the present paper, the unique solvability of (1.4) with $\mathcal{W}^1_q(\Omega) = \widehat{W}^1_{q,0}(\Omega)$, which coincides with $W^1_{q,0}(\Omega)$ for the layer, is recovered thanks to observation by Shibata [27, Remark 1.7].

The purpose of this paper is to provide a new solution formula of (1.1) and to show the $\mathscr{R}$-boundedness of the solution operator families with the resolvent parameter $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ for arbitrary $\gamma_0 > 0$ and $0 < \varepsilon < \pi/2$. From this, we obtain the resolvent estimates for the same λ. As an application, we prove the maximal L_p-L_q regularity for (1.2) with $1 < p, q < \infty$ by using the operator-valued Fourier multiplier theorem due to Weis [43, Theorem 3.4]. In order to derive a new exact solution formula to (1.1), we apply the Fourier transform with respect to tangential variable $x' \in \mathbb{R}^{N-1}$. We can take any $\gamma_0 > 0$, see (1.3), by taking advantage of use of this formula, while γ_0 was taken large enough in [28] for general domains. It is desirable to obtain the result above for $\lambda \in \Sigma_{\varepsilon,0}$, which is the crucial step toward analysis of large time behavior of solutions to (1.2). This issue will be discussed elsewhere. The condition $\gamma_0 > 0$ is needed for several steps, but the most essential part is the estimate of the determinant, $\det \mathbf{L}$, in the solution formula, see Lemma 6.1. The solution formula itself is valid for $\lambda = 0$ as well, however, when $\lambda = 0$, we see that $|\det \mathbf{L}|^{-1}$ is too singular at the origin $\xi' = 0$ in the Fourier side, see Remark 6.1 for details. Our strategy follows [35] and [25], and it is based on a series of technical lemmas which guarantees the $\mathscr{R}$-boundedness from pointwise estimates of the kernel by considering the solution formula as a singular integral operator. As compared with Neumann-Dirichlet boundary condition, however, the kernel decays only with the order $|x'|^{-(N-1)}$ unlike the case of $\mathbb{R}^N$ and, in the Fourier side, our symbol in the solution formula possesses higher singularity at the origin $\xi' = 0$, see Remark 4.1. This is because the same boundary conditions on both sides of $\partial\Omega$ imply that $\det \mathbf{L}$ involves similar rows and, therefore, is degenerate for $\xi' \to 0$. Moreover, the estimate of $|\det \mathbf{L}|^{-1}$ is not homogeneous in the sense that the rate for $\xi' \to 0$ is different from the one for $|\xi'| \to \infty$. In order to overcome those difficulties, we carry out a cut-off procedure and then employ analysis developed by Saito [25, Lemma 5.5], see Lemma 5.4. To deal with rather singular symbols mentioned above, as in [25], after fixing the normal variable, we regard the solution formula as a singular integral on $\mathbb{R}^{N-1}$ and deduce an estimate uniformly with respect to the normal variable. Then we handle the integral in the normal direction with the aid of the boundedness of the domain in that direction.

Problems (1.1) and (1.2) arise from a free boundary problem for the Navier-Stokes equations describing the motion of incompressible viscous fluid flow with free surfaces without taking account of surface tension:

$$
(1.5)\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \operatorname{Div}\mathbf{S}(\mathbf{v},\pi) = 0, \quad \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega(t),\ 0 < t < T, \\ \mathbf{S}(\mathbf{v},\pi)\nu_t = 0, \quad \mathbf{v}\cdot\nu_t = V_n & \text{on } \partial\Omega(t),\ 0 < t < T, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases}
$$

Here, $\mathbf{v} = (v_1(x,t), \cdots, v_N(x,t))^\mathsf{T}$ is the velocity vector field and $\pi = \pi(x,t)$ is the pressure in a time-dependent domain $\Omega(t)$, while $\mathbf{v}_0 = (v_{01}(x), \cdots, v_{0N}(x))^\mathsf{T}$ is a given initial velocity in the initial domain Ω. By V_n we denote the velocity of the evolution of $\partial\Omega(t)$ and ν_t stands for the unit outer normal to $\partial\Omega(t)$. The novelty of the problem (1.5) is that both upper and lower boundaries are free ones to be determined. If we replace the boundary condition on the lower boundary by the non-slip one, the problem is considered in an asymptotic layer

$$\Omega(t) = \{x = (x', x_N) \in \mathbb{R}^N \mid -b(x') < x_N < \eta(x', t)\}$$

with a fixed bottom, where the only free boundary is the upper surface. In this setting, there are extensive studies and they will be mentioned in order. In the L_2-framework, the existence of solutions locally in time was established by Beale [14] without surface tension, whereas by Allain [11, 12] and by Tani [37] with surface tension. Here, in the latter case, the boundary condition on the free surface should be

$$\mathbf{S}(\mathbf{v}, \pi)\nu_t = \sigma\mathcal{H}\nu_t, \quad \mathbf{v} \cdot \nu_t = V_n,$$

with $\mathcal{H}$ being the doubled mean curvature of $\partial\Omega(t)$, where $\sigma > 0$ is a constant representing the coefficient of surface tension. Also, in [39] and [40], Teramoto showed the local well-posedness especially in an inclined layer without and with surface tension, respectively. The global well-posedness was proved by Beale [15] with surface tension and by Tani and Tanaka [38] with and without surface tension. Beale and Nishida [16] and Hataya and Kawashima [22] studied the large time behavior of the solution obtained in the study of Beale [15]. Hataya [21] established the existence of a global solution with some decay properties under the periodic boundary condition in the horizontal direction. In the L_q-framework, Abels [6] obtained the local well-posedness without surface tension. In the L_p-L_q setting, Saito [26] proved the well-posedness globally in time without surface tension. Once we have the maximal L_p-L_q regularity for (1.2) as a corollary of the $\mathscr{R}$-boundedness of the solution operator families of (1.1), a fix-point argument as in [29, 26, 33] leads to the local well-posedness of (1.5) in L_p-in-time and L_q-in-space setting for $2 < p < \infty$ and $N < q < \infty$, although this result is covered by the theory for general domains under the unique solvability of (1.4) which was established by Shibata without and with taking account of surface tension in [29] and in [31, 30], respectively. Nevertheless, for completeness of the present paper, we give the statement of a unique existence of a local solution to (1.5) without proof, see Theorem 2.3.

This paper is organized as follows: in Section 2, we state our main results on the $\mathscr{R}$-boundedness for (1.1), maximal regularity for (1.2) and local solvability of (1.5). In Section 3, we reduce (1.1) to the problem in which the data are prescribed only on the boundary. The solution formula for the latter problem derived in Section 4 is a novelty of the present paper. In Section 5, we introduce some technical lemmas to prove the $\mathscr{R}$-boundedness from estimates of symbols and, finally, Section 6 is devoted to completion of the proof.

2. Main Results

In this section, we introduce notation and several function spaces, which are used throughout this paper, and then provide our main results.

We denote the sets of all natural numbers, real numbers, and complex numbers by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, and set $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. For multi-index $\alpha = (\alpha_1, \cdots, \alpha_N) \in \mathbb{N}_0^N$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_N^{\alpha_N}$, where $x = (x_1, \cdots, x_N)$ and $\partial_J = \partial/\partial x_J$ for $1 \leq J \leq N$. Given scalar function f and N-vector function $\mathbf{g} = (g_1, \cdots, g_N)$, we set

$$\nabla f = (\partial_1 f, \cdots, \partial_N f)^{\mathsf{T}}, \qquad \nabla\mathbf{g} = (\partial_j g_i)_{1\leq i,j\leq N}$$

$$\nabla^2 f = (\partial^\alpha f \mid |\alpha| = 2), \qquad \nabla^2\mathbf{g} = (\partial^\alpha g_J \mid |\alpha| = 2,\ J = 1, \cdots, N).$$

Let $D = \mathbb{R}^n$, $\mathbb{R}^n_+$, Ω. For scalar functions f, g and N-vector functions $\mathbf{f}$, $\mathbf{g}$, we write

$$(f,g)_D = \int_D f(x)g(x)\,dx, \qquad (\mathbf{f},\mathbf{g})_D = \int_D \mathbf{f}(x)\cdot\mathbf{g}(x)\,dx,$$
$$\langle f,g\rangle_{\partial D} = \int_{\partial D} f(x)g(x)\,d\sigma, \qquad \langle \mathbf{f},\mathbf{g}\rangle_{\partial D} = \int_{\partial D} \mathbf{f}(x)\cdot\mathbf{g}(x)\,d\sigma,$$

where $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^N a_ib_i$ for $\mathbf{a} = (a_1,\cdots,a_N)^{\mathsf{T}}$ and $\mathbf{b} = (b_1,\cdots,b_N)^{\mathsf{T}}$, and $d\sigma$ is the surface element of ∂D. Let $1 \le q \le \infty$ and $m \in \mathbb{N}_0$. The symbols $L_q(D)$ and $W^m_q(D)$ stand for the Lebesgue space and Sobolev space with their associated norms $\|\cdot\|_{L_q(\Omega)}$ and $\|\cdot\|_{W^m_q(\Omega)}$, respectively. Here, $W^0_q(D) = L_q(D)$. We denote by $C^\infty_0(D)$ the set of all $C^\infty(D)$ functions whose supports are compact and contained in D. We define

$$\begin{aligned}
&\widehat{W}^1_q(D) = \{\theta \in L_{q,\mathrm{loc}}(\overline{D}) \mid \nabla\theta \in L_q(\Omega)^N\},\\
&W^1_{q,0}(D) = \{\varphi \in W^1_q(D) \mid \varphi|_{\partial\Omega} = 0\},\\
&\dot{W}^1_{q,0}(D) = \{\varphi \in \widehat{W}^1_q(D) \mid \varphi|_{\partial\Omega} = 0\},\\
&\widehat{W}^1_{q,0}(D) = \text{ the closure of } W^1_{q,0}(D) \text{ in } \dot{W}^1_{q,0}(D) \text{ with respect to } \|\nabla\cdot\|_{L_q(\Omega)},\\
&\widehat{W}^{-1}_q(D) = \text{ the dual space of } \widehat{W}^1_{q',0}(D),
\end{aligned}$$

where q' is the dual exponent of q, that is, $1/q + 1/q' = 1$.

Remark 2.1. *Similarly to* [27, Theorem A.3 (4)], *we have* $\dot{W}^1_{q,0}(\Omega) = \widehat{W}^1_{q,0}(\Omega)$ *for the layer* Ω *under consideration. Indeed, given* $\varphi \in \dot{W}^1_{q,0}(\Omega)$, *we consider an extension* $E_0\varphi \in \dot{W}^1_{q,0}(\mathbb{R}^N_+)$ *of* φ *by setting 0 outside* Ω*:*

$$E_0\varphi(x) = \begin{cases} \varphi(x), & \text{when } x \in \Omega,\\ 0, & \text{when } x \notin \Omega, \end{cases} \tag{2.1}$$

so that the question is reduced to the case of the half space. Then we obtain the desired property by [27, Lemma A.1] *and* [27, Lemma A.2].

Given functions $f = f(x)$ and $g = g(\xi)$ on $\mathbb{R}^N$, the Fourier transform and its inverse transform are denoted by $\mathscr{F}_x$ and $\mathscr{F}^{-1}_\xi$, that is,

$$\mathscr{F}_x[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx, \qquad \mathscr{F}^{-1}_\xi[g](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

Also, the partial Fourier transform with respect to $x' = (x_1,\cdots,x_{N-1})$ and its inverse transform are defined by

$$\mathscr{F}_{x'}[f](\xi',x_N) = \widehat{f}(\xi',x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x',x_N)\,dx',$$
$$\mathscr{F}^{-1}_{\xi'}[g](x',x_N) = \mathscr{F}^{-1}_{\xi'}[g(\xi',x_N)](x') = \frac{1}{(2\pi)^{N-1}}\int_{\mathbb{R}^{N-1}} e^{ix'\cdot\xi'} g(\xi',x_N)\,d\xi'.$$

Given Banach spaces X and Y, we denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X to Y, and we write $\mathcal{L}(X) = \mathcal{L}(X,X)$ to shorten notation. For $d \in \mathbb{N}$ and Banach space X with norm $\|\cdot\|_X$, d-product of X is denoted by X^d, nevertheless we continue to write $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for abbreviation. We often write $\gamma = \operatorname{Re}\lambda$ and $\tau = \operatorname{Im}\lambda$ for the resolvent parameter λ in the sector

$$\Sigma_{\varepsilon,\gamma_0} = \{\lambda \in \mathbb{C}\setminus\{0\} \mid |\arg\lambda| \le \pi - \varepsilon,\ |\lambda| > \gamma_0\} \quad (0 < \varepsilon < \pi/2,\ \gamma_0 \ge 0).$$

Finally, the letter C denotes generic constants and $C_{a,b,\cdots}$ stands for constants depending on the quantities $a, b, \cdots$. Both constants C and $C_{a,b,\cdots}$ may change from line to line.

The notion of the $\mathscr{R}$-boundedness of operator families is defined as follows.

Definition 2.1. *Let X and Y be Banach spaces. The family $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathscr{R}$-bounded if there exist $1 \le p < \infty$ and $C > 0$ such that for any $m \in \mathbb{N}$, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, $\{x_j\}_{j=1}^m \subset X$ and sequence $\{r_j\}_{j=1}^m$ of independent, symmetric and $\{\pm 1\}$-valued random variables on $(0,1)$, there holds the estimate*

$$\Big\{ \int_0^1 \Big\| \sum_{j=1}^m r_j(u) T_j x_j \Big\|_Y^p \, du \Big\}^{1/p} \le C \Big\{ \int_0^1 \Big\| \sum_{j=1}^m r_j(u) x_j \Big\|_X^p du \Big\}^{1/p}. \tag{2.2}$$

The infimum of such C is called $\mathscr{R}$-bound and denoted by $\mathscr{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$, or $\mathscr{R}_{\mathcal{L}(X)}(\mathcal{T})$ if $X = Y$.

Remark 2.2.

a) *It is well known that* (2.2) *holds for any $p \in [1,\infty)$ if it holds for some $p \in [1,\infty)$ by Kahane's inequality (cf.* [23, Theorem 3.11]*).*
b) *We get the uniform boundedness of $\mathcal{T}$ from the $\mathscr{R}$-boundedness of $\mathcal{T}$ by letting $m = 1$ in* (2.2)*.*

We are in a position to state our main result on the $\mathscr{R}$-boundedness of the solution operator families of the resolvent problem (1.1). Set

$$X_q(\Omega) = \{(F_1, F_2, F_3, F_4, F_5, F_6) \mid \\ F_1, F_4, F_5 \in L_q(\Omega)^N, F_2 \in \widehat{W}_q^{-1}(\Omega), F_3 \in L_q(\Omega), F_6 \in L_q(\Omega)^{N^2}\}.$$

Theorem 2.1. *For all $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, there exist operators $\mathcal{U}(\lambda) = (\mathcal{U}_1(\lambda), \cdots, \mathcal{U}_N(\lambda))$ and $\mathcal{P}(\lambda)$ (precisely, they are given by* (3.17)*) satisfying $\mathcal{U}(\lambda) \in \mathcal{L}(X_q(\Omega), W_q^2(\Omega)^N)$ and $\mathcal{P}(\lambda) \in \mathcal{L}(X_q(\Omega), \widehat{W}_q^1(\Omega))$ for $1 < q < \infty$ such that the following assertions hold:*

a) *For any $1 < q < \infty$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and the data*
$$(\mathbf{f}, g, \mathbf{h}) \in L_q(\Omega)^N \times (\widehat{W}_q^{-1}(\Omega) \cap W_q^1(\Omega)) \times W_q^1(\Omega)^N,$$
the pair $(\mathbf{u}, \theta) \in W_q^2(\Omega)^N \times \widehat{W}_q^1(\Omega)$ given by
$$(\mathbf{u}, \theta) = (\mathcal{U}(\lambda), \mathcal{P}(\lambda))(\mathbf{f}, \lambda g, \lambda^{1/2} g, \nabla g, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h})$$
is a solution of (1.1)*. Additionally, the solution of* (1.1) *is unique, that is, if $(\mathbf{u}, \theta) \in W_q^2(\Omega)^N \times \widehat{W}_q^1(\Omega)$ satisfies* (1.1) *with $(\mathbf{f}, g, \mathbf{h}) = (0, 0, 0)$, then $\mathbf{u} = 0$ and $\theta = 0$ a.e.*
b) *For any $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, $\ell = 0, 1$ and $1 \le m, n, J \le N$, there hold*
$$\begin{aligned}
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2} \partial_m \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m \partial_n \mathcal{U}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m \mathcal{P}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta},
\end{aligned}$$

where $\Sigma_{\varepsilon,\gamma_0}$ *is given by* (1.3) *and* $\lambda = \gamma + i\tau$.

By this theorem and Remark 2.2 b), we immediately obtain the resolvent estimates:

Proposition 2.1. *Let* $1 < q < \infty$, $0 < \varepsilon < \pi/2$ *and* $\gamma_0 > 0$. *There exists a constant* $C = C_{N,q,\varepsilon,\gamma_0,\mu,\delta} > 0$ *such that for any* $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $\mathbf{f} \in L_q(\Omega)^N$, $g \in \widehat{W}_q^{-1}(\Omega) \cap W_q^1(\Omega)$ *and* $\mathbf{h} \in W_q^1(\Omega)^N$, *the solution* $(\mathbf{u}, p)$ *given in Theorem 2.1 satisfies*

$$\begin{aligned}&\|(\lambda\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \nabla^2\mathbf{u}, \nabla p)\|_{L_q(\Omega)}\\&\le C(\|\mathbf{f}\|_{L_q(\Omega)} + \|\lambda g\|_{\widehat{W}_q^{-1}(\Omega)} + \|(\lambda^{1/2}g, \nabla g)\|_{L_q(\Omega)} + \|(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})\|_{L_q(\Omega)}).\end{aligned}$$

As an important corollary to Theorem 2.1, we establish the maximal L_p-L_q regularity for the nonstationary Stokes problem (1.2) with the aid of the operator-valued Fourier multiplier theorem due to Weis [43]. To describe the statement precisely, we introduce some function spaces. Let I be an interval in $\mathbb{R}$, X a Banach space, $1 < p < \infty$, $m \in \mathbb{N}_0$ and $\gamma_0 > 0$. We denote the X-valued Bochner and Sobolev spaces by $L_p(I; X)$ and $W_p^m(I; X)$, respectively, and let

$$\begin{aligned}W^m_{p,\gamma_0}(I;X) &= \{f \in W^m_{p,\mathrm{loc}}(\overline{I};X) \mid e^{-\gamma_0 t}f(t) \in W^m_p(I;X)\},\\ W^m_{p,0,\gamma_0}(\mathbb{R};X) &= \{f \in W^m_{p,\gamma_0}(\mathbb{R};X) \mid f(t) = 0 \text{ for } t < 0\},\\ L_{p,\gamma_0}(I;X) &= W^0_{p,\gamma_0}(I;X), \quad L_{p,0,\gamma_0}(\mathbb{R};X) = W^0_{p,0,\gamma_0}(\mathbb{R};X),\\ H^{1/2}_{p,0,\gamma_0}(\mathbb{R};X) &= \{f \in L_{p,0,\gamma_0}(\mathbb{R};X) \mid e^{-\gamma t}\Lambda^{1/2}_\gamma f(t) \in L_p(\mathbb{R};X) \text{ for all } \gamma \ge \gamma_0\}.\end{aligned}$$

Here, we have set

$$[\Lambda_\gamma^s f](t) = \mathscr{L}_\gamma^{-1}[\lambda^s \mathscr{L}[f](\lambda)](t)$$

for $s > 0$ and $\gamma > 0$ where $\lambda = \gamma + i\tau \in \mathbb{C}$; for functions f, g with $f(t) = 0$ $(t < 0)$, we define the Laplace transform $\mathscr{L}[f]$ of f and its inverse transform $\mathscr{L}_\gamma^{-1}[g]$ of g by

$$\mathscr{L}[f](\lambda) = \int_{-\infty}^\infty e^{-\lambda t} f(t)\,dt, \qquad \mathscr{L}_\gamma^{-1}[g](t) = \frac{1}{2\pi}e^{\gamma t}\int_{-\infty}^\infty e^{i\tau t} g(\gamma + i\tau)\,d\tau.$$

Note that we have $W^1_{p,0,\gamma_0}(\mathbb{R}; \widehat{W}_q^{-1}(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W_q^1(\Omega)) \subset H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; L_q(\Omega))$; this is proved in [29, Appendix A] by applying Weis' operator-valued Fourier multiplier theorem after extending functions to be defined on $\mathbb{R}^N$.

The maximal L_p-L_q regularity theorem for (1.2) is stated in the following theorem. This can be proved by the same way as in [25, Theorem 2.1] and we may omit the proof.

Theorem 2.2. *Let* $1 < p, q < \infty$ *and* $\gamma_0 > 0$. *Then, for every data*

$$\begin{aligned}\mathbf{F} \in L_{p,0,\gamma_0}(\mathbb{R}; L_q(\Omega)^N), \quad &G \in W^1_{p,0,\gamma_0}(\mathbb{R}; \widehat{W}_q^{-1}(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}; W_q^1(\Omega)),\\ &\mathbf{H} \in H^{1/2}_{p,0,\gamma_0}(\mathbb{R}; L_q(\Omega)^N) \cap L_{p,0,\gamma_0}(\mathbb{R}; W_q^1(\Omega)^N),\end{aligned}$$

problem (1.2) *admits a solution* $(\mathbf{U}, \Theta)$ *of class*

$$\begin{aligned}&\mathbf{U} \in W^1_{p,\gamma_0}(0,\infty; L_q(\Omega)^N) \cap L_{p,\gamma_0}(0,\infty; W_q^2(\Omega)^N),\\ &\Theta \in L_{p,\gamma_0}(0,\infty; \widehat{W}_q^1(\Omega))\end{aligned} \tag{2.3}$$

satisfying the estimate

$$
\begin{aligned}
&\|e^{-\gamma t}(\partial_t \mathbf{U}, \gamma \mathbf{U}, \Lambda_\gamma^{1/2}\nabla \mathbf{U}, \nabla^2 \mathbf{U}, \nabla \Theta)\|_{L_p(0,\infty;L_q(\Omega))} \\
&\leq C_{\gamma_0}\{\|e^{-\gamma t}(\mathbf{F}, \Lambda_\gamma^{1/2} G, \nabla G, \Lambda_\gamma^{1/2}\mathbf{H}, \nabla \mathbf{H})\|_{L_p(\mathbb{R};L_q(\Omega))} + \|e^{-\gamma t}\partial_t G\|_{L_p(\mathbb{R};\widehat{W}_q^{-1}(\Omega))}\}
\end{aligned}
$$

for any $\gamma \geq \gamma_0$ *with some constant* C_{γ_0} *independent of* γ*. Moreover, the solution of* (1.2) *is unique, that is, if* $(\mathbf{U}, \Theta)$ *of class* (2.3) *is a solution to* (1.2) *with* $(\mathbf{F}, G, \mathbf{H}) = (0,0,0)$*, then,* $(\mathbf{U}, \Theta)(x,t) = (0,0)$ *a.e.* $(x,t) \in \Omega \times (0,\infty)$*.*

Using a fixed-point argument based on this theorem, we can prove the local well-posedness of the free boundary problem (1.5). We first derive a quasilinear problem in the fixed layer from (1.5) and next introduce further notation. Since $\Omega(t)$ is unknown, we rewrite the equation (1.5) in the Lagrange coordinates $y \in \Omega$ instead of the Euler ones $x \in \Omega(t)$ by using the Lagrange transform:

$$
x = y + \int_0^t \mathbf{u}(y,s)\,ds \equiv \mathbf{X_u}(y,t)
$$

$$
\mathbf{u} = (u_1(y,t), \cdots, u_N(y,t)) = \mathbf{v}(\mathbf{X_u}(y,t),t), \quad \theta(y,t) = \pi(\mathbf{X_u}(y,t),t).
$$

Then the pair $(\mathbf{u}, \theta)$ obeys the following problem (cf. [33, Appendix A]):

$$
(2.4)\begin{cases} \partial_t \mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u},\theta) = \mathbf{f}(\mathbf{u}), \quad \mathrm{div}\,\mathbf{u} = g(\mathbf{u}) = \mathrm{div}\,\mathbf{g}(\mathbf{u}) & \text{in } \Omega \times (0,T), \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{h}(\mathbf{u}) & \text{on } \partial\Omega \times (0,T), \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases}
$$

Here, nonlinear terms $\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \cdots, f_N(\mathbf{u}))$, $g(\mathbf{u})$, $\mathbf{g}(\mathbf{u}) = (g_1(\mathbf{u}), \cdots, g_N(\mathbf{u}))$ and $\mathbf{h}(\mathbf{u}) = (h_1(\mathbf{u}), \cdots, h_N(\mathbf{u}))$ are given by

$$
\begin{aligned}
f_i(\mathbf{u}) &= \sum_j \mathbf{V}^1_{ij}\left(\int_0^t \nabla \mathbf{u}\,ds\right)\partial_t u_j + \sum_{j,k,\ell} \mathbf{V}^2_{ijk\ell}\left(\int_0^t \nabla \mathbf{u}\,ds\right)\partial_\ell \partial_k u_j \\
&\quad + \sum_{j,k,\ell,m,n} \mathbf{V}^3_{ijk\ell mn}\left(\int_0^t \nabla \mathbf{u}\,ds\right)\int_0^t \partial_\ell \partial_k u_j\,ds\,\partial_n u_m, \\
g(\mathbf{u}) &= \sum_{j,k} \mathbf{V}^4_{jk}\left(\int_0^t \nabla \mathbf{u}\,ds\right)\partial_k u_j, \quad g_i(\mathbf{u}) = \sum_j \mathbf{V}^5_{ij}\left(\int_0^t \nabla \mathbf{u}\,ds\right) u_j, \\
h_i(\mathbf{u}) &= \sum_{j,k} \mathbf{V}^6_{ijk}\left(\int_0^t \nabla \mathbf{u}\,ds\right)\partial_k u_j
\end{aligned}
$$

with some polynomials $\mathbf{V}^1_{ij}$, $\mathbf{V}^2_{ijk\ell}$, $\mathbf{V}^3_{ijk\ell mn}$, $\mathbf{V}^4_{jk}$, $\mathbf{V}^5_{ij}$ and $\mathbf{V}^6_{ijk}$ such that

$$
\mathbf{V}^1_{ij}(\mathbf{O}),\ \mathbf{V}^2_{ijk\ell}(\mathbf{O}),\ \mathbf{V}^4_{jk}(\mathbf{O}),\ \mathbf{V}^5_{ij}(\mathbf{O}),\ \mathbf{V}^6_{ijk}(\mathbf{O}) = 0.
$$

As the linearized problem associated with (2.4), we have to study the Stokes initial value problem

$$
(2.5) \qquad \begin{cases} \partial_t \mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u},\theta) = \mathbf{F}, \quad \mathrm{div}\,\mathbf{u} = G & \text{in } \Omega \times (0,\infty), \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{H} & \text{on } \partial\Omega \times (0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases}
$$

By virtue of Theorem 2.2 for (1.2) with zero initial condition, it suffices to solve the case where $(\mathbf{F}, G, \mathbf{H}) = (0,0,0)$. To show the generation of an analytic semigroup, we follow the ideas in [20, Section 4], see also [34, p. 159, 160]. By applying div to the first equation and by taking the N-th component of the boundary condition in (2.5) with $(\mathbf{F}, G, \mathbf{H}) = (0, 0, 0)$, we have

$$\begin{cases} \Delta\theta = 0 & \text{in } \Omega, \\ \theta = 2\mu\partial_N u_N - \operatorname{div} \mathbf{u} & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

since $\operatorname{div} \mathbf{u} = 0$. Set $\tilde\theta = \theta - (2\mu\partial_N u_N - \operatorname{div} \mathbf{u})$. Then we get

$$\begin{cases} \Delta\tilde\theta = -\operatorname{div} \nabla(2\mu\partial_N u_N - \operatorname{div} \mathbf{u}) & \text{in } \Omega, \\ \tilde\theta = 0 & \text{on } \partial\Omega, \end{cases}$$

whose weak formulation is given by (1.4) with $\mathbf{f} = -\nabla(2\mu\partial_N u_N - \operatorname{div} \mathbf{u})$. By Proposition 2.1 together with [27, Remark 1.7], we have the following proposition, which guarantees the unique solvablity of the weak Dirichlet problem with $\mathcal{W}^1_q(\Omega) = \widehat{W}^1_{q,0}(\Omega)$.

Proposition 2.2. *For all $\mathbf{f} \in L_q(\Omega)^N$, there exists a unique solution*

$$\Pi_0\mathbf{f} := \theta \in \widehat{W}^1_{q,0}(\Omega)$$

of (1.4) *with $\mathcal{W}^1_q(\Omega) = \widehat{W}^1_{q,0}(\Omega)$ along with*

$$\|\nabla\theta\|_{L_q(\Omega)} \leq C_{N,q}\|\mathbf{f}\|_{L_q(\Omega)}.$$

Thus, the solution θ to (2.6) is given by

$$\theta = \Pi\mathbf{u} := (2\mu\partial_N u_N - \operatorname{div} \mathbf{u}) - \Pi_0(\nabla(2\mu\partial_N u_N - \operatorname{div} \mathbf{u})),$$
$$\Pi : W^2_q(\Omega) \to W^1_q(\Omega) + \widehat{W}^1_{q,0}(\Omega).$$

In this way, the problem (2.5) with $(\mathbf{F}, G, \mathbf{H}) = (0, 0, 0)$ is reduced to

$$\begin{cases} \partial_t\mathbf{u} - \operatorname{Div} \mathbf{S}(\mathbf{u}, \Pi(\mathbf{u})) = 0, \quad \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \times (0, \infty), \\ \mathbf{S}(\mathbf{u}, \Pi(\mathbf{u}))\nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \mathbf{u}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega. \end{cases} \tag{2.7}$$

Note that the second equation $\operatorname{div} \mathbf{u} = 0$ can be recovered if $\mathbf{v}_0$ is taken from

$$J_q(\Omega) = \{\mathbf{u} \in L_q(\Omega)^N \mid \operatorname{div} \mathbf{u} = 0\},$$

by the uniqueness of solutions to the initial value problem for the heat equation subject to the Dirichlet boundary condition, which $\operatorname{div} \mathbf{u}$ obeys, see [20, p. 243]. We then define the Stokes operator A_q by

$$D(A_q) = \{\mathbf{u} \in J_q(\Omega) \cap W^2_q(\Omega)^N \mid \mathbf{S}(\mathbf{u}, \Pi(\mathbf{u}))\nu = 0 \text{ on } \partial\Omega\},\ A_q\mathbf{u} = -\operatorname{Div} \mathbf{S}(\mathbf{u}, \Pi(\mathbf{u})).$$

Then the system (2.7) is formulated as

$$\partial_t\mathbf{u} + A_q\mathbf{u} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{v}_0.$$

By Proposition 2.1 and by the argument in [26, Lemma 4.4] and [34, Lemma 3.7], we get the following proposition.

Proposition 2.3. *The operator $-A_q$ generates an analytic semigroup $\{e^{-tA_q}\}_{t\geq 0}$ of class C^0 on $J_q(\Omega)$ for $1 < q < \infty$.*

Hence, $(e^{-tA_q}\mathbf{v}_0, \Pi(e^{-tA_q}\mathbf{v}_0))$ is a solution of (2.5) with $(\mathbf{F}, G, \mathbf{H}) = (0, 0, 0)$. Let $(\mathbf{U}, \Theta)$ be the solution obtained in Theorem 2.2. Then $\mathbf{u} = e^{-tA_q}\mathbf{v}_0 + \mathbf{U}$ and $\theta = \Pi(e^{-tA_q}\mathbf{v}_0) + \Theta$ solve the problem (2.5).

Let $1 < p, q < \infty$. We define the Besov space $B^{2(1-1/p)}_{q,p}(\Omega)$ by $B^{2(1-1/p)}_{q,p}(\Omega) = (L_q(\Omega), W^2_q(\Omega))_{1-1/p,p}$ by use of real interpolation functor $(\cdot, \cdot)_{1-1/p,p}$. Moreover, let $\mathcal{D}_{q,p}(\Omega) = (J_q(\Omega), D(A_q))_{1-1/p,p}$. The following theorem provides the local well-posedness for the nonlinear problem (2.4). The proof may be omitted since similar arguments can be found in [29, 26, 33].

Theorem 2.3. *Let $2 < p < \infty$ and $N < q < \infty$. For all $R > 0$, there exists $T = T(R) > 0$ such that for any initial data $\mathbf{v}_0 \in \mathcal{D}_{q,p}(\Omega) \subset B^{2(1-1/p)}_{q,p}(\Omega)^N$ with $\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq R$, the problem (2.4) admits a unique solution*

$$\mathbf{u} \in W^1_p(0,T; L_q(\Omega)^N) \cap L_p(0,T; W^2_q(\Omega)^N)$$

with some pressure term $\theta \in L_p(0,T; \widehat{W}^1_q(\Omega))$ satisfying the following estimate:

$$\|\partial_t \mathbf{u}\|_{L_p(0,T;L_q(\Omega))} + \|\mathbf{u}\|_{L_p(0,T;W^2_q(\Omega))} \leq M_0 R$$

with some constant M_0 independent of T and R.

3. Reduction to the problem only with boundary data

In this section, we reduce the problem (1.1) to the case where $\mathbf{f} = 0$ and $g = 0$, and the main theorem to the corresponding theorem.

We begin with the following properties of $\mathscr{R}$-bounded families.

Lemma 3.1 ([17, Proposition 3.4])**.** *Let X, Y and Z be Banach spaces.*

i) *Given $T \in \mathcal{L}(X,Y)$, the singleton $\{T\}$ is $\mathscr{R}$-bounded in $\mathcal{L}(X,Y)$.*
ii) *Let $\mathcal{S}$ and $\mathcal{T}$ be $\mathscr{R}$-bounded families on $\mathcal{L}(X,Y)$. Then $\mathcal{S}+\mathcal{T} = \{S+T \mid S \in \mathcal{S},\ T \in \mathcal{T}\}$ is also $\mathscr{R}$-bounded on $\mathcal{L}(X,Y)$ and*
$$\mathscr{R}_{\mathcal{L}(X,Y)}(\mathcal{S}+\mathcal{T}) \leq \mathscr{R}_{\mathcal{L}(X,Y)}(\mathcal{S}) + \mathscr{R}_{\mathcal{L}(X,Y)}(\mathcal{T}).$$
iii) *Let $\mathcal{S}$ and $\mathcal{T}$ be $\mathscr{R}$-bounded families on $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then $\mathcal{T}\mathcal{S} = \{TS \mid T \in \mathcal{T},\ S \in \mathcal{S}\}$ is $\mathscr{R}$-bounded on $\mathcal{L}(X,Z)$ and*
$$\mathscr{R}_{\mathcal{L}(X,Z)}(\mathcal{T}\mathcal{S}) \leq \mathscr{R}_{\mathcal{L}(Y,Z)}(\mathcal{T})\mathscr{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

First, we reduce (1.1) to the case $g = 0$. Let $\varphi \in C^\infty(\mathbb{R})$ be a cut-off function satisfying

$$0 \leq \varphi(x_N) \leq 1, \quad \varphi(x_N) = \left\{ \begin{array}{ll} 0, & |x_N| \leq 1/3, \\ 1, & |x_N| \geq 2/3, \end{array} \right.$$

and set

$$\varphi_\delta(x_N) = \varphi(x_N/\delta), \quad \varphi_0(x_N) = 1 - \varphi_\delta(x_N). \tag{3.1}$$

Then, given function f defined on Ω, we define

$$\begin{aligned} f^o_0(x) &= \begin{cases} \varphi_0(x_N) f(x', x_N), & x_N > 0, \\ -\varphi_0(-x_N) f(x', -x_N), & x_N < 0, \end{cases} \\ f^o_\delta(x) &= \begin{cases} \varphi_\delta(x_N) f(x', x_N), & x_N < \delta, \\ -\varphi_\delta(2\delta - x_N) f(x', 2\delta - x_N), & x_N > \delta. \end{cases} \end{aligned} \tag{3.2}$$

The following lemma gives a solution to the divergence equation.

Lemma 3.2 ([25, Theorem 1.2])**.** *We define the operator $\mathcal{V}_0 g = (\mathcal{V}_{01} g, \cdots, \mathcal{V}_{0N} g)^\mathsf{T}$ by*

$$\mathcal{V}_{0J} g(x) = -\mathscr{F}^{-1}_\xi \left[\frac{i\xi_J}{|\xi|^2} \mathscr{F}_x[g^*](\xi) \right](x) \quad (J = 1, \cdots, N)$$

where $g^*(x) = g_0^o(x) + g_\delta^o(x) \in \widehat{W}_q^{-1}(\mathbb{R}^N) \cap W_q^1(\mathbb{R}^N)$*. Then, for* $1 < q < \infty$ *and* $g \in \widehat{W}_q^{-1}(\Omega) \cap W_q^1(\Omega)$*, the operator* $\mathcal{V}_0 \in \mathcal{L}(\widehat{W}_q^{-1}(\Omega) \cap W_q^1(\Omega), W_q^2(\Omega)^N)$ *satisfies the following properties:*

a) $\mathbf{v} = \mathcal{V}_0 g$ *solves the divergence equation* $div\, \mathbf{v} = g$.

b) *We have the estimates*

$$\|\mathcal{V}_0 g\|_{L_q(\Omega)} \le C\|g\|_{\widehat{W}_q^{-1}(\Omega)}, \quad \|\nabla \mathcal{V}_0 g\|_{L_q(\Omega)} \le C\|g\|_{L_q(\Omega)}, \quad \|\nabla^2 \mathcal{V}_0 g\|_{L_q(\Omega)} \le C\|\nabla g\|_{L_q(\Omega)}.$$

Set $\mathbf{u} = \mathcal{V}_0 g + \mathbf{v}$ in (1.1). We have

$$\begin{cases} \lambda \mathbf{v} - \mathrm{Div}\, \mathbf{S}(\mathbf{v}, \theta) = \tilde{\mathbf{f}}, \quad \mathrm{div}\, \mathbf{v} = 0 & \text{in } \Omega, \\ \mathbf{S}(\mathbf{v}, \theta)\nu = \mathbf{h} - \mu \mathbf{D}(\mathcal{V}_0 g)\nu & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

where

$$\tilde{\mathbf{f}} = \mathbf{f} - \lambda \mathcal{V}_0 g + \mathrm{Div}\,(\mu \mathbf{D}(\mathcal{V}_0 g)). \tag{3.4}$$

Next, we reduce the system (3.3) to the case where the data are only on the boundary. For this purpose, given function $\mathbf{f}$ defined on Ω, we consider the problem in the whole space

$$\lambda \mathbf{v} - \mathrm{Div}\, \mathbf{S}(\mathbf{v}, \pi) = E\mathbf{f}, \quad \mathrm{div}\, \mathbf{v} = 0 \quad \text{in } \mathbb{R}^N. \tag{3.5}$$

Here, $E\mathbf{f} = (\bar{f}_1, \cdots, \bar{f}_N)$ with $\bar{f}_j = f_{j0}^e + f_{j\delta}^o$ ($j = 1, \cdots, N-1$) and $\bar{f}_N = f_{N0}^o + f_{N\delta}^e$, where we have set, in addition to (3.2),

$$f_0^e(x) = \begin{cases} \varphi_0(x_N) f(x', x_N) & x_N > 0, \\ \varphi_0(-x_N) f(x', -x_N) & x_N < 0, \end{cases}$$

$$f_\delta^e(x) = \begin{cases} \varphi_\delta(x_N) f(x', x_N) & x_N < \delta, \\ \varphi_\delta(2\delta - x_N) f(x', 2\delta - x_N) & x_N > \delta. \end{cases}$$

The following lemma on the $\mathscr{R}$-boundedness of the solution operator families for (3.5) is proved by Saito [25].

Lemma 3.3 ([25, Lemma 2.7]). *For all* $\lambda \in \mathbb{C} \setminus (-\infty, 0]$*, there exist operators* $S_0(\lambda) = (S_{01}(\lambda), \cdots, S_{0N}(\lambda))$ *and* T_0 *satisfying* $S_0(\lambda) \in \mathcal{L}(L_q(\Omega)^N, W_q^2(\Omega)^N)$ *and* $T_0 \in \mathcal{L}(L_q(\Omega)^N, W_q^1(\Omega))$ *for* $1 < q < \infty$ *such that the following assertions hold:*

a) *For any* $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ *and* $\mathbf{f} \in L_q(\Omega)^N$*, the pair*

$$(\mathbf{v}, \pi) = (S_0(\lambda)\mathbf{f}, T_0\mathbf{f}) \in W_q^2(\Omega)^N \times W_q^1(\Omega)$$

is a unique solution of (3.5).

b) *For any* $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\ell = 0, 1$ *and* $1 \le m, n, J \le N$*, there hold*

$$\begin{aligned} \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda S_{0J}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\le C_{N,q,\varepsilon,\mu,\delta}, \\ \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma S_{0J}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\le C_{N,q,\varepsilon,\mu,\delta}, \\ \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2} \partial_m S_{0J}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\le C_{N,q,\varepsilon,\mu,\delta}, \\ \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m \partial_n S_{0J}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\le C_{N,q,\varepsilon,\mu,\delta}, \\ \|T_0\|_{L_q(\Omega)} + \|\nabla T_0\|_{L_q(\Omega)} &\le C_{N,q,\mu,\delta}, \end{aligned} \tag{3.6}$$

where $\Sigma_{\varepsilon,0}$ *is given by* (1.3) *with* $\gamma_0 = 0$*, and* $\lambda = \gamma + i\tau$.

Let $\mathbf{v} = S_0(\lambda)\tilde{\mathbf{f}} + \mathbf{w}$ and $\theta = T_0\tilde{\mathbf{f}} + \pi$ in (3.3). We have

$$\begin{cases} \lambda\mathbf{w} - \mathrm{Div}\,\mathbf{S}(\mathbf{w},\pi) = 0, \quad \mathrm{div}\,\mathbf{w} = 0 & \text{in } \Omega, \\ \mathbf{S}(\mathbf{w},\pi)\nu = \tilde{\mathbf{h}} & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

where

$$\tilde{\mathbf{h}} = \mathbf{h} - \mu\mathbf{D}(\mathcal{V}_0 g)\nu - \mathbf{S}(S_0(\lambda)\tilde{\mathbf{f}}, T_0\tilde{\mathbf{f}})\nu. \tag{3.8}$$

Then, as justified later in this section, it suffices to prove the $\mathscr{R}$-boundedness of the solution operator families of (1.1) with $\mathbf{f} = 0$ and $g = 0$:

$$\begin{cases} \lambda\mathbf{u} - \mathrm{Div}\,\mathbf{S}(\mathbf{u},\theta) = 0, \quad \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u},\theta)\nu = \mathbf{h} & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Theorem 3.1. *For all $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, there exist operators $\mathcal{S}(\lambda) = (\mathcal{S}_1(\lambda), \cdots, \mathcal{S}_N(\lambda))$ and $\mathcal{T}(\lambda)$ (precisely, they are given by* (6.24) *and* (6.31)*) satisfying $\mathcal{S}(\lambda) \in \mathcal{L}(L_q(\Omega)^{N+N^2}, W^2_q(\Omega)^N)$ and $\mathcal{T}(\lambda) \in \mathcal{L}(L_q(\Omega)^{N+N^2}, \widehat{W}^1_q(\Omega))$ for $1 < q < \infty$ such that the following assertions hold:*

a) *For any $1 < q < \infty$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $\mathbf{h} = (\mathbf{h}', h_N) = (h_1, \cdots, h_N) \in W^1_q(\Omega)^N$, the pair $(\mathbf{u}, \theta) \in W^2_q(\Omega)^N \times \widehat{W}^1_q(\Omega)$ given by*

$$\mathbf{u} = \mathcal{S}(\lambda)(\lambda^{1/2}\mathbf{h}', h_N, \nabla\mathbf{h}), \quad \theta = \mathcal{T}(\lambda)(\lambda^{1/2}\mathbf{h}', h_N, \nabla\mathbf{h}) \tag{3.10}$$

is a solution of (3.9)*.*

b) *For any $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, $\ell = 0, 1$ and $1 \le m, n, J \le N$, there hold*

$$\begin{aligned} \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\ \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\ \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2}\partial_m\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\ \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\partial_n\mathcal{S}_J(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\ \mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\mathcal{T}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\le C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \end{aligned} \tag{3.11}$$

where $\Sigma_{\varepsilon,\gamma_0}$ is given by (1.3) *and $\lambda = \gamma + i\tau$.*

Remark 3.1. *It is reasonable (and possible by this theorem on account of $\gamma_0 > 0$) to show the $\mathscr{R}$-boundedness for*

$$\mathbf{u} = \mathcal{S}(\lambda)(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h}), \quad \theta = \mathcal{T}(\lambda)(\lambda^{1/2}\mathbf{h}, \nabla\mathbf{h})$$

instead of (3.10)*. In view of* (3.6) *and* (3.8)*, however, this is not useful to show Theorem 2.1 since one cannot control $\lambda^{1/2}T_0\tilde{\mathbf{f}}\nu$ that appears in $\lambda^{1/2}\tilde{\mathbf{h}}$. This is why we provide Theorem 3.1 with the solution operators* (3.10)*. Note that the pressure $T_0\tilde{\mathbf{f}}$ is present only in the N-th component of $\tilde{\mathbf{h}}$.*

In the remainder of this section, we prove that our main theorem (Theorem 2.1) follows from Theorem 3.1, which will be established in Section 6. To show the uniqueness of solutions by duality, we use the following lemma, see for instance [25, Lemma 7.1].

Lemma 3.4. *Let $1 < q < \infty$ and let q' be its dual exponent. For given $\mathbf{u} \in W^2_q(\Omega)^N$, $\mathbf{v} \in W^2_{q'}(\Omega)^N$, $\theta \in W^1_q(\Omega)$ and $\pi \in W^1_{q'}(\Omega)$, we have the following formula:*

$$(\mathbf{u}, \mathit{Div}\,\mathbf{S}(\mathbf{v},\pi))_\Omega = (\mathit{Div}\,\mathbf{S}(\mathbf{u},\theta), \mathbf{v})_\Omega + \langle \mathbf{u}, \mathbf{S}(\mathbf{v},\pi)\nu\rangle_{\partial\Omega} - \langle \mathbf{S}(\mathbf{u},\theta)\nu, \mathbf{v}\rangle_{\partial\Omega}$$

$$+(div\,\mathbf{u},\pi)_\Omega-(\theta,div\,\mathbf{v})_\Omega.$$

Proof of Theorem 2.1 from Theorem 3.1. By summarizing the aforementioned arguments, the solution of (1.1) is given by

(3.12)
$$(\mathbf{u},\,\theta)=\big(\mathcal{V}_0 g+S_0(\lambda)\tilde{\mathbf{f}}+\mathcal{S}(\lambda)(\lambda^{1/2}\tilde{\mathbf{h}}',\tilde{h}_N,\nabla\tilde{\mathbf{h}}),\,T_0\tilde{\mathbf{f}}+\mathcal{T}(\lambda)(\lambda^{1/2}\tilde{\mathbf{h}}',\tilde{h}_N,\nabla\tilde{\mathbf{h}})\big).$$

We define the operator $\widetilde{\mathcal{V}}_0=(\widetilde{\mathcal{V}}_0^{ijk})_{1\le i,j,k\le N}$ so that $\widetilde{\mathcal{V}}_0(\nabla g)=\nabla\mathbf{D}(\mathcal{V}_0 g)=(\partial_k D_{ij}(\mathcal{V}_0 g))_{1\le i,j,k\le N}$ by

$$\begin{aligned}
&\widetilde{\mathcal{V}}_0^{\ell mk}(\nabla g):=\widetilde{\mathcal{V}}_0^{m\ell k}(\nabla g):=\partial_k D_{\ell m}(\mathcal{V}_0 g)=\partial_k D_{m\ell}(\mathcal{V}_0 g)\\
&=\partial_k\left(-\partial_\ell\mathscr{F}_\xi^{-1}\left[\frac{i\xi_m}{|\xi|^2}\mathscr{F}_x[g^*](\xi)\right](x)-\partial_m\mathscr{F}_\xi^{-1}\left[\frac{i\xi_\ell}{|\xi|^2}\mathscr{F}_x[g^*](\xi)\right](x)\right)\\
&=-2\partial_k\mathscr{F}_\xi^{-1}\left[\frac{i\xi_\ell}{|\xi|^2}\mathscr{F}_x[(\partial_m g)^*](\xi)\right](x)\\
&=2\partial_k\mathcal{V}_{0\ell}(\partial_m g),\\
&\widetilde{\mathcal{V}}_0^{NNk}(\nabla g):=\partial_k D_{NN}(\mathcal{V}_0 g)\\
&=2\partial_k\left(\mathrm{div}\,\mathcal{V}_0 g-\sum_{j=1}^{N-1}\partial_j\mathcal{V}_{0j}g\right)=2\partial_k g-2\sum_{j=1}^{N-1}\partial_k\mathcal{V}_{0j}(\partial_j g)
\end{aligned}$$

for $1\le k,\ell\le N$ and $1\le m\le N-1$. By Lemma 3.2, we get

$$\widetilde{\mathcal{V}}_0\in\mathcal{L}(L_q(\Omega)^N,L_q(\Omega)^{N^3}).\tag{3.13}$$

Additionally, we define $M_{\lambda^{-1/2}}$ and an extension $\tilde\nu=(0,\cdots,0,\tilde\nu_N)$ of ν by

$$\begin{aligned}
&M_{\lambda^{-1/2}}f=\lambda^{-1/2}f\quad(f\in L_q(\Omega)),\\
&\tilde\nu(x)=\varphi_\delta(x_N)\nu(x',\delta)+\varphi_0(x_N)\nu(x',0).
\end{aligned}\tag{3.14}$$

By the Kahane's contraction principle, see [24, Proposition 2.5], if $\gamma_0>0$, we have

$$\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{M_{\lambda^{-1/2}}\mid\lambda\in\Sigma_{\varepsilon,\gamma_0}\})\le 2\gamma_0^{-1/2}.\tag{3.15}$$

In view of (3.4) and (3.8), we define

$$\begin{aligned}
&[R_0(\mathbf{f},\lambda g,\nabla g)]_i\\
&=[\tilde{\mathbf{f}}]_i\\
&=[\mathbf{f}]_i-\mathcal{V}_{0i}\lambda g+\sum_{j=1}^N\mu\widetilde{\mathcal{V}}_0^{ijj}(\nabla g),\\
&R_1'(\lambda)(\mathbf{f},\lambda g,\lambda^{1/2}g,\nabla g,\lambda^{1/2}\mathbf{h},\nabla\mathbf{h})\\
&=\lambda^{1/2}\tilde{\mathbf{h}}'\\
&=\lambda^{1/2}h_j-\mu[\mathbf{D}(\mathcal{V}_0\lambda^{1/2}g)\tilde\nu]'-\mu[\lambda^{1/2}\mathbf{D}(S_0(\lambda)R_0(\mathbf{f},\lambda g,\nabla g))\tilde\nu]',\\
&R_{1N}(\lambda)(\mathbf{f},\lambda g,\lambda^{1/2}g,\nabla g,\lambda^{1/2}\mathbf{h},\nabla\mathbf{h})\\
&=\tilde h_N\\
&=M_{\lambda^{-1/2}}\lambda^{1/2}h_N-\mu M_{\lambda^{-1/2}}[\mathbf{D}(\mathcal{V}_0\lambda^{1/2}g)\tilde\nu]_N\\
&-\mu M_{\lambda^{-1/2}}[\lambda^{1/2}\mathbf{D}(S_0(\lambda)R_0(\mathbf{f},\lambda g,\nabla g))\tilde\nu]_N+T_0R_0(\mathbf{f},\lambda g,\nabla g)\tilde\nu_N,
\end{aligned}$$

$$
\begin{aligned}
&R_2(\lambda)(\mathbf{f}, \lambda g, \lambda^{1/2} g, \nabla g, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h})\\
&= \nabla \tilde{\mathbf{h}}\\
&= \nabla \mathbf{h} - \nabla\big(\mu \mathbf{D}(\mathcal{V}_0 g)\tilde{\nu}\big) - \nabla\big(\mathbf{S}(S_0(\lambda)R_0(\mathbf{f}, \lambda g, \nabla g), T_0 R_0(\mathbf{f}, \lambda g, \nabla g))\tilde{\nu}\big)\\
&= \nabla \mathbf{h} - \mu \widetilde{\mathcal{V}}_0(\nabla g)\tilde{\nu} - \mu M_{\lambda^{-1/2}}\mathbf{D}(\mathcal{V}_0 \lambda^{1/2} g)\nabla\tilde{\nu}\\
&- \nabla \mathbf{S}(S_0(\lambda)R_0(\mathbf{f}, \lambda g, \nabla g), T_0 R_0(\mathbf{f}, \lambda g, \nabla g))\tilde{\nu}\\
&- M_{\lambda^{-1/2}}\mu\lambda^{1/2}\mathbf{D}(S_0(\lambda)R_0(\mathbf{f}, \lambda g, \nabla g))\nabla\tilde{\nu} + T_0 R_0(\mathbf{f}, \lambda g, \nabla g)\nabla\tilde{\nu},
\end{aligned}
$$

where $[\mathbf{g}]_J$ is the J-th component of N-vector function $\mathbf{g}$ for $1 \le J \le N$ and $[\mathbf{g}]' = ([\mathbf{g}]_1, \cdots, [\mathbf{g}]_{N-1})$. By Lemma 3.1, Lemma 3.2, Lemma 3.3, (3.13) and (3.15),

$$
\begin{aligned}
&R_0 \in \mathcal{L}(L_q(\Omega)^N \times \widehat{W}_q^{-1}(\Omega) \times L_q(\Omega)^N, L_q(\Omega)^N),\\
&\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{R_1'(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) \le C_{N,q,\varepsilon,\mu,\delta},\\
&\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{R_{1N}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) \le C_{N,q,\varepsilon,\mu,\delta},\\
&\mathscr{R}_{\mathcal{L}(X_q(\Omega), L_q(\Omega))}(\{R_2(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) \le C_{N,q,\varepsilon,\mu,\delta}.
\end{aligned}
\tag{3.16}
$$

In view of (3.12), we define $\mathcal{U}(\lambda)$ and $\mathcal{P}(\lambda)$ by

(3.17)

$$
\begin{aligned}
&\mathcal{U}(\lambda)(\mathbf{f}, \lambda g, \lambda^{1/2} g, \nabla g, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h})\\
&= \mathcal{V}_0 g + S_0(\lambda)R_0(\lambda)(\mathbf{f}, \lambda g, \nabla g) + \mathcal{S}(\lambda)\big(R_1'(\lambda), R_{1N}(\lambda), R_2(\lambda)\big)(\mathbf{f}, \lambda g, \lambda^{1/2} g, \nabla g, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h}),\\
&\mathcal{P}(\lambda)(\mathbf{f}, \lambda g, \lambda^{1/2} g, \nabla g, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h})\\
&= T_0 R_0(\lambda)(\mathbf{f}, \lambda g, \nabla g) + \mathcal{T}(\lambda)\big(R_1'(\lambda), R_{1N}(\lambda), R_2(\lambda)\big)(\mathbf{f}, \lambda g, \lambda^{1/2} g, \nabla g, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h}).
\end{aligned}
$$

Then, from Lemma 3.1, Lemma 3.2, Lemma 3.3, Theorem 3.1 and (3.16), we obtain the existence and the $\mathscr{R}$-boundedness of the solution operator families.

Thus, the proof will be completed by showing the uniqueness of solutions to (1.1). Assume that $(\mathbf{u}, \theta) \in W_q^2(\Omega)^N \times \widehat{W}_q^1(\Omega)$ satisfies the following equation:

$$
\begin{cases}
\lambda \mathbf{u} - \operatorname{Div} \mathbf{S}(\mathbf{u}, \theta) = 0, \quad \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega,\\
\mathbf{S}(\mathbf{u}, \theta)\nu = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.18}
$$

We find $\theta \in L_q(\Omega)$ by using the N-th component of the boundary condition as follows:

$$
\begin{aligned}
\|\theta\|_{L_q(\Omega)} &= \left\| 2\mu \partial_N u_N(x', 0) + \int_0^{x_N} \partial_N \theta(x', y_N)\, dy_N \right\|_{L_q(\Omega)}\\
&\le 2\mu \delta^{1/q} \|\partial_N u_N(x', 0)\|_{L_q(\mathbb{R}^{N-1})} + \delta \|\partial_N \theta\|_{L_q(\Omega)}\\
&\le C(\|u\|_{W_q^2(\Omega)} + \|\partial_N \theta\|_{L_q(\Omega)}) < \infty.
\end{aligned}
$$

Thus, given $\varphi \in C_0^\infty(\Omega)$, we take a solution $(\mathbf{v}, \pi) \in W_{q'}^2(\Omega)^N \times W_{q'}^1(\Omega)$ of the problem

$$
\begin{cases}
\overline{\lambda}\mathbf{v} - \operatorname{Div} \mathbf{S}(\mathbf{v}, \pi) = \varphi, \quad \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega,\\
\mathbf{S}(\mathbf{v}, \pi)\nu = 0 & \text{on } \partial\Omega,
\end{cases}
$$

and employ Lemma 3.4 to see that

$$
(\mathbf{u}, \varphi)_\Omega = (\mathbf{u}, \overline{\lambda}\mathbf{v} - \operatorname{Div} \mathbf{S}(\mathbf{v}, \pi))_\Omega = (\lambda \mathbf{u} - \operatorname{Div} \mathbf{S}(\mathbf{u}, \theta), \mathbf{v})_\Omega = 0,
$$

which implies $\mathbf{u} = 0$. Hence, $\nabla\theta = 0$ by the first equation of (3.18), and so $\theta = 0$ by the boundary condition, which proves the uniqueness. $\square$

4. Solution formulas for the problem only with boundary data

In this section, we give the solution formulas of (3.9) in order to prove Theorem 3.1. By applying the partial Fourier transform with respect to x' to (3.9), we get

$$\begin{cases} \mu(B^2-\partial_N^2)\widehat{u}_j(\xi',x_N)+i\xi_j\widehat{\theta}(\xi',x_N)=0 & 0<x_N<\delta, \\ \mu(B^2-\partial_N^2)\widehat{u}_N(\xi',x_N)+\partial_N\widehat{\theta}(\xi',x_N)=0 & 0<x_N<\delta, \\ i\xi'\cdot\widehat{u}'(\xi',x_N)+\partial_N\widehat{u}_N(\xi',x_N)=0 & 0<x_N<\delta, \\ \mu(\partial_N\widehat{u}_j(\xi',x_N)+i\xi_j\widehat{u}_N(\xi',x_N))\nu_N(x_N)=\widehat{h}_j(\xi',x_N) & x_N\in\{0,\delta\}, \\ (2\mu\partial_N\widehat{u}_N(\xi',x_N)-\widehat{\theta}(\xi',x_N))\nu_N(x_N)=\widehat{h}_N(\xi',x_N) & x_N\in\{0,\delta\} \end{cases} \tag{4.1}$$

for $1\leq j\leq N-1$, where ν_N means the N-th component of the unit outer normal to $\partial\Omega$, that is, $\nu_N(\delta)=1$ and $\nu_N(0)=-1$. Here, we have set

$$A=|\xi'|,\quad B=\sqrt{\mu^{-1}\lambda+A^2} \tag{4.2}$$

with $\mathrm{Re}\,B>0$. To simplify the first and fourth equations of (4.1), we multiply them by $-i\xi_j$ and add up the resultant formulas. Then, if we set

$$\widehat{u}_d(\xi',x_N)=i\xi'\cdot\widehat{u}'(\xi',x_N),\quad \widehat{h}_d(\xi',x_N)=i\xi'\cdot\widehat{h}'(\xi',x_N),$$

we obtain the following ordinary differential equations with only three unknowns $\widehat{u}_d(\xi',\cdot)$, $\widehat{u}_N(\xi',\cdot)$ and $\widehat{\theta}(\xi',\cdot)$:

$$\begin{cases} \mu(B^2-\partial_N^2)\widehat{u}_d(\xi',x_N)-A^2\widehat{\theta}(\xi',x_N)=0 & 0<x_N<\delta, \\ \mu(B^2-\partial_N^2)\widehat{u}_N(\xi',x_N)+\partial_N\widehat{\theta}(\xi',x_N)=0 & 0<x_N<\delta, \\ \widehat{u}_d(\xi',x_N)+\partial_N\widehat{u}_N(\xi',x_N)=0 & 0<x_N<\delta, \\ \mu(\partial_N\widehat{u}_d(\xi',x_N)-A^2\widehat{u}_N(\xi',x_N))\nu_N(x_N)=\widehat{h}_d(\xi',x_N) & x_N\in\{0,\delta\}, \\ (2\mu\partial_N\widehat{u}_N(\xi',x_N)-\widehat{\theta}(\xi',x_N))\nu_N(x_N)=\widehat{h}_N(\xi',x_N) & x_N\in\{0,\delta\}. \end{cases} \tag{4.3}$$

In order to solve this system, we add the second equation multiplied by ∂_N to the first equation in (4.3) and then we have by the third equation

$$(\partial_N^2-A^2)\widehat{\theta}(\xi',x_N)=0. \tag{4.4}$$

Multiplying $(A^2-\partial_N^2)$ to the first and the second equations implies that

$$\begin{aligned} (A^2-\partial_N^2)(B^2-\partial_N^2)\widehat{u}_d(\xi',x_N)&=0, \\ (A^2-\partial_N^2)(B^2-\partial_N^2)\widehat{u}_N(\xi',x_N)&=0. \end{aligned} \tag{4.5}$$

Thus, the solution to (4.3) can be given by

$$\begin{aligned} \widehat{u}_d(\xi',x_N)&=\sum_{\ell=1,2}(\alpha_{\ell d}^0e^{-Ad_\ell(x_N)}+\beta_{\ell d}^0e^{-Bd_\ell(x_N)}), \\ \widehat{u}_N(\xi',x_N)&=\sum_{\ell=1,2}(\alpha_{\ell N}^0e^{-Ad_\ell(x_N)}+\beta_{\ell N}^0e^{-Bd_\ell(x_N)}), \\ \widehat{\theta}(\xi',x_N)&=\sum_{\ell=1,2}\gamma_\ell e^{-Ad_\ell(x_N)} \end{aligned} \tag{4.6}$$

with some coefficients $\alpha_{\ell d}^0$, $\beta_{\ell d}^0$, $\alpha_{\ell N}^0$, $\beta_{\ell N}^0$ and γ_ℓ depending on λ and ξ', where

$$d_\ell(x_N)=\begin{cases} \delta-x_N, & \ell=1, \\ x_N, & \ell=2. \end{cases} \tag{4.7}$$

We obtain the following relations of these coefficients by inserting (4.6) to the first, second and third equations of (4.3):

$$
\begin{aligned}
\lambda\alpha^0_{\ell d} - A^2\gamma_\ell = 0, \quad \lambda\alpha^0_{\ell N} + (-1)^{\ell-1}A\gamma_\ell = 0,\\
\alpha^0_{\ell d} + (-1)^{\ell-1}A\alpha^0_{\ell N} = 0, \quad \beta^0_{\ell d} + (-1)^{\ell-1}B\beta^0_{\ell N} = 0
\end{aligned}
\tag{4.8}
$$

for $\ell = 1, 2$. Then we have the following system for $\alpha^0_{\ell N}$ and $\beta^0_{\ell N}$ if we insert (4.6) to the fourth and fifth equations of (4.3) and rewrite them by using (4.8):

$$
\begin{aligned}
-2A^2(\alpha^0_{1N} + a\alpha^0_{2N}) - D_0(\beta^0_{1N} + b\beta^0_{2N}) &= \mu^{-1}\widehat{h}_d(\xi', \delta),\\
-2A^2(a\alpha^0_{1N} + \alpha^0_{2N}) - D_0(b\beta^0_{1N} + \beta^0_{2N}) &= -\mu^{-1}\widehat{h}_d(\xi', 0),\\
-D_0(-\alpha^0_{1N} + a\alpha^0_{2N}) - 2AB(-\beta^0_{1N} + b\beta^0_{2N}) &= \mu^{-1}A\widehat{h}_N(\xi', \delta),\\
-D_0(-a\alpha^0_{1N} + \alpha^0_{2N}) - 2AB(-b\beta^0_{1N} + \beta^0_{2N}) &= -\mu^{-1}A\widehat{h}_N(\xi', 0)
\end{aligned}
\tag{4.9}
$$

where

$$
a = e^{-A\delta}, \quad b = e^{-B\delta}, \quad D_0 = B^2 + A^2. \tag{4.10}
$$

Now, we rewrite (4.6) as

$$
\begin{aligned}
\widehat{u}_d(\xi', x_N) &= \sum_{\ell=1,2} (\mu_{\ell d}\mathcal{M}(d_\ell(x_N)) + \beta_{\ell d}e^{-Bd_\ell(x_N)}),\\
\widehat{u}_N(\xi', x_N) &= \sum_{\ell=1,2} (\mu_{\ell N}\mathcal{M}(d_\ell(x_N)) + \beta_{\ell N}e^{-Bd_\ell(x_N)}),\\
\widehat{\theta}(\xi', x_N) &= \sum_{\ell=1,2} \gamma_\ell e^{-Ad_\ell(x_N)}
\end{aligned}
\tag{4.11}
$$

with some coefficients $\mu_{\ell d}$, $\beta_{\ell d}$, $\mu_{\ell N}$ and $\beta_{\ell N}$ depending on λ and ξ' so that $(B-A)^{-1}$ does not appear, where

$$
\mathcal{M}(x_N) = \frac{e^{-Bx_N} - e^{-Ax_N}}{B - A}. \tag{4.12}
$$

Then, by comparing (4.6) and (4.11), we get

$$
\alpha^0_{\ell d} = -\frac{\mu_{\ell d}}{B - A}, \quad \beta^0_{\ell d} = \frac{\mu_{\ell d}}{B - A} + \beta_{\ell d}, \quad \alpha^0_{\ell N} = -\frac{\mu_{\ell N}}{B - A}, \quad \beta^0_{\ell N} = \frac{\mu_{\ell N}}{B - A} + \beta_{\ell N}. \tag{4.13}
$$

And thus, (4.9) yields

$$
\mathbf{L}\begin{bmatrix} \mu_{1N} \\ \beta_{1N} \\ \mu_{2N} \\ \beta_{2N} \end{bmatrix} = \begin{bmatrix} \mu^{-1}\widehat{h}_d(\xi', \delta) \\ -\mu^{-1}\widehat{h}_d(\xi', 0) \\ \mu^{-1}A\widehat{h}_N(\xi', \delta) \\ -\mu^{-1}A\widehat{h}_N(\xi', 0) \end{bmatrix} \tag{4.14}
$$

where (i,j) component L_{ij} of $\mathbf{L}$ is given by

$$
\begin{aligned}
&L_{11} = -(B+A), && L_{12} = -D_0,\\
&L_{13} = -(B+A)a - D_0\mathcal{M}(\delta), && L_{14} = -D_0 a - D_0(B-A)\mathcal{M}(\delta),\\
&L_{21} = -(B+A)a - D_0\mathcal{M}(\delta), && L_{22} = -D_0 a - D_0(B-A)\mathcal{M}(\delta),\\
&L_{23} = -(B+A), && L_{24} = -D_0,\\
&L_{31} = -(B-A), && L_{32} = 2AB,\\
&L_{33} = (B-A)a - 2AB\mathcal{M}(\delta), && L_{34} = -2ABa - 2AB(B-A)\mathcal{M}(\delta),\\
&L_{41} = -(B-A)a + 2AB\mathcal{M}(\delta), && L_{42} = 2ABa + 2AB(B-A)\mathcal{M}(\delta),\\
&L_{43} = -(B-A), && L_{44} = 2AB,
\end{aligned}
$$

where A and B are defined in (4.2), a and D_0 are defined by (4.10), and $\mathcal{M}(\delta)$ is given by (4.12) with $x_N = \delta$. Also, the coefficient γ_ℓ can be obtained by

$$\gamma_\ell = \frac{\mu(B+A)}{A}\mu_{\ell N} \tag{4.15}$$

from (4.8) and (4.13).

Thus, if we solve (4.14), we obtain the solution formula of u_N and θ by (4.11) and (4.15) as follows:

$$
\begin{aligned}
u_N(x) &= \sum_{k=1}^4\sum_{\ell=1}^2 \mathscr{F}^{-1}_{\xi'}\left[\left\{\frac{L_{k,2\ell-1}}{\det\mathbf{L}}\mathcal{M}(d_\ell(x_N)) + \frac{L_{k,2\ell}}{\det\mathbf{L}}e^{-Bd_\ell(x_N)}\right\}r_k\right](x'),\\
\theta(x) &= \sum_{k=1}^4\sum_{\ell=1}^2 \mathscr{F}^{-1}_{\xi'}\left[\left\{\frac{\mu(B+A)}{A}\frac{L_{k,2\ell-1}}{\det\mathbf{L}}e^{-Ad_\ell(x_N)}\right\}r_k\right](x'),
\end{aligned}
\tag{4.16}
$$

where $L_{i,j}$ denotes the (i,j) cofactor of $\mathbf{L}$ and r_k stands for the right-hand side:

$$r_1 = \mu^{-1}\widehat{h}_d(\xi',\delta), \quad r_2 = -\mu^{-1}\widehat{h}_d(\xi',0), \quad r_3 = \mu^{-1}A\widehat{h}_N(\xi',\delta), \quad r_4 = -\mu^{-1}A\widehat{h}_N(\xi',0).$$

Here, the determinant of $\mathbf{L}$ is given by

$$\det\mathbf{L} = \frac{1}{(B-A)^2}\prod_{+,-}\{(B^2+A^2)^2(1\pm a)(1\mp b) - 4A^3B(1\mp a)(1\pm b)\} \tag{4.17}$$

and the cofactor $L_{i,j}$ is

$$
\begin{aligned}
L_{1,1} &= -2ABD_3(1-a^2) - 16A^4B^2a\mathcal{M}(\delta) - 2AB(B^2-A^2)D_2\mathcal{M}(\delta)^2,\\
L_{1,2} &= -(B-A)D_3(1-a^2) - 8A^3B(B-A)a\mathcal{M}(\delta) + 2AB(B+A)D_2\mathcal{M}(\delta)^2,\\
L_{1,3} &= 2ABD_3a(1-a^2) + 4AB(D_0^2 - (B-A)D_3a^2)\mathcal{M}(\delta) - 2AB(B-A)^2D_3a\mathcal{M}(\delta)^2,\\
L_{1,4} &= (B-A)D_3a(1-a^2) - (D_0D_1 + (B^2-4AB+A^2)D_3a^2)\mathcal{M}(\delta) + 2AB(B-A)D_3a\mathcal{M}(\delta)^2,\\
L_{2,1} &= 2ABD_3a(1-a^2) + 4AB(D_0^2 - (B-A)D_3a^2)\mathcal{M}(\delta) - 2AB(B-A)^2D_3a\mathcal{M}(\delta)^2,\\
L_{2,2} &= (B-A)D_3a(1-a^2) - (D_0D_1 + (B^2-4AB+A^2)D_3a^2)\mathcal{M}(\delta) + 2AB(B-A)D_3a\mathcal{M}(\delta)^2,\\
L_{2,3} &= -2ABD_3(1-a^2) - 16A^4B^2a\mathcal{M}(\delta) - 2AB(B^2-A^2)D_2\mathcal{M}(\delta)^2,\\
L_{2,4} &= -(B-A)D_3(1-a^2) - 8A^3B(B-A)a\mathcal{M}(\delta) + 2AB(B+A)D_2\mathcal{M}(\delta)^2,\\
L_{3,1} &= -D_0D_3(1-a^2) + 2D_0^3a\mathcal{M}(\delta) + (B^2-A^2)D_0D_2\mathcal{M}(\delta)^2,
\end{aligned}
\tag{4.18}
$$

$L_{3,2} = (B+A)D_3(1-a^2) - 2(B+A)D_0^2 a\mathcal{M}(\delta) - (B+A)D_0D_2\mathcal{M}(\delta)^2,$
$L_{3,3} = -D_0D_3a(1-a^2) + 2D_0(4A^3B + (B-A)D_3a^2)\mathcal{M}(\delta) + (B-A)^2D_0D_3a\mathcal{M}(\delta)^2,$
$L_{3,4} = (B+A)D_3a(1-a^2) - 2(A^2D_1 + B^2D_3a^2)\mathcal{M}(\delta) - (B-A)D_0D_3a\mathcal{M}(\delta)^2,$
$L_{4,1} = D_0D_3a(1-a^2) - 2D_0(4A^3B + (B-A)D_3a^2)\mathcal{M}(\delta) - (B-A)^2D_0D_3a\mathcal{M}(\delta)^2,$
$L_{4,2} = -(B+A)D_3a(1-a^2) + 2(A^2D_1 + B^2D_3a^2)\mathcal{M}(\delta) + (B-A)D_0D_3a\mathcal{M}(\delta)^2,$
$L_{4,3} = D_0D_3(1-a^2) - 2D_0^3a\mathcal{M}(\delta) - (B^2-A^2)D_0D_2\mathcal{M}(\delta)^2,$
$L_{4,4} = -(B+A)D_3(1-a^2) + 2(B+A)D_0^2a\mathcal{M}(\delta) + (B+A)D_0D_2\mathcal{M}(\delta)^2,$

where A and B are defined by (4.2), a, b and D_0 are defined by (4.10), $\mathcal{M}(\delta)$ is given by (4.12) with $x_N = \delta$ and

$$
\begin{aligned}
&D_1 = -B^3 + AB^2 + 3A^2B + A^3, \quad D_2 = B^3 - AB^2 + 3A^2B + A^3,\\
&D_3 = B^3 + AB^2 + 3A^2B - A^3.
\end{aligned} \tag{4.19}
$$

Remark 4.1. *The singularity appearing in the symbol of the solution formula* (4.16) *is higher than that for Neumann-Dirichlet boundary condition. In fact, the third and fourth rows of* $\mathbf{L}$ *coincide if* $\xi' = 0$ *and, thereby,* $\det \mathbf{L} \to 0$ *as* $\xi' \to 0$*. This is because the upper and lower boundary conditions, which cause the third and fourth rows, respectively, are the same. In contrast to that, any two rows of* $\mathbf{L}$ *do not coincide when* $\xi' = 0$ *and* $\det \mathbf{L} \not\to 0$ *as* $\xi' \to 0$ *in the case of Neumann-Dirichlet boundary condition.*

Once we have u_N and θ, we can obtain u_j with $j = 1, \cdots, N-1$ from the j-th component of the first equation and the boundary condition in (3.9), that is, as the solution of the following equation:

$$
\begin{cases}
\lambda u_j - \mu \Delta u_j = -\partial_j \theta & \text{in } \Omega,\\
\partial_N u_j = \mu^{-1}\nu_N h_j - \partial_j u_N & \text{on } \partial\Omega.
\end{cases} \tag{4.20}
$$

We first prove the $\mathscr{R}$-boundedness for $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$ by analyzing solution formulas (4.16) with the aid of lemmas in Section 5. By combining those with (4.20), we then show the $\mathscr{R}$-boundedness for $\mathcal{S}_j(\lambda)$ with $j = 1, \cdots, N-1$, which is performed in Section 6.

5. Technical lemmas

In this section, we introduce some lemmas to establish Theorem 3.1. To this end, first of all, we introduce some classes of symbols. Let $m(\lambda, \xi')$ be a function defined on $\Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$ with $0 < \varepsilon < \pi/2$ and $\gamma_0 \geq 0$. For all $s \in \mathbb{R}$ and $k = 1, 2$, $m(\lambda, \xi')$ is said to be a multiplier of order s with type k if it satisfies the following two conditions:

i) $m(\lambda, \xi')$ is infinitely many times differentiable with respect to ξ' and τ where $\lambda = \gamma + i\tau$.
ii) For any $\ell = 0, 1$, multi-index $\alpha' \in \mathbb{N}_0^{N-1}$ and $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$,
$$\left|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} m(\lambda, \xi')\right| \leq C_{\alpha'} \left(|\lambda|^{1/2} + |\xi'|\right)^{s-|\alpha'|}$$
when $k = 1$, and
$$\left|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} m(\lambda, \xi')\right| \leq C_{\alpha'} \left(|\lambda|^{1/2} + |\xi'|\right)^{s} |\xi'|^{-|\alpha'|}$$

when $k=2$.

We set

$$\mathbb{M}_{s,k,\varepsilon,\gamma_0} = \{m(\lambda,\xi') \mid m(\lambda,\xi') \text{ is a multiplier of order } s \text{ with type } k\}. \tag{5.1}$$

By the definition and the Leibniz rule, we have the following properties (see [35]).

Lemma 5.1. *Let $s, s_1, s_2 \in \mathbb{R}$, $k_1, k_2 = 1, 2$, $0 < \varepsilon < \pi/2$ and $\gamma_0 \geq 0$.*

a) *We have $\mathbb{M}_{s,1,\varepsilon,\gamma_0} \subset \mathbb{M}_{s,2,\varepsilon,\gamma_0}$.*
b) *Given $m_j \in \mathbb{M}_{s_j,k_j,\varepsilon,\gamma_0}$ $(j=1,2)$, we have $m_1 m_2 \in \mathbb{M}_{s_1+s_2,\max\{k_1,k_2\},\varepsilon,\gamma_0}$.*

Let A, B and $\mathcal{M}(x_N)$ be given by (4.2) and (4.12). We set

$$k_i(x_N) = \begin{cases} e^{-Bx_N}, & i=1, \\ e^{-Ax_N}, & i=2, \\ B\mathcal{M}(x_N), & i=3, \end{cases} \qquad (0 < x_N < \delta). \tag{5.2}$$

Lemma 5.2. *Let $0 < \varepsilon < \pi/2$ and $s \in \mathbb{R}$. For $x_N, y_N \in (0,\delta)$, $m \in \mathbb{N}$, $i, i_1, i_2 = 1,2,3$, $\ell = 0,1$, $(\lambda,\xi') \in \Sigma_{\varepsilon,0} \times (\mathbb{R}^{N-1} \setminus \{0\})$ and multi-index $\alpha' \in \mathbb{N}_0^{N-1}$, we have the following estimates:*

$$\begin{aligned} c_{\varepsilon,\mu}(|\lambda|^{1/2} + A) \leq Re\, B \leq |B| \leq C_\mu(|\lambda|^{1/2} + A), \\ c_{\varepsilon,\mu}(|\lambda|^{1/2} + A)^3 \leq |D_3| \leq C_\mu(|\lambda|^{1/2} + A)^3, \end{aligned} \tag{5.3}$$

$$|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} A^s| \leq C_{\alpha',s} A^{s-|\alpha'|}, \quad B^s \in \mathbb{M}_{s,1,\varepsilon,0}, \quad D_3^s \in \mathbb{M}_{3s,2,\varepsilon,0}, \tag{5.4}$$

$$\begin{aligned} &|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} k_1(x_N)| \leq C_{\alpha',\varepsilon,\mu}(|\lambda|^{1/2} + A)^{-|\alpha'|} e^{-c_{\varepsilon,\mu}(|\lambda|^{1/2}+A)x_N}, \\ &|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} k_i(x_N)| \leq C_{\alpha',\varepsilon,\mu} A^{-|\alpha'|} e^{-c_{\varepsilon,\mu}Ax_N}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} &|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} k_1(x_N)k_1(y_N)| \leq C_{\alpha',\varepsilon,\mu}(|\lambda|^{1/2} + A)^{-|\alpha'|} e^{-c_{\varepsilon,\mu}(|\lambda|^{1/2}+A)(x_N+y_N)}, \\ &|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} k_{i_1}(x_N)k_{i_2}(y_N)| \leq C_{\alpha',\varepsilon,\mu} A^{-|\alpha'|} e^{-c_{\varepsilon,\mu}A(x_N+y_N)}. \end{aligned} \tag{5.6}$$

Here, $D_3 = B^3 + AB^2 + 3A^2B - A^3$, which is defined in (4.19).

Proof. The estimates (5.3), (5.4) and (5.5) are proved in [32, Lemma 4.4], [35, Lemma 5,2], [35, Lemma 5,3], respectively. The inequality (5.6) is obtained by (5.5) and the Leibniz rule as in [25, Lemma 5.2]. □

In view of Lemma 3.1 ii), we prove the $\mathscr{R}$-boundedness for each term of the solution formula (4.16). Let φ_0 and φ_δ be given by (3.1), and define

$$\Phi_i(y_N) = \begin{cases} \varphi_\delta(y_N), & i=1, \\ \varphi_0(y_N), & i=2, \\ \varphi_0'(y_N) = -\varphi_\delta'(y_N), & i=3. \end{cases} \tag{5.7}$$

In the same fashion as in [25], we have the following lemma. The symbols of the operators and the range of γ_0 are slightly generalized as compared with those in [25].

Lemma 5.3 ([25, Lemma 5.3])**.** *Let $0 < \varepsilon < \pi/2$, $\gamma_0 \geq 0$ and $m_k \in \mathbb{M}_{0,k,\varepsilon,\gamma_0}$ $(k = 1,2)$. We define the operators*

$$[K_1(\lambda)h](x) = \int_0^\delta \mathscr{F}_{\xi'}^{-1}\left[\Phi_i(y_N) m_1(\lambda,\xi') B k_1(d_{\ell_1}(x_N)) k_1(d_{\ell_2}(y_N)) \widehat{h}(\xi', y_N)\right](x')\, dy_N,$$

(5.8)

$$[K_2(\lambda)h](x) = \int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\Phi_i(y_N)m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N))\widehat{h}(\xi',y_N)\right](x')\,dy_N$$

for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $i, i_1, i_2 = 1,2,3$ *and* $\ell_1, \ell_2 = 1,2$, *where* $d_\ell(x_N)$ *is given by* (4.7). *Then, for any* $1 < q < \infty$ *and* $\ell = 0,1$, *there hold*

$$\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell K_1(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \le C_{N,q,\varepsilon,\gamma_0,m_1},$$

$$\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell K_2(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \le C_{N,q,\varepsilon,\gamma_0,m_2}.$$

Here, $\Sigma_{\varepsilon,\gamma_0}$ *is given by* (1.3) *and* $\lambda = \gamma + i\tau$.

In addition, by the argument in [25, Lemma 5.5], we have the lemma to treat operator families with higher singularity at the origin in the Fourier side. As in Lemma 5.3, the symbol of the operator and the range of γ_0 can be slightly generalized.

Lemma 5.4 ([25, Lemma 5.5])**.** *Let* $0 < \varepsilon < \pi/2$ *and* $\gamma_0 \ge 0$. *Recall that* $\mathbb{M}_{s,k,\varepsilon,\gamma_0}$, $\Phi_i(y_N)$, $k_i(x_N)$, $d_\ell(x_N)$ *and* $\Sigma_{\varepsilon,\gamma_0}$ *are given by* (5.1), (5.7), (5.2), (4.7) *and* (1.3), *respectively, and write* $\lambda = \gamma + i\tau$ *for* $\lambda \in \Sigma_{\varepsilon,\gamma_0}$. *Assume that* $m_2 \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}$. *We define the operator*

(5.9)

$$[K_3(\lambda)h](x) = \int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\Phi_i(y_N)m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N))\widehat{h}(\xi',y_N)\right](x')\,dy_N$$

for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $i, i_1, i_2 = 1,2,3$ *and* $\ell_1, \ell_2 = 1,2$. *Then, for any* $1 < q < \infty$ *and* $\ell = 0,1$, *we have*

$$\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell K_3(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \le C_{N,q,\varepsilon,\gamma_0,\delta,m_2}.$$

Remark 5.1. *As mentioned in Remark 4.1, our symbol possesses higher singularity at* $\xi' = 0$ *than the case of Neuman-Dirichlet boundary condition. Following* [35], *which studies the case of the half space, as well as* [25, Lemma 5.3], *we recall the idea of the proof of Lemma 5.3 (lower singularity case). We rewrite* $K_j(\lambda)$ $(j = 1,2)$ *as*

$$K_j(\lambda)h = \int_\Omega k^j_\lambda(x'-y',x_N,y_N)h(y',y_N)\,dy$$

where

$$k^j_\lambda(z',x_N,y_N) = \begin{cases} \mathscr{F}^{-1}_{\xi'}\left[\Phi_i(y_N)m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))k_1(d_{\ell_2}(y_N))\right](z') & (j=1),\\ \mathscr{F}^{-1}_{\xi'}\left[\Phi_i(y_N)m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))k_{i_2}(d_{\ell_2}(y_N))\right](z') & (j=2).\end{cases}$$

Then we have

$$|k^j_\lambda(z',x_N,y_N)| \le C|(z',x_N,y_N)|^{-N},$$

which implies desired estimate, see the proof of [25, Lemma 5.3] *for details. In contrast to that, for the higher singularity case, we only know that the kernel* $k^3_\lambda(z',x_N,y_N)$ *of the operator* $K_3(\lambda)$, *which is defined similarly, decays with slower rate*

$$|k^3_\lambda(z',x_N,y_N)| \le C|(z',x_N,y_N)|^{-(N-1)}.$$

However, we can estimate the operator families $\{K_3(\lambda)\}_\lambda$ *by applying Lemma 5.5 below only to the tangential direction; the lemma is proved by* [19, Theorem 3.2], *see also* [18, Theorem 3.3]. *This is performed in the proof of Lemma 5.4.*

Lemma 5.5. *Let Λ be a set and $\{m_\lambda\}_{\lambda\in\Lambda} \subset C^\infty(\mathbb{R}^d \setminus \{0\})$ satisfy*

$$|\partial_\xi^\alpha m_\lambda(\xi)| \le C_\alpha |\xi|^{-|\alpha|}$$

for any $\lambda \in \Lambda$, $\xi \in \mathbb{R}^d \setminus \{0\}$ and multi-index $\alpha \in \mathbb{N}_0^d$. Then, for $q \in (1,\infty)$, the operator family $\{T_\lambda \mid \lambda \in \Lambda\}$ given by $T_\lambda f = \mathscr{F}_\xi^{-1} m_\lambda \mathscr{F}_x f$ satisfies

$$\mathscr{R}_{\mathcal{L}(L_q(\mathbb{R}^d))}(\{T_\lambda \mid \lambda \in \Lambda\}) \le C_{N,q} \max_{|\alpha|\le N+2} C_\alpha.$$

Proof of Lemma 5.4. Following Saito [25, Lemmma 5.5], we prove the estimate (2.2) in the definition of the $\mathscr{R}$-boundedness with $p = q$. Let $m \in \mathbb{N}$, $\{\lambda_j\}_{j=1}^m \subset \Sigma_{\varepsilon,\gamma_0}$ and $\{h_j\}_{j=1}^m \subset L_q(\Omega)$, and let $\{r_j\}_{j=1}^m$ be a sequence of independent, symmetric and $\{\pm 1\}$-valued random variables on $(0,1)$. Define the operator $K_{30}(\lambda; x_N, y_N)$ for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$ and $x_N, y_N \in (0,\delta)$ by

$$K_{30}(\lambda; x_N, y_N)h_0(x') = \mathscr{F}_{\xi'}^{-1}\left[\Phi_i(y_N) m_2(\lambda,\xi') k_{i_1}(d_{\ell_1}(x_N)) k_{i_2}(d_{\ell_2}(y_N)) \widehat{h_0}(\xi')\right](x')$$

so that

$$[K_3(\lambda)h](x) = \int_0^\delta K_{30}(\lambda; x_N, y_N)[h(\cdot, y_N)](x')\, dy_N.$$

Note $\Phi_i(y_N)$ is constant with respect to (λ,ξ') and bounded with respect to y_N, and so, by (5.6) and Lemma 5.1,

$$|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} (\Phi_i(y_N) m_2(\lambda,\xi') k_{i_1}(d_{\ell_1}(x_N)) k_{i_2}(d_{\ell_2}(y_N)))| \le C_{N,q,\varepsilon,\gamma_0} |\xi'|^{-|\alpha'|}$$

for $(\lambda,\xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$. Therefore, by Lemma 5.5 and the Hölder inequality, we have

$$\begin{aligned}
&\int_0^1 \left\| \sum_{j=1}^m r_j(u) K_3(\lambda_j) h_j \right\|_{L_q(\Omega)}^q du \\
&= \int_0^1 \int_0^\delta \int_{\mathbb{R}^{N-1}} \left| \int_0^\delta \sum_{j=1}^m r_j(u) K_{30}(\lambda_j; x_N, y_N) h_j(\cdot, y_N)\, dy_N \right|^q dx'\, dx_N\, du \\
&\le \delta^{q-1} \int_0^1 \int_0^\delta \int_{\mathbb{R}^{N-1}} \int_0^\delta \left| \sum_{j=1}^m r_j(u) K_{30}(\lambda_j; x_N, y_N) h_j(\cdot, y_N) \right|^q dy_N\, dx'\, dx_N\, du \\
&\le \delta^{q-1} \int_0^1 \int_0^\delta \int_0^\delta \left\| \sum_{j=1}^m r_j(u) K_{30}(\lambda_j; x_N, y_N) h_j(\cdot, y_N) \right\|_{L_q(\mathbb{R}^{N-1})}^q dy_N\, dx_N\, du \\
&\le C_{N,q,\varepsilon,\gamma_0} \delta^{q-1} \int_0^1 \int_0^\delta \int_0^\delta \left\| \sum_{j=1}^m r_j(u) h_j(\cdot, y_N) \right\|_{L_q(\mathbb{R}^{N-1})}^q dy_N\, dx_N\, du \\
&= C_{N,q,\varepsilon,\gamma_0} \delta^q \int_0^1 \left\| \sum_{j=1}^m r_j(u) h_j \right\|_{L_q(\Omega)}^q du,
\end{aligned}$$

which completes the proof. □

By the lemmas above, the same argument as in [25, Lemma 5.4] implies the following lemma.

Lemma 5.6. *Let $0 < \varepsilon < \pi/2$, $\gamma_0 \geq 0$ and $m_k \in \mathbb{M}_{-2,k,\varepsilon,\gamma_0}$, and let $K_j(\lambda)$ ($j = 1, 2, 3$) be the operators given in Lemmas 5.3 and 5.4. For any $1 < q < \infty$, $j = 1, 2, 3$, $\ell = 0, 1$ and $1 \leq m, n \leq N$, there hold*

$$
\begin{aligned}
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda K_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},\\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma K_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},\\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2}\partial_m K_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},\\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\partial_n K_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},
\end{aligned}
$$

where $k = 1$ when $j = 1$ and $k = 2$ when $j = 2, 3$. If $i_1 = 2$ (that is, $k_{i_1}(d_{\ell_1}(x_N)) = e^{-Ad_{\ell_1}(x_N)}$) in (5.8) and (5.9), and if $m_2 \in \mathbb{M}_{-2,2,\varepsilon,\gamma_0}$ is replaced by

$$
m_2 \in A^{-1}\mathbb{M}_{0,2,\varepsilon,\gamma_0} = \{A^{-1}m(\lambda,\xi') \mid m(\lambda,\xi') \in \mathbb{M}_{0,2,\varepsilon,\gamma_0}\}, \tag{5.10}
$$

then, for $1 < q < \infty$, $j = 2, 3$, $\ell = 0, 1$ and $1 \leq m \leq N$, we have

$$
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m K_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_2}.
$$

Now, we state the lemma which plays a crucial role in showing Theorem 3.1 from the estimate of the symbol.

Lemma 5.7. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$. Recall that $\mathbb{M}_{s,k,\varepsilon,\gamma_0}$, $A^{-1}\mathbb{M}_{0,2,\varepsilon,\gamma_0}$, A, $d_\ell(x_N)$, $k_i(x_N)$ and $\Sigma_{\varepsilon,\gamma_0}$ are given by (5.1), (5.10), (4.2), (4.7), (5.2) and (1.3), respectively, and write $\lambda = \gamma + i\tau$ for $\lambda \in \Sigma_{\varepsilon,\gamma_0}$. Assume that $m_k \in \mathbb{M}_{-2,k,\varepsilon,\gamma_0}$ ($k = 1, 2$). For all $j = 1, 2, 3$, $\lambda \in \Sigma_{\varepsilon,\gamma_0}$, $i_1 = 1, 2, 3$ and $\ell_1, \ell_2 = 1, 2$, there exist operators*

$$
\mathcal{K}_1(\lambda) = \mathcal{K}^1_{1,\ell_1,\ell_2}(\lambda, m_1), \quad \mathcal{K}_2(\lambda) = \mathcal{K}^{i_1}_{2,\ell_1,\ell_2}(\lambda, m_2), \quad \mathcal{K}_3(\lambda) = \mathcal{K}^{i_1}_{3,\ell_1,\ell_2}(\lambda, m_2)
$$

(precisely, they are given by (5.14), (5.17) and (5.18), respectively) satisfying $\mathcal{K}_j(\lambda) \in \mathcal{L}(L_q(\Omega)^{1+N}, W^2_q(\Omega))$ for $1 < q < \infty$ such that the following two assertions hold:

a) *Given $h \in C^\infty_0(\mathbb{R}^N)$, we have the formula*

$$
[\mathcal{K}_1(\lambda)(\lambda^{1/2}h, \nabla h)](x) = \mathscr{F}^{-1}_{\xi'}\left[m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))\widehat{h}(\xi', d_{\ell_2}(0))\right](x'), \tag{5.11}
$$

$$
\begin{aligned}
[\mathcal{K}_2(\lambda)(h, \nabla h)](x) &= \mathscr{F}^{-1}_{\xi'}\left[m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))\widehat{h}(\xi', d_{\ell_2}(0))\right](x'),\\
[\mathcal{K}_3(\lambda)(h, \nabla h)](x) &= \mathscr{F}^{-1}_{\xi'}\left[m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))\widehat{h}(\xi', d_{\ell_2}(0))\right](x').
\end{aligned} \tag{5.12}
$$

b) *For any $1 < q < \infty$, $j = 1, 2, 3$, $\ell = 0, 1$ and $1 \leq m, n \leq N$, there hold*

$$
\begin{aligned}
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda \mathcal{K}_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},\\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma \mathcal{K}_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},\\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2}\partial_m \mathcal{K}_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},\\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\partial_n \mathcal{K}_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta,m_k},
\end{aligned}
$$

where $k = 1$ when $j = 1$ and $k = 2$ when $j = 2, 3$.

Furthermore, if $m_2 \in A^{-1}\mathbb{M}_{0,2,\varepsilon,\gamma_0}$ instead of $m_2 \in \mathbb{M}_{-2,2,\varepsilon,\gamma_0}$ and if $i_1 = 2$, then, the operators $\mathcal{K}_2(\lambda) = \mathcal{K}^2_{2,\ell_1,\ell_2}(\lambda, m_2)$ and $\mathcal{K}_3(\lambda) = \mathcal{K}^2_{3,\ell_1,\ell_2}(\lambda, m_2)$ satisfy the following assertions:

a) *Given $h \in C^\infty_0(\mathbb{R}^N)$, we have the formula (5.12).*

b) *For any* $1 < q < \infty$, $j = 2, 3$, $\ell = 0, 1$ *and* $1 \le m \le N$, *there holds*

$$\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m \mathcal{K}_j(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \le C_{N,q,\varepsilon,\gamma_0,\delta,m_2}.$$

Remark 5.2. *It might not look natural to deduce the* $\mathscr{R}$*-boundedness for* (5.12) *in place of*

$$[\mathcal{K}_2(\lambda)(\lambda^{1/2}h, \nabla h)](x) = \mathscr{F}^{-1}_{\xi'}\left[m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))\widehat{h}(\xi', d_{\ell_2}(0))\right](x'),$$
$$[\mathcal{K}_3(\lambda)(\lambda^{1/2}h, \nabla h)](x) = \mathscr{F}^{-1}_{\xi'}\left[m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))\widehat{h}(\xi', d_{\ell_2}(0))\right](x').$$

However, the $\mathscr{R}$*-boundedness for* (5.12) *is needed in order to estimate the pressure term in showing Theorem 2.1 from Theorem 3.1, see Remark 3.1. On the other hand, there is no need to prove the* $\mathscr{R}$*-boundedness for*

$$[\mathcal{K}_1(\lambda)(h, \nabla h)](x) = \mathscr{F}^{-1}_{\xi'}\left[m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))\widehat{h}(\xi', d_{\ell_2}(0))\right](x')$$

since we use the operator $\mathcal{K}_1(\lambda)$ *only in Lemma 6.5 below to estimate the solution* u_j *of* (4.20).

Proof of Lemma 5.7. This lemma is proved by a trick due to Volevich [41] and Lemma 5.6. We write $\varphi_{d_1(0)} = \varphi_\delta$ and $\varphi_{d_2(0)} = \varphi_0$, see (4.7). For $h \in C_0^\infty(\mathbb{R}^N)$, we rewrite the right-hand side of (5.11) as

(5.13)

$$\begin{aligned}
&\mathscr{F}^{-1}_{\xi'}\left[m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))\widehat{h}(\xi', d_{\ell_2}(0))\right] \\
&= -\int_0^\delta \partial_{y_N}\mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{h}(\xi', y_N)\right]dy_N \\
&= -\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi'_{d_{\ell_2}(0)}(y_N)m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{h}(\xi', y_N)\right]dy_N \\
&\quad +(-1)^{\ell_2}\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_1(\lambda,\xi')B^2k_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{h}(\xi', y_N)\right]dy_N \\
&\quad -\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{\partial_N h}(\xi', y_N)\right]dy_N \\
&= -\frac{1}{\gamma_0^{1/2}}\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi'_{d_{\ell_2}(0)}(y_N)\frac{\gamma_0^{1/2}}{\lambda^{1/2}}m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{\lambda^{1/2}h}(\xi', y_N)\right]dy_N \\
&\quad +(-1)^{\ell_2}\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)\frac{\lambda^{1/2}}{\mu B}m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{\lambda^{1/2}h}(\xi', y_N)\right]dy_N \\
&\quad -(-1)^{\ell_2}\sum_{j'=1}^{N-1}\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)\frac{i\xi_{j'}}{B}m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{\partial_{j'}h}(\xi', y_N)\right]dy_N \\
&\quad -\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_1(\lambda,\xi')Bk_1(d_{\ell_1}(x_N))e^{-Bd_{\ell_2}(y_N)}\widehat{\partial_N h}(\xi', y_N)\right]dy_N,
\end{aligned}$$

where we have used $B = B^2/B = \lambda/(\mu B) + \sum_{j'=1}^{N-1}(-i\xi_{j'}/B)i\xi_{j'}$. We thus define $\mathcal{K}_1(\lambda)$ by

$$\mathcal{K}_1(\lambda)(\lambda^{1/2}h, \nabla h) = (\text{the right-hand side of } (5.13)). \tag{5.14}$$

Similarly, by using the relation $A = A^2/A = \sum_{j'=1}^{N-1}(-i\xi_{j'}/A)i\xi_{j'}$, we rewrite the right-hand sides of (5.12) as

(5.15)
$$
\begin{aligned}
&\mathscr{F}^{-1}_{\xi'}\left[m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))\widehat{h}(\xi',d_{\ell_2}(0))\right]\\
&= -\int_0^\delta \partial_{y_N}\mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{h}(\xi',y_N)\right]\,dy_N\\
&= -\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi'_{d_{\ell_2}(0)}(y_N)m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{h}(\xi',y_N)\right]\,dy_N\\
&\quad -(-1)^{\ell_2}\sum_{j'=1}^{N-1}\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)\frac{i\xi_{j'}}{A}m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{\partial_{j'}h}(\xi',y_N)\right]\,dy_N\\
&\quad -\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_2(\lambda,\xi')Ak_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{\partial_N h}(\xi',y_N)\right]\,dy_N,
\end{aligned}
$$

(5.16)
$$
\begin{aligned}
&\mathscr{F}^{-1}_{\xi'}\left[m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))\widehat{h}(\xi',d_{\ell_2}(0))\right]\\
&= -\int_0^\delta \partial_{y_N}\mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{h}(\xi',y_N)\right]\,dy_N\\
&= -\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi'_{d_{\ell_2}(0)}(y_N)m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{h}(\xi',y_N)\right]\,dy_N\\
&\quad -(-1)^{\ell_2}\sum_{j'=1}^{N-1}\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)\frac{i\xi_{j'}}{A}m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{\partial_{j'}h}(\xi',y_N)\right]\,dy_N\\
&\quad -\int_0^\delta \mathscr{F}^{-1}_{\xi'}\left[\varphi_{d_{\ell_2}(0)}(y_N)m_2(\lambda,\xi')k_{i_1}(d_{\ell_1}(x_N))e^{-Ad_{\ell_2}(y_N)}\widehat{\partial_N h}(\xi',y_N)\right]\,dy_N
\end{aligned}
$$

Then we define

$$\mathcal{K}_2(\lambda)(h,\nabla h) = (\text{the right-hand side of (5.15)}), \tag{5.17}$$

$$\mathcal{K}_3(\lambda)(h,\nabla h) = (\text{the right-hand side of (5.16)}). \tag{5.18}$$

If m_1 belongs to $\mathbb{M}_{-2,1,\varepsilon,\gamma_0}$,

$$\frac{\gamma_0^{1/2}}{\lambda^{1/2}}m_1(\lambda,\xi'),\ \ \frac{\lambda^{1/2}}{\mu B}m_1(\lambda,\xi'),\ \ \frac{i\xi_{j'}}{B}m_1(\lambda,\xi')$$

also belong to $\mathbb{M}_{-2,1,\varepsilon,\gamma_0}$ by (5.4) and Lemma 5.1. Similarly, if m_2 is in $\mathbb{M}_{-2,2,\varepsilon,\gamma_0}$ or $A^{-1}\mathbb{M}_{0,2,\varepsilon,\gamma_0}$,

$$\frac{i\xi_{j'}}{A}m_2(\lambda,\xi')$$

is also in $\mathbb{M}_{-2,2,\varepsilon,\gamma_0}$ or $A^{-1}\mathbb{M}_{0,2,\varepsilon,\gamma_0}$, respectively. Thus, the assertion for $K_j(\lambda)$ ($j=1,2,3$) in Lemma 5.6 together with Lemma 3.1 ii) implies the conclusion for $\mathcal{K}_j(\lambda)$. □

6. Proof of Theorem 3.1

In this section, we first prove the $\mathscr{R}$-boundedness of $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$ by deduction of estimates of symbols in the solution formula (4.16). We then discuss the other solution operators $\mathcal{S}_j(\lambda)$ $(j = 1, \cdots, N-1)$ by analyzing the Laplace resolvent problem (4.20).

We begin with analysis of $\det \mathbf{L}$ given by (4.17).

Lemma 6.1. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 > 0$, and let $\alpha' \in \mathbb{N}_0^{N-1}$ be any multi-index. There exists a constant $C_{N,\varepsilon,\gamma_0,\alpha'} > 0$ such that for any $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$ and $\ell = 0, 1$, the following estimate holds:*

$$\left| (\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} \frac{1}{\det \mathbf{L}} \right| \leq C_{N,\varepsilon,\gamma_0,\alpha'} (|\lambda|^{1/2} + A)^{-6} \left(1 + \frac{1}{A} \right) A^{-|\alpha'|}. \tag{6.1}$$

Here, $\Sigma_{\varepsilon,\gamma_0}$ and A are given by (1.3) *and* (4.2)*, respectively, and $\lambda = \gamma + i\tau$.*

Remark 6.1. *We need the condition $\gamma_0 > 0$ to obtain* (6.1) *for $A \leq 1$, see* (6.17)*,* (6.18) *and* (6.21) *below. In fact, if $\lambda = 0$, the singularity is too high at $\xi' = 0$ such as*

$$|\det \mathbf{L}|^{-1} \sim A^{-10}, \quad \forall \xi' \in \mathbb{R}^{N-1} \setminus \{0\} \text{ with } |\xi'| \leq 1$$

(though the proof is omitted), while

$$|\det \mathbf{L}|^{-1} \sim (|\lambda|^{1/2} + A)^{-6} A^{-1}, \quad \forall (\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\}) \text{ with } |\xi'| \leq 1$$

from (6.1)*,* (6.19) *and* (6.20) *((6.19) and (6.20) are valid for $\beta' = 0$). Here, $M_1 \sim M_2$ means that $c_{\varepsilon,\gamma_0} M_2 \leq M_1 \leq C_{\varepsilon,\gamma_0} M_2$ for all λ and ξ' with some constants $c_{\varepsilon,\gamma_0}, C_{\varepsilon,\gamma_0} > 0$ independent of λ and ξ'.*

To prove Lemma 6.1, we first show the following lemma.

Lemma 6.2. *Let $0 < \varepsilon < \pi/2$ and $\gamma_0 \geq 0$. Let m and $f > 0$ be functions defined on $\Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$. Assume that, for any $\ell = 0, 1$ and multi-index $\beta' \in \mathbb{N}_0^{N-1}$ with $|\beta'| \geq 1$, there exists $C_{\beta'} > 0$ such that*

$$|m(\lambda, \xi')| \geq f(\lambda, \xi'), \tag{6.2}$$

$$|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\beta'} m(\lambda, \xi')| \leq C_{\beta'} f(\lambda, \xi') A^{-|\beta'|} \tag{6.3}$$

for $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$. Then, for any $\ell = 0, 1$ and multi-index $\alpha' \in \mathbb{N}_0^{N-1}$, we have the estimate

$$|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} m(\lambda, \xi')^{-1}| \leq C_{\alpha'} f(\lambda, \xi')^{-1} A^{-|\alpha'|} \tag{6.4}$$

for $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$. This assertion still holds if A is replaced by $(|\lambda|^{1/2} + A)$ in (6.3) *and* (6.4)*.*

Proof. We give the proof of (6.4) for $\ell = 0$, since we can show the case $\ell = 1$ and the case where A is replaced by $(|\lambda|^{1/2} + A)$ similarly. By the Faà di Bruno's formula (cf. [13, Lemma 2.3]), for any multi-index $\alpha' \in \mathbb{N}_0^{N-1}$ with $|\alpha'| \geq 1$ and $(\lambda, \xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1} \setminus \{0\})$,

$$\begin{aligned}
\left| \partial_{\xi'}^{\alpha'} \frac{1}{m(\lambda, \xi')} \right| &= \left| \sum_{\ell=1}^{|\alpha'|} \frac{(-1)^\ell \ell!}{m(\lambda, \xi')^{\ell+1}} \sum_{\substack{\alpha_1' + \cdots + \alpha_\ell' = \alpha' \\ |\alpha_i'| \geq 1}} c_{\alpha_1' \cdots \alpha_\ell'} (\partial_{\xi'}^{\alpha_1'} m(\lambda, \xi')) \cdots (\partial_{\xi'}^{\alpha_\ell'} m(\lambda, \xi')) \right| \\
&\leq \sum_{\ell=1}^{|\alpha'|} \frac{C_{\alpha'}}{|m(\lambda, \xi')|} \sum_{\substack{\alpha_1' + \cdots + \alpha_\ell' = \alpha' \\ |\alpha_i'| \geq 1}} \left(\frac{|\partial_{\xi'}^{\alpha_1'} m(\lambda, \xi')|}{|m(\lambda, \xi')|} \right) \cdots \left(\frac{|\partial_{\xi'}^{\alpha_\ell'} m(\lambda, \xi')|}{|m(\lambda, \xi')|} \right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\ell=1}^{|\alpha'|} \frac{C_{\alpha'}}{f(\lambda,\xi')} \sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha' \\ |\alpha'_i|\geq 1}} \left(\frac{f(\lambda,\xi')A^{-|\alpha'_1|}}{f(\lambda,\xi')}\right)\cdots\left(\frac{f(\lambda,\xi')A^{-|\alpha'_\ell|}}{f(\lambda,\xi')}\right) \\
&\leq C_{\alpha'} f(\lambda,\xi')^{-1} A^{-|\alpha'|}.
\end{aligned}
$$

Hence, the proof is complete since the case $\alpha' = 0$ is obvious by (6.2). □

Proof of Lemma 6.1. The proof is divided into three steps:

(i) For any $(\lambda,\xi') \in (\mathbb{C}\setminus(-\infty,0]) \times (\mathbb{R}^{N-1}\setminus\{0\})$, $\det \mathbf{L} \neq 0$.
(ii) For any $0<\varepsilon<\pi/2$ and $\gamma_0>0$, there exists $c_{\varepsilon,\gamma_0}>0$ such that

$$
|\det \mathbf{L}| \geq c_{\varepsilon,\gamma_0}(|\lambda|^{1/2}+A)^6 \min\{1,A\} \ (\forall(\lambda,\xi') \in \Sigma_{\varepsilon,\gamma_0}\times(\mathbb{R}^{N-1}\setminus\{0\})). \tag{6.5}
$$

(iii) Conclusion (6.1).

(i) Following the argument in [1, Lemma 2.2], we first prove $\det \mathbf{L} \neq 0$ under the assumption $\operatorname{Im}\lambda \neq 0$. We argue by contradiction. Assume that $\det \mathbf{L} = 0$ with some $(\lambda,\xi') \in (\mathbb{C}\setminus\mathbb{R}) \times (\mathbb{R}^{N-1}\setminus\{0\})$. Then there exists

$$
\mathbf{x} := (\mu_{1N},\mu_{2N},\beta_{1N},\beta_{2N})^{\mathsf{T}} \neq 0 \tag{6.6}
$$

such that $\mathbf{L}\mathbf{x}=0$. We define

$$
\widehat{u}_d(x_N) = \widehat{u}_d(\xi',x_N) \text{ in (4.11)}, \quad \widehat{u}_N(x_N) = \widehat{u}_N(\xi',x_N) \text{ in (4.11)}, \\
\widehat{\theta}(x_N) = \widehat{\theta}(\xi',x_N) \text{ in (4.11)}
$$

with coefficients

$$
\mu_{\ell d} = (-1)^\ell A\mu_{\ell N}, \quad \beta_{\ell d} = (-1)^\ell(\mu_{\ell N}+B\beta_{\ell N}), \quad \gamma_\ell = \frac{\mu(B+A)}{A}\mu_{\ell N},
$$

which can be deduced by (4.8) and (4.13). Note that $\widehat{u}_d, \widehat{u}_N, \widehat{\theta} \in C^\infty([0,\delta])$ by the definition (4.11). Then they obey (4.3) with zero data, that is,

$$
\begin{cases}
\mu(B^2-\partial_N^2)\widehat{u}_d(x_N) - A^2\widehat{\theta}(x_N) = 0 & 0\leq x_N\leq\delta, \\
\mu(B^2-\partial_N^2)\widehat{u}_N(x_N) + \partial_N\widehat{\theta}(x_N) = 0 & 0\leq x_N\leq\delta, \\
\widehat{u}_d(x_N) + \partial_N\widehat{u}_N(x_N) = 0 & 0\leq x_N\leq\delta, \\
\mu(\partial_N\widehat{u}_d(x_N) - A^2\widehat{u}_N(x_N))\nu_N(x_N) = 0 & x_N\in\{0,\delta\}, \\
(2\mu\partial_N\widehat{u}_N(x_N) - \widehat{\theta}(x_N))\nu_N(x_N) = 0 & x_N\in\{0,\delta\}
\end{cases} \tag{6.7}
$$

and satisfy

$$
(\partial_N^2 - A^2)\widehat{\theta}(x_N) = 0 \quad (0\leq x_N\leq\delta), \tag{6.8}
$$

$$
(A^2-\partial_N^2)(B^2-\partial_N^2)\widehat{u}_N(x_N) = 0 \quad (0\leq x_N\leq\delta) \tag{6.9}
$$

as in (4.4) and (4.5). By inserting the third equation to the fourth one, we also get

$$
(\partial_N^2 + A^2)\widehat{u}_N(x_N) = 0 \quad (x_N\in\{0,\delta\}). \tag{6.10}
$$

Multiplying (6.9) by $\overline{\widehat{u}_N(x_N)}$, integrating the resultant formula over $(0,\delta)$, and integration by parts yield

(6.11)
$$\begin{aligned}
&\big(\partial_N^3\widehat{u}_N(\delta)\overline{\widehat{u}_N(\delta)} - \partial_N^3\widehat{u}_N(0)\overline{\widehat{u}_N(0)}\big)\\
&-\big(\partial_N^2\widehat{u}_N(\delta)\overline{\partial_N\widehat{u}_N(\delta)} - \partial_N^2\widehat{u}_N(0)\overline{\partial_N\widehat{u}_N(0)}\big) + \|\partial_N^2\widehat{u}_N\|_{L_2(0,\delta)}^2\\
&-(B^2+A^2)\big(\partial_N\widehat{u}_N(\delta)\overline{\widehat{u}_N(\delta)} - \partial_N\widehat{u}_N(0)\overline{\widehat{u}_N(0)}\big) + (B^2+A^2)\|\partial_N\widehat{u}_N\|_{L_2(0,\delta)}^2\\
&+A^2B^2\|\widehat{u}_N\|_{L_2(0,\delta)}^2\\
&=0.
\end{aligned}$$

From the second equation of (6.7) multiplied by ∂_N, from (6.8) and from the fifth equation in (6.7), we get

$$\partial_N^3\widehat{u}_N = B^2\partial_N\widehat{u}_N + \mu^{-1}\partial_N^2\widehat{\theta} = B^2\partial_N\widehat{u}_N + \mu^{-1}A^2\widehat{\theta} = (B^2+2A^2)\partial_N\widehat{u}_N \text{ on } \{0,\delta\}.$$

By this and (6.10), the equation (6.11) implies

$$\begin{aligned}
&2A^2\mathrm{Re}\,\big(\partial_N\widehat{u}_N(\delta)\overline{\widehat{u}_N(\delta)} - \partial_N\widehat{u}_N(0)\overline{\widehat{u}_N(0)}\big)\\
&+\|\partial_N^2\widehat{u}_N\|_{L_2(0,\delta)}^2 + (B^2+A^2)\|\partial_N\widehat{u}_N\|_{L_2(0,\delta)}^2 + A^2B^2\|\widehat{u}_N\|_{L_2(0,\delta)}^2 = 0.
\end{aligned}$$

If we take the imaginary part, we have $\widehat{u}_N = 0$ by $\mathrm{Im}\,\lambda \neq 0$, but this contradicts the assumption (6.6).

Next, we show $\det\mathbf{L} \neq 0$ even if $\lambda > 0$. If we set

$$x = \sqrt{1+\lambda/(\mu A^2)},$$

we have $B = Ax$ and, so, by the definition (4.17) of $\det\mathbf{L}$,

$$(B-A)^2\det\mathbf{L} = A^8\prod_{+,-} f_A^{\pm}(x) \tag{6.12}$$

with

$$f_A^{\pm}(x) = (x^2+1)^2(1\pm e^{-A\delta})(1\mp e^{-A\delta x}) - 4x(1\mp e^{-A\delta})(1\pm e^{-A\delta x}).$$

Also, we rewrite $f_A^-(x)$ as

$$\begin{aligned}
f_A^-(x) &= \big((x^2+1)^2-4x\big)(1-e^{-A\delta(x+1)}) - \big((x^2+1)^2+4x\big)(e^{-A\delta}-e^{-A\delta x})\\
&= (x-1)(x^3+x^2+3x-1)A\delta(x+1)\int_0^1 e^{-A\delta(x+1)t}\,dt\\
&\quad-(x^4+2x^2+4x+1)A\delta(x-1)\int_0^1 e^{-A\delta(1+(x-1)t)}\,dt.
\end{aligned}$$

Then we have $f_A^{\pm}(x) = p_1^{\pm}p_2^{\pm}p_3^{\pm} - q_1^{\pm}q_2^{\pm}q_3^{\pm}$ with $p_1^{\pm} \geq q_1^{\pm} > 0$ and $p_i^{\pm} > q_i^{\pm} > 0$ for $i = 2,3$, where

$$\begin{aligned}
&p_1^+ = (x^2+1)^2, &&p_2^+ = 1+e^{-A\delta}, &&p_3^+ = 1-e^{-A\delta x},\\
&q_1^+ = 4x, &&q_2^+ = 1+e^{-A\delta x}, &&q_3^+ = 1-e^{-A\delta},\\
&p_1^- = (x-1)A\delta, &&p_2^- = (x^3+x^2+3x-1)(x+1), &&p_3^- = \int_0^1 e^{-A\delta(1+x)t}\,dt,\\
&q_1^- = (x-1)A\delta, &&q_2^- = x^4+2x^2+4x+1, &&q_3^- = \int_0^1 e^{-A\delta(1+(x-1)t)}\,dt.
\end{aligned}$$

In fact, to verify $p_3^- > q_3^-$, we observe

$$p_3^- - q_3^- = \int_0^1 (e^{-A\delta(1+x)t} - e^{-A\delta(1+(x-1)t)})\,dt = \int_0^1 \Big(\int_0^1 A\delta(1-2t)e^{-A\delta(s+(1+x-2s)t)}\,ds\Big)\,dt$$

$$= A\delta\int_0^1 e^{-A\delta s}\int_0^1 (1-2t)e^{-\alpha(s)t}\,dt\,ds = A\delta\int_0^1 e^{-A\delta s}\int_0^{1/2}(1-2t)(e^{-\alpha(s)t} - e^{-\alpha(s)(1-t)})\,dt\,ds,$$

where we have set $\alpha(s) = A\delta(1+x-2s) > 0$ for $s \in (0,1)$. Since the integrand of the right-hand side is positive, we get $p_3^- - q_3^- > 0$. Other inequalities are verified easily. Hence, we get $f_A^{\pm}(x) > 0$ for $x > 1$ and $A > 0$, which implies $\det \mathbf{L} \neq 0$ for $(\lambda, \xi') \in (0,\infty) \times (\mathbb{R}^{N-1} \setminus \{0\})$ by (6.12).

(ii) We shall show the estimate (6.5) of $\det \mathbf{L}$ in this step. Set

$$\ell_\pm(A,B) = \frac{1}{(B-A)}\{(B^2+A^2)^2(1 \pm e^{-A\delta})(1 \mp e^{-B\delta}) - 4A^3B(1 \mp e^{-A\delta})(1 \pm e^{-B\delta})\}$$

so that

$$\det \mathbf{L} = \ell_+(A,B)\ell_-(A,B).$$

Then it is sufficient to prove

$$|\ell_+(A,B)| \geq c(|\lambda|^{1/2}+A)^3, \quad |\ell_-(A,B)| \geq c(|\lambda|^{1/2}+A)^3 \min\{1,A\}. \tag{6.13}$$

We first show (6.13) for $(\lambda,\xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1}\setminus\{0\})$ such that $|\xi'| > M$ for some $M > 0$. We rewrite $\ell_\pm(A,B)$ as

(6.14)

$$\begin{aligned}
&\ell_\pm(A,B)\\
&= \frac{1}{(B-A)}\Big\{\big((B^2+A^2)^2 - 4A^3B\big)(1-e^{-(B+A)\delta}) \mp \big((B^2+A^2)^2+4A^3B\big)(e^{-B\delta} - e^{-A\delta})\Big\}\\
&= \frac{(B-A)(B^3+AB^2+3A^2B-A^3)}{(B-A)}(1-e^{-(B+A)\delta}) \mp \big((B^2+A^2)^2+4A^3B\big)\frac{e^{-B\delta}-e^{-A\delta}}{B-A}\\
&= D_3(1-e^{-(B+A)\delta}) \mp (B^4+2A^2B^2+4A^3B+A^4)\mathcal{M}(\delta),
\end{aligned}$$

where D_3 and $\mathcal{M}(\delta)$ is given by (4.19) and (4.12) with $x_N = \delta$. Thus, by (5.3), (5.5) with $\ell, \alpha' = 0$ and

$$|e^{-B\delta}| = e^{-\mathrm{Re}\,B\delta} \leq e^{-c_{\varepsilon,\mu}(|\lambda|^{1/2}+A)\delta} \leq 1,$$

we have

$$\begin{aligned}
|\ell_\pm(A,B)| &\geq |D_3| - |D_3 e^{-B\delta}e^{-A\delta}| - |B^{-1}(B^4+2A^2B^2+4A^3B+A^4)B\mathcal{M}(\delta)|\\
&\geq 2c(|\lambda|^{1/2}+A)^3 - C_1(|\lambda|^{1/2}+A)^3e^{-M\delta} - C_2(|\lambda|^{1/2}+A)^3e^{-M\delta}\\
&\geq c(|\lambda|^{1/2}+A)^3
\end{aligned}$$

for $(\lambda,\xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1}\setminus\{0\})$ with $A > M$, by taking $M > 0$ so large that $(C_1+C_2)e^{-M\delta} \leq c$.

Next, we consider the case where $(\lambda,\xi') \in \Sigma_{\varepsilon,\gamma_0} \times (\mathbb{R}^{N-1}\setminus\{0\})$ such that $|\xi'|/|\lambda|^{1/2} < \eta$ for some $\eta > 0$. Define D and $n_\pm(D,A,B)$ by

$$D = \frac{A}{B}$$

(6.15)

$$\begin{aligned}\ell_\pm(A,B) &= \frac{B^3}{1-D}\{(1+2D^2+D^4)(1\pm e^{-A\delta})(1\mp e^{-B\delta}) - 4D^3(1\mp e^{-A\delta})(1\pm e^{-B\delta})\} \\ &=: \frac{B^3}{1-D} n_\pm(D,A,B).\end{aligned}$$

Since $|1+e^{-B\delta}|$ is continuous and nonzero for $(\lambda,\xi')\in\Sigma_{\varepsilon,\gamma_0}\times(\mathbb{R}^{N-1}\setminus\{0\})$, it has the minimum value c_0. Then, since

$$1-e^{-A\delta}\begin{cases}\geq 1-e^{-\delta} & \text{for } A\geq 1,\\ = A\delta\displaystyle\int_0^1 e^{-A\delta t}\,dt \geq A\delta e^{-\delta} & \text{for } 0<A\leq 1,\end{cases}$$

$$|D| \leq \frac{A}{c_{\varepsilon,\mu}(|\lambda|^{1/2}+A)} \leq \begin{cases}\eta/c_{\varepsilon,\mu}\leq 1/2,\\ A/(c_{\varepsilon,\mu}\gamma_0^{1/2})\end{cases} \tag{6.16}$$

(by taking η so small if necessary), we get $1-e^{-A\delta}\geq 2c\min\{1,A\}$ for some $c>0$ and $|D|\leq C_{\gamma_0}\min\{1,A\}$ for some $C_{\gamma_0}>0$. Then we obtain

(6.17)

$$\begin{aligned}&|n_-(D,A,B)| \\ &\geq |(1-e^{-A\delta})(1+e^{-B\delta})| - |(2D^2+D^4)(1-e^{-A\delta})(1+e^{-B\delta})| - 4|D^3(1+e^{-A\delta})(1-e^{-B\delta})| \\ &\geq 2cc_0\min\{1,A\} - C_1C_{\gamma_0}(\eta/c_{\varepsilon,\mu})\min\{1,A\} - C_2C_{\gamma_0}(\eta/c_{\varepsilon,\mu})^2\min\{1,A\} \\ &\geq cc_0\min\{1,A\}\end{aligned}$$

if we take $\eta>0$ so small that $C_1C_{\gamma_0}(\eta/c_{\varepsilon,\mu})+C_2C_{\gamma_0}(\eta/c_{\varepsilon,\mu})^2\leq cc_0$. Moreover, by (6.16) and

$$|e^{-B\delta}|\leq e^{-c(|\lambda|^{1/2}+A)\delta}\leq e^{-c\gamma_0^{1/2}\delta},$$

we have

(6.18)

$$\begin{aligned}&|n_+(D,A,B)| \\ &\geq |(1+e^{-A\delta})(1-e^{-B\delta})| - |(2D^2+D^4)(1+e^{-A\delta})(1-e^{-B\delta})| - 4|D^3(1-e^{-A\delta})(1+e^{-B\delta})| \\ &\geq (1-e^{-c\gamma_0^{1/2}\delta}) - 12(\eta/c_{\varepsilon,\mu})^2 - 8(\eta/c_{\varepsilon,\mu})^3 \\ &\geq (1-e^{-c\gamma_0^{1/2}\delta})/2\end{aligned}$$

if we take $\eta>0$ so small that $12(\eta/c_{\varepsilon,\mu})^2+8(\eta/c_{\varepsilon,\mu})^3\leq(1-e^{-c\gamma_0^{1/2}\delta})/2$. Thus, by (5.3) and (6.15)–(6.18), we obtain (6.13) in this case.

Finally, the estimate (6.13) also holds on the remainder region

$$D_r=\{(\lambda,\xi')\in\overline{\Sigma_{\varepsilon,\gamma_0}}\times(\mathbb{R}^{N-1}\setminus\{0\}) \mid |\xi'|\leq M,\ |\xi'|/|\lambda|^{1/2}\geq\eta\}$$

by step (i) since $\det\mathbf{L}$ is a continuous function of (λ,ξ') on D_r and D_r is compact.

(iii) By Lemma 6.2 and step (ii), we will obtain (6.1) if we prove (6.3) with $f(\lambda,\xi')=c_{\varepsilon,\gamma_0}(|\lambda|^{1/2}+A)^6\min\{1,A\}$. Thus, by the Leibniz rule, it suffices to show

$$|(\tau\partial_\tau)^\ell\partial_{\xi'}^{\beta'}\ell_+(A,B)|\leq C_{\varepsilon,\beta'}(|\lambda|^{1/2}+A)^3A^{-|\beta'|}, \tag{6.19}$$

$$|(\tau\partial_\tau)^\ell\partial_{\xi'}^{\beta'}\ell_-(A,B)|\leq C_{\varepsilon,\gamma_0,\beta'}(|\lambda|^{1/2}+A)^3\min\{1,A\}A^{-|\beta'|} \tag{6.20}$$

for $(\lambda,\xi')\in\Sigma_{\varepsilon,\gamma_0}\times(\mathbb{R}^{N-1}\setminus\{0\})$ and multi-index $\beta'\in\mathbb{N}_0^{N-1}$ with $|\beta'|\le 1$. The estimate (6.20) for $A\ge 1$ and (6.19) are obtained by (6.14) and Lemma 5.2. We shall show (6.20) for $A\le 1$. By (6.14),

$$\begin{aligned}\ell_-(A,B)&=(B^3+AB^2+3A^2B-A^3)(1-e^{-(B+A)\delta})\\&\quad+(B^3(B-A)+(AB^3+2A^2B^2+4A^3B+A^4))\mathcal{M}(\delta)\\&=B^3((1-e^{-(B+A)\delta})+(e^{-B\delta}-e^{-A\delta}))\\&\quad+A\{(B^2+3AB-A^2)(1-e^{-(B+A)\delta})+(B^3+2AB^2+4A^2B+A^3)\mathcal{M}(\delta)\}.\end{aligned}$$

Then, if we rewrite the first term of the right-hand side as

$$\begin{aligned}&B^3((1-e^{-(B+A)\delta})+(e^{-B\delta}-e^{-A\delta}))\\&=B^3(1-e^{-A\delta})(1+e^{-B\delta})=B^3A\delta\int_0^1e^{-A\delta t}\,dt(1+e^{-B\delta}),\end{aligned}$$

by Lemma 5.2, we get

(6.21)

$$\begin{aligned}&|(\tau\partial_\tau)^\ell\partial_{\xi'}^{\beta'}\ell_-(A,B)|\\&\le\int_0^1\left|(\tau\partial_\tau)^\ell\partial_{\xi'}^{\beta'}\left(B^3A\delta e^{-A\delta t}(1+e^{-B\delta})\right)\right|dt\\&+\left|(\tau\partial_\tau)^\ell\partial_{\xi'}^{\beta'}\left(A\{(B^2+3AB-A^2)(1-e^{-(B+A)\delta})+(B^3+2AB^2+4A^2B+A^3)\mathcal{M}(\delta)\}\right)\right|\\&\le C_{\beta',\varepsilon}\{(|\lambda|^{1/2}+A)^3A\delta+A(|\lambda|^{1/2}+A)^2\}A^{-|\beta'|}\\&\le C_{\beta',\varepsilon}(\delta+\gamma_0^{-1/2})(|\lambda|^{1/2}+A)^3A^{1-|\beta'|}.\end{aligned}$$

This proves (6.20) for $A\le 1$ and, thus, the proof is complete. □

From now on, in order to prove the assertions for $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$ in Theorem 3.1, we rewrite the solution formula (4.16) of u_N and θ. We will construct $\mathcal{S}_j(\lambda)$ $(j=1,\cdots,N-1)$ together with the $\mathscr{R}$-boundedness at the end of this section. As one can see from (6.1), the estimate of $\det\mathbf{L}$ is inhomogeneous in the sense that

$$|\det\mathbf{L}|^{-1}=\begin{cases}O(|\xi'|^{-1}), & \text{as } \xi'\to 0,\\ O(1), & \text{as } |\xi'|\to\infty\end{cases}$$

for fixed $\lambda\in\Sigma_{\varepsilon,\gamma_0}$. In order to overcome this difficulty, we divide each term of the solution formula into two parts: the part with the same singularity as that for the case of the Neumann-Dirichlet boundary condition and the one with higher singularity. Let $\zeta_0,\zeta_1\in C^\infty(\mathbb{R}^{N-1})$ be cut-off functions such that

$$0\le\zeta_0(\xi')\le 1,\quad \zeta_0(\xi')=\begin{cases}1, & |\xi'|\ge 2,\\ 0, & |\xi'|\le 1,\end{cases}\qquad \zeta_1(\xi')=A(1-\zeta_0(\xi'))$$

so that $1=\zeta_0(\xi')+\zeta_1(\xi')/A$. By using this, we rewrite (4.16):

(6.22)

$$u_N=\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\mu B\det\mathbf{L}}A+\zeta_1\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\mu B\det\mathbf{L}}\right\}B\mathcal{M}(d_\ell(x_N))\widehat{h}_j(\xi',\delta)\right]$$

$$
\begin{aligned}
&+\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{i\xi_j}{A}\frac{L_{1,2\ell}}{\mu\det\mathbf{L}}A+\zeta_1\frac{i\xi_j}{A}\frac{L_{1,2\ell}}{\mu\det\mathbf{L}}\right\}e^{-Bd_\ell(x_N)}\widehat{h}_j(\xi',\delta)\right]\\
&-\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\mu B\det\mathbf{L}}A+\zeta_1\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\mu B\det\mathbf{L}}\right\}B\mathcal{M}(d_\ell(x_N))\widehat{h}_j(\xi',0)\right]\\
&-\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{i\xi_j}{A}\frac{L_{2,2\ell}}{\mu\det\mathbf{L}}A+\zeta_1\frac{i\xi_j}{A}\frac{L_{2,2\ell}}{\mu\det\mathbf{L}}\right\}e^{-Bd_\ell(x_N)}\widehat{h}_j(\xi',0)\right]\\
&+\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{L_{3,2\ell-1}}{\mu B\det\mathbf{L}}A+\zeta_1\frac{L_{3,2\ell-1}}{\mu B\det\mathbf{L}}\right\}B\mathcal{M}(d_\ell(x_N))\widehat{h}_N(\xi',\delta)\right]\\
&+\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{L_{3,2\ell}}{\mu\det\mathbf{L}}A+\zeta_1\frac{L_{3,2\ell}}{\mu\det\mathbf{L}}\right\}e^{-Bd_\ell(x_N)}\widehat{h}_N(\xi',\delta)\right]\\
&-\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{L_{4,2\ell-1}}{\mu B\det\mathbf{L}}A+\zeta_1\frac{L_{4,2\ell-1}}{\mu B\det\mathbf{L}}\right\}B\mathcal{M}(d_\ell(x_N))\widehat{h}_N(\xi',0)\right]\\
&-\sum_{\ell=1}^{2}\mathscr{F}_{\xi'}^{-1}\left[\left\{\zeta_0\frac{L_{4,2\ell}}{\mu\det\mathbf{L}}A+\zeta_1\frac{L_{4,2\ell}}{\mu\det\mathbf{L}}\right\}e^{-Bd_\ell(x_N)}\widehat{h}_N(\xi',0)\right]\\
&=\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\Bigg\{\mathcal{K}^3_{2,\ell,1}\left(\lambda,\zeta_0\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\mu B\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\\
&\qquad\qquad+\mathcal{K}^3_{3,\ell,1}\left(\lambda,\zeta_1\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\mu B\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\Bigg\}\\
&+\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\Bigg\{\mathcal{K}^1_{2,\ell,1}\left(\lambda,\zeta_0\frac{i\xi_j}{A}\frac{L_{1,2\ell}}{\mu\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\\
&\qquad\qquad+\mathcal{K}^1_{3,\ell,1}\left(\lambda,\zeta_1\frac{i\xi_j}{A}\frac{L_{1,2\ell}}{\mu\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\Bigg\}\\
&-\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\Bigg\{\mathcal{K}^3_{2,\ell,2}\left(\lambda,\zeta_0\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\mu B\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\\
&\qquad\qquad+\mathcal{K}^3_{3,\ell,2}\left(\lambda,\zeta_1\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\mu B\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\Bigg\}\\
&-\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\Bigg\{\mathcal{K}^1_{2,\ell,2}\left(\lambda,\zeta_0\frac{i\xi_j}{A}\frac{L_{2,2\ell}}{\mu\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\\
&\qquad\qquad+\mathcal{K}^1_{3,\ell,2}\left(\lambda,\zeta_1\frac{i\xi_j}{A}\frac{L_{2,2\ell}}{\mu\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\Bigg\}\\
&+\sum_{\ell=1}^{2}\left\{\mathcal{K}^3_{2,\ell,1}\left(\lambda,\zeta_0\frac{L_{3,2\ell-1}}{\mu B\det\mathbf{L}}\right)(h_N,\nabla h_N)+\mathcal{K}^3_{3,\ell,1}\left(\lambda,\zeta_1\frac{L_{3,2\ell-1}}{\mu B\det\mathbf{L}}\right)(h_N,\nabla h_N)\right\}\\
&+\sum_{\ell=1}^{2}\left\{\mathcal{K}^1_{2,\ell,1}\left(\lambda,\zeta_0\frac{L_{3,2\ell}}{\mu\det\mathbf{L}}\right)(h_N,\nabla h_N)+\mathcal{K}^1_{3,\ell,1}\left(\lambda,\zeta_1\frac{L_{3,2\ell}}{\mu\det\mathbf{L}}\right)(h_N,\nabla h_N)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{\ell=1}^{2}\left\{\mathcal{K}^3_{2,\ell,2}\left(\lambda,\zeta_0\frac{L_{4,2\ell-1}}{\mu B\det\mathbf{L}}\right)(h_N,\nabla h_N)+\mathcal{K}^3_{3,\ell,2}\left(\lambda,\zeta_1\frac{L_{4,2\ell-1}}{\mu B\det\mathbf{L}}\right)(h_N,\nabla h_N)\right\}\\
&-\sum_{\ell=1}^{2}\left\{\mathcal{K}^1_{2,\ell,2}\left(\lambda,\zeta_0\frac{L_{4,2\ell}}{\mu\det\mathbf{L}}\right)(h_N,\nabla h_N)+\mathcal{K}^1_{3,\ell,2}\left(\lambda,\zeta_1\frac{L_{4,2\ell}}{\mu\det\mathbf{L}}\right)(h_N,\nabla h_N)\right\},
\end{aligned}
$$

(6.23)

$$
\begin{aligned}
\theta &= \sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\mathscr{F}^{-1}_{\xi'}\left[\left\{\zeta_0\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\det\mathbf{L}}A+\zeta_1\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\det\mathbf{L}}\right\}e^{-Ad_\ell(x_N)}\widehat{h}_j(\xi',\delta)\right]\\
&-\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\mathscr{F}^{-1}_{\xi'}\left[\left\{\zeta_0\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\det\mathbf{L}}A+\zeta_1\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\det\mathbf{L}}\right\}e^{-Ad_\ell(x_N)}\widehat{h}_j(\xi',0)\right]\\
&+\sum_{\ell=1}^{2}\mathscr{F}^{-1}_{\xi'}\left[\left\{\zeta_0\frac{(B+A)}{A}\frac{L_{3,2\ell-1}}{\det\mathbf{L}}A+\zeta_1\frac{(B+A)}{A}\frac{L_{3,2\ell-1}}{\det\mathbf{L}}\right\}e^{-Ad_\ell(x_N)}\widehat{h}_N(\xi',\delta)\right]\\
&-\sum_{\ell=1}^{2}\mathscr{F}^{-1}_{\xi'}\left[\left\{\zeta_0\frac{(B+A)}{A}\frac{L_{4,2\ell-1}}{\det\mathbf{L}}A+\zeta_1\frac{(B+A)}{A}\frac{L_{4,2\ell-1}}{\det\mathbf{L}}\right\}e^{-Ad_\ell(x_N)}\widehat{h}_N(\xi',0)\right]\\
&=\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\left\{\mathcal{K}^2_{2,\ell,1}\left(\lambda,\zeta_0\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\right.\\
&\qquad\qquad\left.+\mathcal{K}^2_{3,\ell,1}\left(\lambda,\zeta_1\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{1,2\ell-1}}{\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\right\}\\
&-\sum_{j=1}^{N-1}\sum_{\ell=1}^{2}\left\{\mathcal{K}^2_{2,\ell,2}\left(\lambda,\zeta_0\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\right.\\
&\qquad\qquad\left.+\mathcal{K}^2_{3,\ell,2}\left(\lambda,\zeta_1\frac{(B+A)}{A}\frac{i\xi_j}{A}\frac{L_{2,2\ell-1}}{\det\mathbf{L}}\right)(M_{\lambda^{-\frac12}}\lambda^{\frac12}h_j,\nabla h_j)\right\}\\
&+\sum_{\ell=1}^{2}\left\{\mathcal{K}^2_{2,\ell,1}\left(\lambda,\zeta_0\frac{(B+A)}{A}\frac{L_{3,2\ell-1}}{\det\mathbf{L}}\right)(h_N,\nabla h_N)+\mathcal{K}^2_{3,\ell,1}\left(\lambda,\zeta_1\frac{(B+A)}{A}\frac{L_{3,2\ell-1}}{\det\mathbf{L}}\right)(h_N,\nabla h_N)\right\}\\
&-\sum_{\ell=1}^{2}\left\{\mathcal{K}^2_{2,\ell,2}\left(\lambda,\zeta_0\frac{(B+A)}{A}\frac{L_{4,2\ell-1}}{\det\mathbf{L}}\right)(h_N,\nabla h_N)+\mathcal{K}^2_{3,\ell,2}\left(\lambda,\zeta_1\frac{(B+A)}{A}\frac{L_{4,2\ell-1}}{\det\mathbf{L}}\right)(h_N,\nabla h_N)\right\},
\end{aligned}
$$

where $\mathcal{K}^{i_1}_{j,\ell_1,\ell_2}$ and $M_{\lambda^{-1/2}}$ are defined in Lemma 5.7 and (3.14), respectively. Then we define the operators $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$ by

$$
\begin{aligned}
\mathcal{S}_N(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h})&=(\text{the right-hand side of (6.22)}),\\
\mathcal{T}(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h})&=(\text{the right-hand side of (6.23)}).
\end{aligned}\tag{6.24}
$$

The following lemma concerns the estimates of the symbols of the solution formula.

Lemma 6.3. *Let $0<\varepsilon<\pi/2$ and $\gamma_0>0$, and also let $\mathbb{M}_{s,k,\varepsilon,\gamma_0}$ and $A^{-1}\mathbb{M}_{0,2,\varepsilon,\gamma_0}$ be given by* (5.1) *and* (5.10)*, respectively.*

a) *We have $\zeta_0(\xi')\in\mathbb{M}_{0,2,\varepsilon,0}$. (And so, $\zeta_1(\xi')/A=1-\zeta_0(\xi')\in\mathbb{M}_{0,2,\varepsilon,0}$.)*
b) *For $j=0,1$, $\zeta_j(\xi')/\det\mathbf{L}\in\mathbb{M}_{-6,2,\varepsilon,\gamma_0}$.*

c) *For* $k = 1, 2, 3, 4$ *and* $\ell = 1, 2$,

$$L_{k,2\ell-1} \in \mathbb{M}_{5,2,\varepsilon,0}, \quad L_{k,2\ell} \in \mathbb{M}_{4,2,\varepsilon,0}.$$

d) *For* $j = 0, 1$, $j' = 1, \cdots, N-1$, $k = 1, 2$, $\tilde{k} = 3, 4$ *and* $\ell = 1, 2$, *there hold*

$$\zeta_j \frac{i\xi_{j'}}{A} \frac{L_{k,2\ell-1}}{\mu B \det \mathbf{L}},\ \zeta_j \frac{i\xi_{j'}}{A} \frac{L_{k,2\ell}}{\mu \det \mathbf{L}},\ \zeta_j \frac{L_{\tilde{k},2\ell-1}}{\mu B \det \mathbf{L}},\ \zeta_j \frac{L_{\tilde{k},2\ell}}{\mu \det \mathbf{L}} \in \mathbb{M}_{-2,2,\varepsilon,\gamma_0},$$

$$\zeta_j \frac{(B+A)}{A} \frac{i\xi_{j'}}{A} \frac{L_{k,2\ell-1}}{\det \mathbf{L}},\ \zeta_j \frac{(B+A)}{A} \frac{L_{\tilde{k},2\ell-1}}{\det \mathbf{L}} \in A^{-1}\mathbb{M}_{0,2,\varepsilon,\gamma_0}.$$

Proof. a) For any multi-index $\alpha' \in \mathbb{N}_0^{N-1}$, if $|\alpha'| \geq 1$, $\partial_{\xi'}^{\alpha'}\zeta_0$ is C^∞ function whose support is in the annulus $\{1 \leq |\xi'| \leq 2\}$. As for $\alpha' = 0$, $\zeta_0(\xi') \in [0, 1]$ for $\xi' \in \mathbb{R}^{N-1}$. These and $(\tau\partial_\tau)\zeta_0(\xi') = 0$ imply

$$|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'}\zeta_0(\xi')| \leq CA^{-|\alpha'|} \quad (\forall \ell = 0, 1,\ \alpha' \in \mathbb{N}_0^{N-1}).$$

b) Note that

$$1 + \frac{1}{A} \leq \begin{cases} 2, & \text{on } \operatorname{supp}\zeta_0 \subset \{|\xi'| \geq 1\}, \\ 3/A, & \text{on } \operatorname{supp}\zeta_1 \subset \{|\xi'| \leq 2\}. \end{cases}$$

Then, by the Leibniz rule, Lemma 5.1, (5.4), a), and Lemma 6.1, we have

$$\begin{aligned} \left|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'} \frac{\zeta_j(\xi')}{\det \mathbf{L}}\right| &\leq C \sum_{\beta' \leq \alpha'} \left|\partial_{\xi'}^{\beta'}\left(A^j \frac{\zeta_j(\xi')}{A^j}\right)\right| \left|(\tau\partial_\tau)^\ell \partial_{\xi'}^{\alpha'-\beta'} \frac{1}{\det \mathbf{L}}\right| \\ &\leq C\mathbf{1}_{\operatorname{supp}\zeta_j} \sum_{\beta' \leq \alpha'} A^{j-|\beta'|}(|\lambda|^{1/2} + A)^{-6}\left(1 + \frac{1}{A}\right)|A|^{-(|\alpha'|-|\beta'|)} \\ &\leq C(|\lambda|^{1/2} + A)^{-6}A^{-|\alpha'|}, \end{aligned}$$

where $\mathbf{1}_{\operatorname{supp}\zeta_j}$ is the charactristic function. c) is implied by (4.18), Lemma 5.1 and Lemma 5.2, and d) is obtained by Lemma 5.1, b), c) and (5.4). $\square$

Now we prove the assertions for $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$.

Proof of the assertions for $\mathcal{S}_N(\lambda)$ *and* $\mathcal{T}(\lambda)$ *in Theorem 3.1.* By Lemma 5.7, Lemma 6.3 d), (3.15) and Lemma 3.1, $(\mathcal{S}(\lambda), \mathcal{T}(\lambda))$ satisfies the $\mathscr{R}$-boundedness properties (3.11), which completes the proof. $\square$

It remains to construct the solution operator $\mathcal{S}_j(\lambda)$ $(1 \leq j \leq N-1)$ and to prove the $\mathscr{R}$-boundedness for it. We first reduce the equation (4.20) to the case where the data are only on boundary. We consider the equation

$$\lambda u_{1j} - \mu\Delta u_{1j} = \tilde{f} \text{ in } \mathbb{R}^N. \tag{6.25}$$

Here,

$$\tilde{f} = -E_0\partial_j\theta, \tag{6.26}$$

where E_0 is an extension operator defined by (2.1). Then, if we set $u_j = u_{1j} + u_{2j}$, we have

$$\begin{cases} \lambda u_{2j} - \mu\Delta u_{2j} = 0 & \text{in } \Omega, \\ \partial_N u_{2j} = \tilde{h} & \text{on } \partial\Omega \end{cases} \tag{6.27}$$

with

$$\tilde{h} = \mu^{-1}\nu_N h_j - \partial_j u_N - \partial_N u_{1j}. \tag{6.28}$$

We obtain the $\mathscr{R}$-boundedness of the solution operator families of (6.25) by the following lemma, see for instance [25, Lemma 2.6].

Lemma 6.4. *For all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, there exists an operator $H_1(\lambda)$ satisfying $H_1(\lambda) \in \mathcal{L}(L_q(\mathbb{R}^N), W_q^2(\mathbb{R}^N))$ for $1 < q < \infty$ such that the following assertions hold:*

a) *For any $1 < q < \infty$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $\tilde{f} \in L_q(\mathbb{R}^N)$, $u_{1j} = H_1(\lambda)\tilde{f} \in W_q^2(\mathbb{R}^N)$ is a unique solution of (6.25).*
b) *For any $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\ell = 0, 1$ and $1 \leq m, n \leq N$, there hold*

$$\begin{aligned}
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda H_1(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\leq C_{N,q,\varepsilon,\mu}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma H_1(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\leq C_{N,q,\varepsilon,\mu}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2}\partial_m H_1(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\leq C_{N,q,\varepsilon,\mu}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\partial_n H_1(\lambda) \mid \lambda \in \Sigma_{\varepsilon,0}\}) &\leq C_{N,q,\varepsilon,\mu},
\end{aligned}$$

where $\Sigma_{\varepsilon,0}$ is given by (1.3) with $\gamma_0 = 0$ and $\lambda = \gamma + i\tau$.

As for (6.27), we show the following lemma to get the $\mathscr{R}$-boundedness of the solution operator families.

Lemma 6.5. *For all $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, there exists an operator $H_2(\lambda)$ (precisely, it is given by (6.30)) satisfying $H_2(\lambda) \in \mathcal{L}(L_q(\Omega)^{N+1}, W_q^2(\Omega))$ for $1 < q < \infty$ such that the following assertions hold:*

a) *For any $1 < q < \infty$, $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $\tilde{h} \in W_q^1(\Omega)$, $u_{2j} = H_2(\lambda)(\lambda^{1/2}\tilde{h}, \nabla\tilde{h}) \in W_q^2(\Omega)$ is a unique solution of (6.27).*
b) *For any $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $\gamma_0 > 0$, $\ell = 0, 1$ and $1 \leq m, n \leq N$, there hold*

$$\begin{aligned}
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda H_2(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \gamma H_2(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \lambda^{1/2}\partial_m H_2(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta}, \\
\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{(\tau\partial_\tau)^\ell \partial_m\partial_n H_2(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) &\leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta},
\end{aligned}$$

where $\Sigma_{\varepsilon,\gamma_0}$ is given by (1.3) and $\lambda = \gamma + i\tau$.

Proof. Applying the partial Fourier transform with respect to x' to (6.27) implies

$$\begin{cases} (B^2 - \partial_N^2)\widehat{u}_{j2}(\xi', x_N) = 0 & 0 < x_N < \delta, \\ \partial_N \widehat{u}_{j2}(\xi', x_N) = \mathscr{F}_{x'}[\tilde{h}](\xi', x_N) & x_N \in \{0, \delta\}. \end{cases}$$

Thus, we set

$$\widehat{u}_{j2}(\xi', x_N) = \sum_{\ell=1}^{2} \beta_\ell e^{-Bd_\ell(x_N)}$$

with some coefficients β_1 and β_2 which depend on (λ, ξ') and obey

$$B \begin{bmatrix} 1 & -e^{-B\delta} \\ e^{-B\delta} & -1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \mathscr{F}_{x'}[\tilde{h}](\xi', \delta) \\ \mathscr{F}_{x'}[\tilde{h}](\xi', 0) \end{bmatrix}.$$

By solving this, we obtain the solution formula for (6.27):

(6.29)
$$\begin{aligned}
u_{2j}(x) &= \mathscr{F}_{\xi'}^{-1}\Bigg[\frac{1}{B^2(1-e^{-2B\delta})}\big\{(e^{-Bd_1(x_N)}+e^{-B\delta}e^{-Bd_2(x_N)})B\mathscr{F}_{x'}[\tilde{h}](\xi',\delta)\\
&\qquad -(e^{-B\delta}e^{-Bd_1(x_N)}+e^{-Bd_2(x_N)})B\mathscr{F}_{x'}[\tilde{h}](\xi',0)\big\}\Bigg](x')\\
&= \mathcal{K}^1_{1,1,1}\left(\lambda,\frac{1}{B^2(1-e^{-2B\delta})}\right)(\lambda^{\frac12}\tilde{h},\nabla\tilde{h}) + \mathcal{K}^1_{1,2,1}\left(\lambda,\frac{e^{-B\delta}}{B^2(1-e^{-2B\delta})}\right)(\lambda^{\frac12}\tilde{h},\nabla\tilde{h})\\
&\quad - \mathcal{K}^1_{1,1,2}\left(\lambda,\frac{e^{-B\delta}}{B^2(1-e^{-2B\delta})}\right)(\lambda^{\frac12}\tilde{h},\nabla\tilde{h}) - \mathcal{K}^1_{1,2,2}\left(\lambda,\frac{1}{B^2(1-e^{-2B\delta})}\right)(\lambda^{\frac12}\tilde{h},\nabla\tilde{h}).
\end{aligned}$$

We thus define the operator $H_2(\lambda)$ by

(6.30) $$H_2(\lambda)(\lambda^{\frac12}\tilde{h},\nabla\tilde{h}) = (\text{the right-hand side of (6.29)}).$$

Since (5.3) and (5.5) imply the assumptions (6.2) and (6.3) of Lemma 6.2 with $f\equiv 1-e^{-2c\gamma_0^{1/2}\delta}$ (constant function):

$$|1-e^{-2B\delta}|\geq 1-e^{-2c\gamma_0^{1/2}\delta},\quad 1-e^{-2B\delta}\in\mathbb{M}_{0,1,\varepsilon,0},$$

we get $(1-e^{-2B\delta})^{-1}\in\mathbb{M}_{0,1,\varepsilon,\gamma_0}$. And so, we have

$$\frac{1}{B^2(1-e^{-2B\delta})},\ \frac{e^{-B\delta}}{B^2(1-e^{-2B\delta})}\in\mathbb{M}_{-2,1,\varepsilon,\gamma_0}$$

from Lemma 5.1 and Lemma 5.2. Then Lemma 5.7 and Lemma 3.1 ii) imply the desired conclusion. □

Let us close the paper with completion of proof of Theorem 3.1.

Proof of the remaining assertions of Theorem 3.1. In view of the arguments above, we define $\mathcal{S}_j(\lambda)$ by

(6.31)
$$\begin{aligned}
&\mathcal{S}_j(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h})\\
&= H_1(\lambda)Q_0(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}) + H_2(\lambda)\big(Q_1(\lambda),Q_2(\lambda)\big)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}),
\end{aligned}$$

where we have set

$$\begin{aligned}
Q_0(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}) &= \tilde{f}\\
&= -E_0\partial_j\mathcal{T}(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}),\\
Q_1(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}) &= \lambda^{1/2}\tilde{h}\\
&= \mu^{-1}\tilde{\nu}_N\lambda^{\frac12}h_j - \lambda^{\frac12}\partial_j\mathcal{S}_N(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h})\\
&\quad - \lambda^{\frac12}\partial_N H_1(\lambda)Q_0(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}),\\
Q_2(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}) &= \nabla\tilde{h}\\
&= \mu^{-1}(\nabla h_j)\tilde{\nu}_N + \mu^{-1}M_{\lambda^{-1/2}}\lambda^{1/2}h_j\nabla\tilde{\nu}_N\\
&\quad - \partial_j\nabla\mathcal{S}_N(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h})\\
&\quad - \partial_N\nabla H_1(\lambda)Q_0(\lambda)(\lambda^{1/2}\mathbf{h}',h_N,\nabla\mathbf{h}).
\end{aligned}$$

Here, $\tilde{f}$ and $\tilde{h}$ are given by (6.26) and (6.28), respectively, E_0 is an extension operator defined by (2.1), $\tilde{\nu}_N$ is the N-th component of $\tilde{\nu}$, and $M_{\lambda^{-1/2}}$ and $\tilde{\nu}$ are defined by (3.14). We have

$$\mathscr{R}_{\mathcal{L}(L_q(\Omega))}(\{Q_i(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\}) \leq C_{N,q,\varepsilon,\gamma_0,\mu,\delta} \tag{6.32}$$

for $i = 0, 1, 2$ from Lemma 3.1, Lemma 6.4, (3.15) and the fact that $\mathcal{S}_N(\lambda)$ and $\mathcal{T}(\lambda)$ satisfy (3.11). Then $\mathcal{S}_j(\lambda)$ satisfies (3.11) by Lemma 3.1, Lemma 6.4, Lemma 6.5 and (6.32). Since $(\mathcal{S}(\lambda), \mathcal{T}(\lambda))$ satisfies a) in Theorem 3.1, by summing up the aforementioned arguments, Theorem 3.1 follows. □

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