A variable-order fractional $p(\cdot)$-Kirchhoff type problem in $\mathbb{R}^N$

Jiabin Zuo[a,b,c], Libo Yang[d], Sihua Liang[e *†]

[a]Faculty of Applied Sciences, Jilin Engineering Normal University, Changchun 130052, P. R. China

[b]College of Science, Hohai University, Nanjing 210098, P. R. China

[c]Departamento de Matemática, Universidade Estadual de Campinas, IMECC, Campinas, SP 13083–859, Brazil

[d]Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huai'an 223003, China

[e] College of Mathematics, Changchun Normal University, Changchun, 130032, P.R. China

Abstract

This paper is concerned with the existence and multiplicity of solutions for the following variable $s(\cdot)$-order fractional $p(\cdot)$-Kirchhoff type problem

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy\right) (-\Delta)^{s(\cdot)}_{p(\cdot)} v(x) + |v(x)|^{\overline{p}(x)-2} v(x) = \mu g(x,v) \quad \text{in } \mathbb{R}^N, \\ v \in W^{s(\cdot),p(\cdot)}(\mathbb{R}^N), \end{cases}$$

where $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is a variable $s(\cdot)$-order $p(\cdot)$-fractional Laplace operator with $s(\cdot) : \mathbb{R}^{2N} \to (0,1)$ and $p(\cdot) : \mathbb{R}^{2N} \to (1,\infty)$, $\overline{p}(x) = p(x,x)$ for $x \in \mathbb{R}^N$, and M is a continuous Kirchhoff-type function, $g(x,v)$ is a Carathéodory function, $\mu > 0$ is a parameter. We obtain that there are at least two distinct solutions for the above problem by applying the generalized abstract critical point theorem. Under the weaker conditions, we also show the existence of one solution and infinitely many solutions by using the mountain pass lemma and fountain theorem, respectively. In particular, the new compact embedding result of the space $W^{s(\cdot),p(\cdot)}(\mathbb{R}^N)$ into $L^{q(\cdot)}_{a(x)}(\mathbb{R}^N)$ will be used to overcome the lack of compactness in $\mathbb{R}^N$. The main feature and difficulty of this paper is the presence of a double non-local term involving two variable parameters.

Keywords: Kirchhoff-type; $p(\cdot)$-fractional Laplacian; Variable-order; Abstract critical point theorem.

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1 Introduction and main results

The evolution of the Laplace operator has taken on a variety of forms so far. Many researchers devoted themselves to the integer order Laplace operators. For instance, Tang *et al*. [32] dealt with the existence of ground state sign-changing solutions for a class of Kirchhoff-type problems involving the Laplace operator in bounded domain. In [33], Tang and Chen obtained a ground state solution of Nehari-Pohozaev type and a least energy solution under some mild assumptions f in the whole space $\mathbb{R}^3$.

* *E-mail address:* zuojiabin88@163.com(J. Zuo), yanglibo80@126.com(L. Yang), liangsihua@163.com(S. Liang).

† Corresponding author at: College of Mathematics, Changchun Normal University, Changchun, 130032, P.R. China.

Then, the integer order Laplace operators were extended to cases involving the p-Laplace operator and $p(x)$-Laplace operator, that is $\Delta \Longrightarrow \Delta_p \Longrightarrow \Delta_{p(x)}$. We still give an example in bounded domain and in the whole space respectively. With the help of a direct variational approach and the theory of the variable exponent Sobolev spaces, the existence and multiplicity of solutions to a class of $p(x)$-Kirchhoff-type problem with Dirichlet boundary data was proved in [6]. In [19], the existence and multiplicity of solutions for a class of $p(x)$-Kirchhoff type Schrödinger problems in $\mathbb{R}^N$ was obtained by means of abstract critical point results. For more about the eigenvalue problem of this operator, we recommend the readers to refer to [25, 31].

In recent years, with the needs of the real world in physics, economics, biology, computing, see [18, 21], more and more mathematicians began to study a non-local operator, i,e., the fractional Laplace operator $(-\Delta)^s$, due to the interest amount of attention towards partial differential equations with nonlocal problems, especially we encourage readers to pay attention to this famous book [27], subsequently the non-local operator was further extended to the fractional p-Laplace operator and fractional $p(x)$-Laplace operator, that is $(-\Delta)^s \Longrightarrow (-\Delta)^s_p \Longrightarrow (-\Delta)^s_{p(x)}$. Again, we point out that Rossi *et al.* [17] extended the Sobolev spaces with variable exponents to include the fractional case, they established a compact embedding theorem of these spaces into variable exponent Lebesgue spaces. As an application, they studied the existence and uniqueness of a solution for a nonlocal problem involving the fractional $p(x)$-Laplacian. In addition, in the whole space, the existence and multiplicity results for a class of fractional $p(x, .)$-Kirchhoff-type problems was established by using many kinds of methods in [2].

On the other hand, the fractional variable order derivatives proposed by Lorenzo and Hartley in [22] were used to introduce different processes of nonlinear diffusion. Subsequently, a new kind of variable-order fractional operator is established in various problems, such as Schrödinger equations, Kirchhoff equations, Choquard equations, etc. From this, a great attention has been devoted to the study of fractional variable order spaces. In particular, see for example, Xiang *et al.* in [39] considered a multiplicity result for a Schrödinger equation driven by the variable $s(\cdot)$-order fractional Laplace operator via variational methods. Zhang *et al.* in [35] investigated the existence of infinitely many solutions for a kind of Kirchhoff type variable $s(\cdot)$-order problem by using four different critical point theorems. In [7], Biswas *et al.* firstly proved a continuous embedding result for the functions in fractional Sobolev spaces with variable order and variable exponents and under the Amvrosetti and Rabinowitz condition ((AR)-condition for short; see [20]), they obtained existence and multiplicity of solutions for the following problem

$$\begin{cases} (-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = \lambda |u(x)|^{\beta(x)-2} u(x) + \left(\int_\Omega \frac{F(y, u(y))}{|x-y|^{\mu(x,y)}} dy \right) f(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We refer the interested readers to [23, 43] about variable-order problems.

However, as far as we know, there are few results concerned with the Kirchhoff type problem driven by a $p(\cdot)$-fractional Laplace operator with variable $s(\cdot)$-order in the whole space $\mathbb{R}^N$. Inspired by above works, in this paper, we are interested in the existence and multiplicity of solutions for the following new double variable order fractional Kirchhoff type problems in $\mathbb{R}^N$:

$$\begin{cases} M\left(\iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy \right) (-\Delta)^{s(\cdot)}_{p(\cdot)} v(x) + |v(x)|^{\overline{p}(x)-2} v(x) = \mu g(x, v), \\ v \in W^{s(\cdot),p(\cdot)}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N > p(x, y)s(x, y)$ for any $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, with $s(\cdot) : \mathbb{R}^{2N} \to (0, 1)$ and $p(\cdot) : \mathbb{R}^{2N} \to (1, \infty)$,

$\overline{p}(x) = p(x,x)$ for $x \in \mathbb{R}^N$ and μ is a real parameter. Here, the main operator $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is the fractional variable $s(\cdot)$-order $p(\cdot)$-Laplacian given by

$$(-\Delta)^{s(\cdot)}_{p(\cdot)} u(x) = P.V. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+p(x,y)s(x,y)}} dy, \quad x \in \mathbb{R}^N,$$

along any $u \in C_0^\infty(\mathbb{R}^N)$, where P.V. denotes the Cauchy principle value.

It is worth mentioning that Kirchhoff in 1883 (see[15]) presented a stationary verion of differential equation, the so-called Kirchhoff equation

$$\rho \frac{\partial^2 v}{\partial t^2} - \Big(\frac{p_0}{l} + \frac{e}{2L}\int_0^L |\frac{\partial v}{\partial x}|^2 dx\Big)\frac{\partial^2 v}{\partial x^2} = 0, \tag{1.2}$$

where ρ, l, e, L, p_0 are positive constants which represent the corresponding physical meanings. (1.2) is a generalization of D'Alembert equation. From then on, much interest has been focused on combining this model with many kinds of problems due to its nonlocal nature; Such as, the existence, multiplicity, and concentration of solutions on (critical) Kirchhoff equations, Kirchhoff system and so on. We only present some very new literature for readers to refer, for instance, Fiscella and Valdinoci firstly proposed a critical Kirchhoff type problem involving a nonlocal operator in [10] and Rădulescu et al. [40] considered the existence and multiplicity of solutions for a Schrödinger-Kirchhoff type problem involving the fractional p-Laplacian and critical exponent in $\mathbb{R}^N$. By means of some appropriate variational arguments, Ambrosio *et al.* [1] investigated the existence and concentration of positive solutions for a class of fractional p-Kirchhoff type problem $\mathbb{R}^3$. Xu *et al.* [36] studied a Kirchhoff-type system with linear weak damping and logarithmic nonlinearities. For more on Kirchhoff, we can also refer to the literature [5, 11, 13, 14, 28, 29, 41, 42, 45].

In order to simplify the notation, throughout this paper, we denote

$$s^- = \min_{(x,y)\in\mathbb{R}^{2N}} s(x,y), \quad s^+ = \max_{(x,y)\in\mathbb{R}^{2N}} s(x,y), \quad p^- = \min_{(x,y)\in\mathbb{R}^{2N}} p(x,y), \quad p^+ = \max_{(x,y)\in\mathbb{R}^{2N}} p(x,y),$$

$$q^- = \min_{x\in\mathbb{R}^N} q(x), \quad p_s^*(x) = \frac{N\overline{p}(x)}{N - \overline{s}(x)\overline{p}(x)} \quad \text{with} \quad \overline{p}(x) = p(x,x), \quad \overline{s}(x) = s(x,x),$$

$$C_+(\mathbb{R}^N) = \Big\{ q \in C(\mathbb{R}^N) : 1 < q(x) \ \text{ for all } \ x \in \mathbb{R}^N \Big\}.$$

We consider problem (1.1) under some mild assumptions. For this, we assume that $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a continuous function satisfying conditions:

(M_1) There exists $\theta \in [1, (p_s^*)^-/p^+)$ such that

$$tM(t) \le \theta \widetilde{M}(t) \quad \textit{for any} \quad t \in \mathbb{R}_0^+,$$

where $\widetilde{M}(t) = \int_0^t M(\tau) d\tau$;

(M_2) *For any* $\tau > 0$ *there exists* $m = m(\tau) > 0$ *such that*

$$M(t) \ge m \quad \textit{for any} \quad t \ge \tau.$$

A classic example of the Kirchhoff function M satisfying $(M_1)-(M_2)$, is given by

$$M(t) = a + bt^{\beta-1}, \quad a, b \geq 0, \quad a+b>0, \quad t \geq 0$$

and

$$\begin{aligned} &\beta \in (1,\infty) \quad if \quad b>0, \\ &\beta = 1 \qquad\quad\ if \quad b = 0. \end{aligned}$$

(M_3) $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is a decreasing function.

Here, let us consider another example that the Kirchhoff function satisfies conditions $(M_1)-(M_3)$, is given by

$$M(t) = 1 + \frac{1}{e+t}, \ t \geq 0.$$

While, let $a : \mathbb{R}^N \to \mathbb{R}$ be a function satisfies the following conditions:

(A_1) $a \in L^{r(x)}(\mathbb{R}^N)$ such that $a(x) \geq 0$, where $a \in C_+(\mathbb{R}^N)$;

(A_2) $a \in C(\mathbb{R}^N \times \mathbb{R})$ such that $a(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $a \neq 0$.

We all know that the (AR)-condition is an important content that the functional associated with the equation satisfies the Palais-Smale condition, which provides a guarantee for ensuring the boundedness of the Palais-Smale sequence. However, this condition is sometimes restrictive about several nonlinearities, so we let g satisfy a weaker condition than the (AR)-condition; see [12, 24, 26]. Accurately, $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory* function and is supposed to satisfy the following assumptions:

(G_1) Let $p, q \in C_+(\mathbb{R}^N)$ and $1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$, there exists $a(x)$ given by (A_1) such that

$$|g(x,t)| \leq a(x)|t|^{q(x)-1} \quad for\ all \quad (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(G_2) There exist $\lambda \in (\theta p^+, \infty)$ and $a(x)$ given by (A_1) such that

$$\lambda G(x,t) \leq tg(x,t) + a(x)|t|^{p^-} \text{ for all } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where $G(x,t) = \int_0^t g(x,\tau)d\tau$;

(G_3) $\lim_{|t|\to\infty} \frac{G(x,t)}{|t|^{\theta p^+}} = \infty$ uniformly for almost $x \in \mathbb{R}^N$;

(G_4) $g(x,-t) = -g(x,t)$ for a.e. $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$;

(G_5) There exists a constant $\lambda \geq 1$ such that $\lambda \mathcal{G}(x,t) \geq \mathcal{G}(x,\nu t)$ for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $\nu \in [0,1]$, where $\mathcal{G}(x,t) = g(x,t)t - p^+G(x,t)$.

Last but not least, we suppose that $s(\cdot) : \mathbb{R}^{2N} \to (0,1)$ and $p(\cdot) : \mathbb{R}^{2N} \to (1,\infty)$ are continuous functions fulfilling

(H_1) $0 < s^- \leq s^+ < 1 < p^- \leq p^+$;

(H_2) $s(\cdot)$ and $p(\cdot)$ are symmetric, that is, $s(x,y) = s(y,x)$ and $p(x,y) = p(y,x)$ for any $(x,y) \in \mathbb{R}^{2N}$.

We can give the definition of (weak) solutions for problem (1.1). The space X will be introduced in the next section.

Definition 1.1. *We say that a function $v \in X$ is a weak solution of problem* (1.1)*, if*

$$M\Big(\delta_{p(\cdot)}(v)\Big)\iint_{\mathbb{R}^{2N}}\frac{|v(x)-v(y)|^{p(x,y)-2}(v(x)-v(y))(\phi(x)-\phi(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\ +\int_{\mathbb{R}^N}|v(x)|^{\overline{p}(x)-2}v(x)\phi(x)dx=\mu\int_{\mathbb{R}^N}g(x,v)\phi dx$$

for all $\phi \in X$, where

$$\delta_{p(\cdot)}(v)=\iint_{\mathbb{R}^{2N}}\frac{1}{p(x,y)}\frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy.$$

Now, we state our main results. The functional I_μ will be covered in the section 3.

Theorem 1.1. *Let $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $s(\cdot)$ and $p(\cdot)$ verify $(H_1)-(H_2)$. Assume that $(M_1)-(M_2)$ and $(G_1)-(G_3)$ are satisfied. Then, there exists $\mu^* > 0$ such that problem* (1.1) *admits at least two distinct weak solutions in X for any $\mu \in (0,\mu^*)$.*

Theorem 1.2. *Let $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $s(\cdot)$ and $p(\cdot)$ verify $(H_1)-(H_2)$. Assume that $(M_1)-(M_3)$, (G_1), (G_3) and (G_5) are satisfied. Then, there exists $\mu^* > 0$ such that problem* (1.1) *admits at least two distinct weak solutions in X for any $\mu \in (0,\mu^*)$.*

Theorem 1.3. *Let $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $s(\cdot)$ and $p(\cdot)$ verify $(H_1)-(H_2)$. Assume that $(M_1)-(M_2)$ and $(G_1)-(G_3)$ are satisfied. Then, for any $\mu > 0$ problem* (1.1) *has a nontrivial weak solution.*

Theorem 1.4. *Let $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $s(\cdot)$ and $p(\cdot)$ verify $(H_1)-(H_2)$. Assume that $(M_1)-(M_3)$, $(G_1),(G_3)$ and (G_5) are satisfied. Then, for any $\mu > 0$ problem* (1.1) *has a nontrivial weak solution.*

Theorem 1.5. *Let $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $s(\cdot)$ and $p(\cdot)$ verify $(H_1)-(H_2)$. Assume that $(M_1)-(M_2)$ and $(G_1)-(G_4)$ are satisfied. Then, for any $\mu > 0$ problem* (1.1) *has a sequence of nontrivial weak solutions $\{v_j\}$ in X such that $I_\mu(v_j) \to \infty$ as $n \to \infty$.*

Theorem 1.6. *Let $N > p(x,y)s(x,y)$ for any $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, where $s(\cdot)$ and $p(\cdot)$ verify $(H_1)-(H_2)$. Assume that $(M_1)-(M_3)$, (G_1) and $(G_3)-(G_5)$ are satisfied. Then, for any $\mu > 0$ problem* (1.1) *has a sequence of nontrivial weak solutions $\{v_j\}$ in X such that $I_\mu(v_j) \to \infty$ as $n \to \infty$.*

The novelty of this article is that the problem (1.1) is studied in the whole space for the first time and in order to overcome the lack of compactness in the whole space $\mathbb{R}^N$, a new compact embedding theorem is established which is different from fractional $p(x,.)$-Kirchhoff-type cases. Moreover, the proof method of Theorem 1.1 and Theorem 1.5 is new in setting of variable-order fractional $p(x,.)$-Laplacian. To the best of our knowledge, there are no works on double variable order fractional Kirchhoff type problems without the (AR)-condition. Thus, our main results generalize reference [2, 5, 9, 38, 44] in several directions.

The paper is structured as follows. In Section 2, we first introduce some preliminary knowledge on variable exponent Lebesgue spaces and variable-order fractional Sobolev spaces with variable exponent, and then we establish a compact embedding of the space $W^{s(\cdot),p(\cdot)}(\mathbb{R}^N)$ into $L^{q(\cdot)}_{a(x)}(\mathbb{R}^N)$. In Section 3, we verify the compactness condition. In Section 4, we prove Theorem 1.1 and Theorem 1.2 by using the abstract critical point theory. In Section 5, with the help of mountain pass lemma and fountain theorem, we give the proof of Theorems 1.3-1.4 and Theorems 1.5-1.6.

2 Functional analytic setup and preliminaries results

In this section, first of all, we review some basic information about the variable exponent spaces and the variable order fractional Sobolev spaces. Here we can refer the recent monograph about variational analysis of the problems with variable exponents by Rădulescu and Repovš [30]. Then, we give some basic lemmas and a new compact embedding theorem that will be used in this paper.

For any $q \in C_+(\mathbb{R}^N)$, the variable exponent Lebesgue space is

$$L^{q(x)}(\mathbb{R}^N) = \left\{ v : \mathbb{R}^N \to \mathbb{R} \ \text{ is measurable } \ \int_{\mathbb{R}^N} |v(x)|^{q(x)} dx < \infty \right\},$$

with the Luxemburg norm

$$\|v\|_{q(x)} = \inf \left\{ \gamma > 0 : \int_{\mathbb{R}^N} \left| \frac{v(x)}{\gamma} \right|^{q(x)} dx \leq 1 \right\}.$$

Then $(L^{q(x)}(\mathbb{R}^N), \|\cdot\|_{q(\cdot)})$ is a separable reflexive Banach space, see [16, Theorem 2.5 and Corollaries 2.7 and 2.12] and [8].

Let $\widetilde{q} \in C_+(\mathbb{R}^N)$ be the conjugate exponent of q, that is

$$\frac{1}{q(x)} + \frac{1}{\widetilde{q}(x)} = 1 \ \text{ for all } x \in \overline{\Omega}.$$

Then we have the following Hölder inequality, whose proof can be found in [16, Theorem 2.1].

Lemma 2.1. *Suppose that $v \in L^{q(\cdot)}(\mathbb{R}^N)$ and $u \in L^{\widetilde{q}(\cdot)}(\mathbb{R}^N)$; then*

$$\Big| \int_{\mathbb{R}^N} vu dx \Big| \leq \Big(\frac{1}{q^-} + \frac{1}{\widetilde{q}^-} \Big) \|v\|_{q(\cdot)} \|u\|_{\widetilde{q}(\cdot)} \leq 2 \|v\|_{q(\cdot)} \|u\|_{\widetilde{q}(\cdot)}.$$

A very important role is played by the modular of the $L^{q(x)}(\mathbb{R}^N)$ space, which defined by

$$\varrho_{q(\cdot)}(v) = \int_{\mathbb{R}^N} |v(x)|^{q(x)} dx,$$

we have the next crucial result given in [7].

Proposition 2.1. *Let $v \in L^{q(\cdot)}(\mathbb{R}^N)$ and $\{v_j\} \subset L^{q(\cdot)}(\mathbb{R}^N)$, then*

$$\begin{aligned}
&(1) \quad \|v\|_{q(\cdot)} < 1 (resp. = 1, > 1) \Leftrightarrow \varrho_{q(\cdot)}(v) < 1 (resp. = 1, > 1),\\
&(2) \quad \|v\|_{q(\cdot)} < 1 \Rightarrow \|v\|_{q(\cdot)}^{q^+} \leq \varrho_{q(\cdot)}(v) \leq \|v\|_{q(\cdot)}^{q^-},\\
&(3) \quad \|v\|_{q(\cdot)} > 1 \Rightarrow \|v\|_{q(\cdot)}^{q^-} \leq \varrho_{q(\cdot)}(v) \leq \|v\|_{q(\cdot)}^{q^+},\\
&(4) \quad \lim_{j\to\infty} \|v_j\|_{q(\cdot)} = 0(\infty) \Leftrightarrow \lim_{j\to\infty} \varrho_{q(\cdot)}(v_j) = 0(\infty),\\
&(5) \quad \lim_{j\to\infty} \|v_j - v\|_{q(\cdot)} = 0 \Leftrightarrow \lim_{j\to\infty} \varrho_{q(\cdot)}(v_j - v) = 0.
\end{aligned}$$

Lemma 2.2. (see, [2, Lemma 2.2]) Let $|v|^{q(x)} \in L^{\beta(x)/q(x)}(\mathbb{R}^N)$, where $q, \beta \in C_+(\mathbb{R}^N)$ and $q(x) \leq \beta(x)$ for all $x \in \mathbb{R}^N$, then $v \in L^{\beta(x)}(\mathbb{R}^N)$ and there exists a number $\overline{q} \in [q^-, q^+]$ such that

$$\Big\| |v|^{q(x)} \Big\|_{\beta(x)/q(x)} = (\|v\|_{\beta(x)})^{\overline{q}}.$$

The variable-order fractional Sobolev spaces with variable exponent via the Gagliardo approach is defined by

$$X = W^{s(\cdot),p(\cdot)}(\mathbb{R}^N) = \left\{ v \in L^{\overline{p}(x)}(\mathbb{R}^N) : \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}} dxdy < \infty \text{ for some } \gamma > 0 \right\}$$

with the norm $\|v\|_X = \|v\|_{\overline{p}(x)} + [v]_{s(\cdot),p(\cdot)}$, where

$$[v]_{s(\cdot),p(\cdot)} = \inf \left\{ \gamma > 0 : \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{\gamma^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}} dxdy < 1 \right\}$$

is a Gagliardo seminorm with variable-order and variable exponent.

The space X is a separable reflexive Banach space, see [4, 7]. We define the convex modular function $\varrho_{p(\cdot)}^{s(\cdot)} : X \to \mathbb{R}$ by

$$\varrho_{p(\cdot)}^{s(\cdot)}(v) = \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy + \int_{\mathbb{R}^N} |v(x)|^{\overline{p}(x)} dx,$$

whose associated norm define by

$$\|v\| = \|v\|_{\varrho_{p(\cdot)}^{s(\cdot)}} = \inf \left\{ \gamma > 0 : \varrho_{p(\cdot)}^{s(\cdot)}\left(\frac{v}{\gamma}\right) \le 1 \right\},$$

which is equivalent to the norm $\|\cdot\|_X$. Note that the norm $\|\cdot\|$ will be used in this paper. Then, similar to Proposition 2.1, we have the next result of [2, Proposition 2.3].

Proposition 2.2. *Let $v \in X$ and $\{v_j\} \subset X$, then*

$$\begin{aligned}
&(1)\quad \|v\|_X < 1 (resp. = 1, > 1) \Leftrightarrow \varrho_{p(\cdot)}^{s(\cdot)}(v) < 1 (resp. = 1, > 1),\\
&(2)\quad \|v\|_X < 1 \Rightarrow \|v\|_X^{p^+} \le \varrho_{p(\cdot)}^{s(\cdot)}(v) \le \|v\|_X^{p^-},\\
&(3)\quad \|v\|_X > 1 \Rightarrow \|v\|_X^{p^-} \le \varrho_{p(\cdot)}^{s(\cdot)}(v) \le \|v\|_X^{p^+},\\
&(4)\quad \lim_{j\to\infty} \|v_j\|_X = 0(\infty) \Leftrightarrow \lim_{j\to\infty} \varrho_{p(\cdot)}^{s(\cdot)}(v_j) = 0(\infty),\\
&(5)\quad \lim_{j\to\infty} \|v_j - v\|_X = 0 \Leftrightarrow \lim_{j\to\infty} \varrho_{p(\cdot)}^{s(\cdot)}(v_j - v) = 0.
\end{aligned}$$

Lemma 2.3. (see, [4, Theorem 3.1]) Let $p(\cdot)$ and $s(\cdot)$ satisfy $(H_1) - (H_2)$, with $N > p(x,y)s(x,y)$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Let $h \in C_+(\overline{\Omega})$ satisfy

$$1 < h^- = \min_{x\in\overline{\Omega}} h(x) \le h(x) < p_s^*(x) = \frac{N\overline{p}(x)}{N - \overline{p}(x)\overline{s}(x)} \quad \text{for any } x \in \overline{\Omega},$$

where $\overline{p}(x) = p(x,x)$ and $\overline{s}(x) = s(x,x)$. Then, there exists a positive constant $C_h = C_h(N, s, p, h, \Omega)$ such that

$$\|v\|_{h(\cdot)} \le C_h \|v\|_{W^{s(\cdot),p(\cdot)}(\Omega)}$$

for any $v \in W^{s(\cdot),p(\cdot)}(\Omega)$. Moreover, the embedding $W^{s(\cdot),p(\cdot)}(\Omega) \hookrightarrow L^{h(\cdot)}(\Omega)$ is compact.

Lemma 2.4. (see, [4, Theorem 3.2]) Let $p(\cdot)$ and $s(\cdot)$ be uniformly continuous functions satisfying $(H_1)-(H_2)$, suppose that $h \in C_+(\mathbb{R}^N)$ is a uniformly continuous such that $\overline{p}(x) \le h(x) < p_s^*(x)$ for $x \in \mathbb{R}^N$. Then, the embedding $X \hookrightarrow L^{h(\cdot)}(\mathbb{R}^N)$ is continuous.

For $q \in C_+(\overline{\Omega})$ and a satisfying (A_2), we define

$$L_a^{q(x)}(\mathbb{R}^N) = \left\{ v : \mathbb{R}^N \to \mathbb{R} \ \text{ is measurable } \ \int_{\mathbb{R}^N} a(x)|v(x)|^{q(x)}dx < \infty \right\}$$

with the norm

$$\|v\|_{L_a^{q(x)}(\mathbb{R}^N)} = \|v\|_{q,a} = \inf\left\{ \gamma > 0 : \int_{\mathbb{R}^N} a(x)\Big|\frac{v(x)}{\gamma}\Big|^{q(x)} dx \le 1 \right\}.$$

It is obvious that $\varrho_{q,a} = \int_{\mathbb{R}^N} a(x)|v(x)|^{q(x)}dx$ is a semimodular, (see, [7, Definition 2.1.1]). Furthermore, $L_a^{q(x)}(\mathbb{R}^N)$ is a Banach space (see [7, Theorem 2.3.13]), which is separable and reflexive (Similar to the proof of Lemma 3.4.4 and Theorem 3.4.7 in [7], we can obtain this property).

Lemma 2.5. (see, [34, Lemma 2.1])

$$\lim_{j\to\infty} \|v_j\|_{q,a} = 0 \Leftrightarrow \lim_{j\to\infty} \varrho_{q,a}(v_j) = 0.$$

We note that the embedding $X \hookrightarrow L^{q(x)}(\mathbb{R}^N)$ is no longer compact, which makes it difficult to verify the Palais-Smale condition. The embedding result below provides a new tool to overcome this difficulty and is very critical in this paper.

Lemma 2.6. *Let $p(\cdot)$ and $s(\cdot)$ satisfy $(H_1)-(H_2)$. Let $q \in C_+(\mathbb{R}^N)$ with $1 < q^- \le q(x) \le q^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$. Suppose that (A_1) holds with h fulfilling*

$$\overline{p}(x) \le \beta(x) = \frac{h(x)q(x)}{h(x)-1} \le p_s^*(x) \quad for\ all\ x \in \mathbb{R}^N.$$

Then, the embedding $X \hookrightarrow L_a^{q(x)}(\mathbb{R}^N)$ is continuous. Furthermore, if $\beta^+ < p_s^*(x)$ for all $x \in \mathbb{R}^N$. Then, $X \hookrightarrow L_a^{q(x)}(\mathbb{R}^N)$ is compact.

Proof. Notice that our work space X is different from the work space X in [2]. Thus, we only need to make a slight modification, that is, according to Lemma 2.4, we know that the embedding $X \hookrightarrow L^{h(\cdot)}(\mathbb{R}^N)$ is also continuous. Next, our discussion is exactly the same as Lemma 2.4 in [2] by combining with Lemmas 2.1-2.3, which we omit here. □

Lemma 2.7. *Let $p(\cdot)$ and $s(\cdot)$ satisfy $(H_1)-(H_2)$. Let $q \in C_+(\mathbb{R}^N)$. Suppose that (A_1) holds. Then for any $v \in X$ there exist two positive constants $\overline{q} \in [q^-, q^+]$ and $C_{q,a}$ such that*

$$\varrho_{q,a}(v) \le C_{q,a}\|v\|^{\overline{q}}.$$

Proof. It follows from Lemma 2.6 that the embedding $X \hookrightarrow L_a^{q(x)}(\mathbb{R}^N)$ is continuous. So, for the rest, our argument is the same as Lemma 2.5 in [2]. □

Now, we study the functional $I : X \to \mathbb{R}$, defined by

$$I(v) = \iint_{\mathbb{R}^{2N}} \frac{1}{p(x,y)} \frac{|v(x) - v(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy + \int_{\mathbb{R}^{2N}} \frac{1}{\overline{p}(x)} |v(x)|^{\overline{p}(x)} dx.$$

We conclude this section presenting a technical result useful to study the compactness condition. The proof of this proposition can be given arguing similarly to [2, Lemma 2.6] and [3, Lemma 4.2], hence, its proof is omitted.

Proposition 2.3. *Let $p(\cdot)$ and $s(\cdot)$ satisfy $(H_1) - (H_2)$. We consider the following functional $\mathcal{L} : X \to X^*$, with X^* the dual space of X, such that*

$$\begin{aligned} \langle \mathcal{L}(v), \phi \rangle =& \langle I^{'}(v), \phi \rangle \qquad (2.1) \\ =& \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p(x,y)-2}(v(x) - v(y))(\phi(x) - \phi(y))}{|x-y|^{N+p(x,y)s(x,y)}} dxdy + \int_{\mathbb{R}^N} |v(x)|^{\overline{p}(x)-2} v(x)\phi(x) dx \end{aligned}$$

for any v, $\phi \in X$. Then:
(i) $\mathcal{L}$ is a bounded and strictly monotone operator;
(ii) $\mathcal{L}$ is a mapping of type (S_+), that is, if $v_j \rightharpoonup v$ in X and $\limsup_{j\to\infty} \mathcal{L}(v_j)(v_j \to v) \leq 0$, then $v_j \to v$ in X;
(iii) $\mathcal{L} : X \to X^*$ is a homeomorphism.

Throughout the paper, for simplicity, we use $\{C_i,\ i \in \mathbb{N}^+\}$ to denote different non-negative or positive constant.

3 Compactness condition

Let us consider the following functional associated to problem (1.1), defined by $\mathcal{I}_\mu : X \to \mathbb{R}$

$$\mathcal{I}_\mu(v) = \mathcal{A}(v) - \mu\mathcal{B}(v),$$

where

$$\mathcal{A}(v) = \widetilde{M}\Big(\delta_{p(\cdot)}(v)\Big) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)} |v(x)|^{\overline{p}(x)} dx, \quad \mathcal{B}(v) = \int_{\mathbb{R}^N} G(x,v) dx.$$

Clearly, according to Lemma 2.6, Proposition 2.3 and (G_1), we know that $\mathcal{I}_\mu$ is well defined. By the continuity of M yields that $\widetilde{M} \in C^1(\mathbb{R},\mathbb{R})$, by Proposition 2.3 we get that $\delta_{p(\cdot)}(v)$ and $v \mapsto \int_{\mathbb{R}^{2N}} (1/\overline{p}(x))|v(x)|^{\overline{p}(x)} dx$ are in $C^1(X,\mathbb{R})$. Since $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory* function, then $v \mapsto \int_{\mathbb{R}^N} G(x,v)dx$ is also in $C^1(X,\mathbb{R})$. Thus $\mathcal{I}_\mu$ is of class C^1 on X. Moreover, we have that its Fréchet derivative is given by

$$\begin{aligned} \langle \mathcal{I}_\mu^{'}(v), \phi \rangle =& M\Big(\delta_{p(\cdot)}(v)\Big) \iint_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p(x,y)-2}(v(x) - v(y))(\phi(x) - \phi(y))}{|x-y|^{N+p(x,y)s(x,y)}} dxdy \\ &+ \int_{\mathbb{R}^N} |v(x)|^{\overline{p}(x)-2} v(x)\phi(x) dx - \mu \int_{\mathbb{R}^N} g(x,v)\phi(x) dx \end{aligned}$$

for all v, $\phi \in X$. Therefore, the weak solutions of problem (1.1) are the critical points of $\mathcal{I}_\mu$.

Definition 3.1. *We say that* $\mathcal{I}_\mu$ *fulfills the Cerami condition at the level* $c \in \mathbb{R}$ *if any sequence* $\{v_j\} \subset X$ *satisfying*

$$\{\mathcal{I}_\mu(v_j)\} \ \ \textit{is bounded} \ \ \textit{and} \ \ (1+\|v\|)\|\mathcal{I}_\mu^{'}(v_j)\|_{X^*} \to 0 \ \ \textit{as} \ \ j \to \infty, \tag{3.1}$$

possesses a convergent subsequence in X.

Lemma 3.1. *Suppose that* $(M_1)-(M_2)$, $(G_1)-(G_3)$ *and* $(H_1)-(H_2)$ *hold. Then, for any* $\mu > 0$ *the functional* $\mathcal{I}_\mu$ *fulfills the Cerami condition at the level* $c \in \mathbb{R}$.

Proof. Let $\{v_j\}$ be a Cerami sequence in X fulfilling (3.1), which implies that

$$\sup|\mathcal{I}_\mu(v_j)| < C_1 \ \text{ and } \ \langle \mathcal{I}_\mu^{'}(v_j), v_j\rangle = o(1), \tag{3.2}$$

where $o(1) \to 0$ as $j \to \infty$.

Step 1. We prove that the sequence $\{v_j\}$ is bounded in X. For this purpose we make use of contradiction, it is supposed that $\|v_j\| \to \infty$ as $j \to \infty$. We define a new sequence $\{z_j\}$ to be denoted by $z_j = v_j/\|v_j\|$. Then, it is clear that $\{z_j\} \subset X$ and $\|z_j\| = 1$. Without loss of generality, according to the reflexivity of the space X, there exists a sub-sequence which is still expressed as $\{z_j\}$, such that

$$z_j \rightharpoonup z \text{ in } \ X, \qquad z_j \to z \ \text{ in } \ L_a^{r(\cdot)}(\mathbb{R}^N), \qquad z_j(x) \to z(x) \ \text{ a.e. in } \ \mathbb{R}^N \tag{3.3}$$

for $1 < r(x) < p_s^*(x)$ thanks to Lemma 2.6.

From Proposition 2.2, (M_1), (M_2), (G_2) and (3.2), we have

$$\begin{aligned}
C_1 + \frac{1}{\lambda}o(1) \geq & \mathcal{I}_\mu(v_j) - \frac{1}{\lambda}\langle \mathcal{I}_\mu^{'}(v_j), v_j\rangle \\
= & \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) - \frac{1}{\lambda}M\Big(\delta_{p(\cdot)}(v_j)\Big)\iint_{\mathbb{R}^{2N}} \frac{|v_j(x)-v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\
& - \mu\int_{\mathbb{R}^N}\left(G(x,v_j) - \frac{1}{\lambda}g(x,v_j)v_j\right)dx + \int_{\mathbb{R}^N}\left(\frac{1}{\overline{p}(x)} - \frac{1}{\lambda}\right)|v_j|^{\overline{p}(x)}dx \\
\geq & \frac{1}{\theta}M\Big(\delta_{p(\cdot)}(v_j)\Big)\delta_{p(\cdot)}(v_j) - \frac{1}{\lambda}M\Big(\delta_{p(\cdot)}(v_j)\Big)\iint_{\mathbb{R}^{2N}} \frac{|v_j(x)-v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\
& - \mu\int_{\mathbb{R}^N}\left(G(x,v_j) - \frac{1}{\lambda}g(x,v_j)v_j\right)dx + \int_{\mathbb{R}^N}\left(\frac{1}{\overline{p}(x)} - \frac{1}{\lambda}\right)|v_j|^{\overline{p}(x)}dx \\
\geq & \left(\frac{1}{\theta p^+} - \frac{1}{\lambda}\right)M\Big(\delta_{p(\cdot)}(v_j)\Big)\iint_{\mathbb{R}^{2N}} \frac{|v_j(x)-v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\
& + \int_{\mathbb{R}^N}\left(\frac{1}{p^+} - \frac{1}{\lambda}\right)|v_j|^{p(x)}dx - \frac{\mu}{\lambda}\int_{\mathbb{R}^N} a(x)|v_j|^{p^-}dx \\
\geq & \left(\frac{1}{\theta p^+} - \frac{1}{\lambda}\right)\min\{1,m\}\|v_j\|^{p^-} - \frac{\mu}{\lambda}\|v_j\|_{p^-,a}^{p^-}.
\end{aligned} \tag{3.4}$$

Dividing both sides of this inequality (3.4) by $\|v_j\|^{p^-}$ and let $j \to \infty$, we have

$$1 \leq \frac{\mu\theta p^+}{(\lambda - \theta p^+)\min\{1,m\}}\|z\|_{p^-,a}^{p^-}. \tag{3.5}$$

On the basis of (3.5), we deduce that $z \neq 0$. Set $\Omega_1 = \{x \in \mathbb{R}^N : z(x) \neq 0\}$.

On the other hand, by (M_2), Proposition 2.2 and (3.2), we have

$$\begin{aligned}
C_1 \geq \mathcal{I}_\mu(v_j) &= \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)}|v_j|^{\overline{p}(x)}dx - \mu \int_{\mathbb{R}^N} G(x, v_j)dx \\
&\geq m\delta_{p(\cdot)}(v_j) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)}|v_j|^{\overline{p}(x)}dx - \mu \int_{\mathbb{R}^N} G(x, v_j)dx \\
&\geq \frac{m}{p^+} \iint_{\mathbb{R}^{2N}} \frac{|v_j(x) - v_j(y)|^{p(x,y)}}{|x - y|^{N+p(x,y)s(x,y)}}dxdy + \frac{1}{p^+} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx - \mu \int_{\mathbb{R}^N} G(x, v_j)dx \\
&\geq \frac{\min\{1, m\}}{p^+}\|v_j\|^{p^-} - \mu \int_{\mathbb{R}^N} G(x, v_j)dx.
\end{aligned} \tag{3.6}$$

Since $\|v_j\| \to \infty$ as $j \to \infty$, we get

$$\int_{\mathbb{R}^N} G(x, v_j) \geq \frac{\min\{1, m\}}{p^+}\|v_j\|^{p^-} - \frac{C_1}{\mu} \to \infty \tag{3.7}$$

as $j \to \infty$. Moreover,

$$\begin{aligned}
\mathcal{I}_\mu(v_j) &= \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)}|v_j|^{\overline{p}(x)}dx - \mu \int_{\mathbb{R}^N} G(x, v_j)dx \\
&\leq \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx - \mu \int_{\mathbb{R}^N} G(x, v_j)dx.
\end{aligned}$$

Thus

$$\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx \geq \mu \int_{\mathbb{R}^N} G(x, v_j)dx + \mathcal{I}_\mu(v_j). \tag{3.8}$$

It follows from (G_3) that there exists $r_0 > 1$ such that $G(x, t) > |t|^{\theta p^+}$ for all $x \in \mathbb{R}^N$ and $|t| > r_0$. From (G_1) and g is a *Carathéodory* function, we know that there exists C_2 such that $|G(x, t)| < C_2$ for all $(x, t) \in \mathbb{R}^N \times [-r_0, r_0]$. Hence, there is a real number C_0 such that $G(x, t) \geq C_0$ for all $(x, t) \in \mathbb{R}^N \times \mathbb{R}$. So,

$$\frac{G(x, v_j) - C_0}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} \geq 0$$

for any $x \in \mathbb{R}^N$ and for all $j \in \mathbb{N}$. It follows from (3.3) that $|v_j(x)| = |z_j(x)|\|v_j\| \to \infty$ as $j \to \infty$ for any $x \in \Omega_1$. In addition, by (G_3), for all $x \in \Omega_1$, we obtain

$$\begin{aligned}
&\lim_{j\to\infty} \frac{G(x, v_j)}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} \\
&\geq \lim_{j\to\infty} \frac{G(x, v_j)}{\widetilde{M}(1)\left(1 + \big(\delta_{p(\cdot)}(v_j)\big)^\theta\right) + \frac{1}{p^-} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} \\
&\geq \lim_{j\to\infty} \frac{G(x, v_j)}{\widetilde{M}(1)\left(1 + \frac{1}{(p^-)^\theta}\right)\|v_j\|^{\theta p^+} + \frac{1}{p^-}\|v_j\|^{\theta p^+}} \\
&= \lim_{j\to\infty} \frac{G(x, v_j)}{\left(\widetilde{M}(1)\left(1 + \frac{1}{(p^-)^\theta}\right) + \frac{1}{p^-}\right)|v_j|^{\theta p^+}}|z_j(x)|^{\theta p^+} \\
&= \infty,
\end{aligned} \tag{3.9}$$

where $\widetilde{M}(\iota) \leq \widetilde{M}(1+\iota^\theta)$ for all $\iota \in \mathbb{R}^+$ Because if $0<\iota<1$, then $\widetilde{M}(\iota)=\int_0^\iota M(t)dt \leq \widetilde{M}(1)$, and if $\iota>1$, then $\widetilde{M}(\iota) \leq \widetilde{M}(1)\iota^\theta$. According to (3.7)-(3.9) and Fatou's lemma, we obtain

$$
\begin{aligned}
\frac{1}{\mu} &= \liminf_{j\to\infty} \frac{\int_{\mathbb{R}^N} G(x,v_j)dx}{\mu \int_{\mathbb{R}^N} G(x,v_j)dx + \mathcal{I}_\mu(v_j)} \geq \liminf_{j\to\infty} \int_{\mathbb{R}^N} \frac{G(x,v_j)}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-}\int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} dx \\
&\geq \liminf_{j\to\infty} \int_{\Omega_1} \frac{G(x,v_j)}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-}\int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} dx - \limsup_{j\to\infty} \int_{\Omega_1} \frac{C_0}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-}\int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} dx \\
&= \liminf_{j\to\infty} \int_{\Omega_1} \frac{G(x,v_j)-C_0}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-}\int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} dx \qquad (3.10)\\
&\geq \int_{\Omega_1} \liminf_{j\to\infty} \frac{G(x,v_j)-C_0}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-}\int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} dx \\
&= \int_{\Omega_1} \liminf_{j\to\infty} \frac{G(x,v_j)}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-}\int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} dx - \int_{\Omega_1} \limsup_{j\to\infty} \frac{C_0}{\widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \frac{1}{p^-}\int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)}dx} dx \\
&= \infty,
\end{aligned}
$$

which is a contradiction. So, $|\Omega_1|=0$, this is impossible due to (3.5). Therefore, $\{v_j\}$ is bounded in X.

Step 2. We will show that $\{v_j\}$ converges strongly in X. It follows from Lemma 2.6, combined with the reflexivity of X, that there exists a subsequence, still denoted by $\{v_j\}$, and $v \in X$ such that

$$
v_j \rightharpoonup v \ \text{ in } \ X, \qquad v_j \to v \ \text{ in } \ L_a^{q(\cdot)}(\mathbb{R}^N), \qquad v_j(x) \to v(x) \ \text{ a.e. in } \ \mathbb{R}^N \tag{3.11}
$$

for any $q \in C_+(\mathbb{R}^N)$, with $1<q(x)<p_s^*(x)$ for $x \in \mathbb{R}^N$. Using (G_1), we get

$$
\begin{aligned}
\Big| \int_{\mathbb{R}^N} g(x,v_j)(v_j-v)dx \Big| &\leq \int_{\mathbb{R}^N} a(x)|v_j|^{q(x)-1}|v_j-v|dx \\
&\leq 2^{q^+-1}\left(\int_{\mathbb{R}^N} a(x)|v_j-v|^{q(x)}dx + \int_{\mathbb{R}^N} a(x)|v|^{q(x)-1}|v_j-v|dx \right).
\end{aligned}
$$

It follows from (3.11) that $\int_{\mathbb{R}^N} a(x)|v|^{q(x)-1}|v_j-v|dx \to 0$ as $j\to\infty$. Again by Lemma 2.5 and strong convergence of sequences, we also obtain $\int_{\mathbb{R}^N} a(x)|v_j-v|^{q(x)}dx \to 0$ as $j \to \infty$. So,

$$
\lim_{j\to\infty} \int_\Omega g(x,v_j)(v_j-v)dx = 0. \tag{3.12}
$$

In view of (3.11), we get

$$
\langle \mathcal{I}_\mu'(v_j), v_j - v \rangle \to 0. \tag{3.13}
$$

Thus, we get

$$
\begin{aligned}
Q_j = M\Big(\delta_{p(\cdot)}(v_j)\Big) \iint_{\mathbb{R}^{2N}} & \frac{|v_j(x)-v_j(y)|^{p(x,y)-2}\left(v_j(x)-v_j(y)\right)\left((v_j(x)-v_j(y))-(v(x)-v(y))\right)}{|x-y|^{N+p(x,y)s(x,y)}}dxdy \\
&+ \int_{\mathbb{R}^N} |v_j(x)|^{\overline{p}(x)-2}v_j(x)(v_j(x)-v(x))dx \to 0 \ \text{ as } j\to\infty.
\end{aligned}
$$

By (M_1) and (2.1), we have

$$Q_j \geq m\langle \mathcal{L}(v_j), v_j - v\rangle.$$

It follows from $\lim_{j\to\infty} Q_j = 0$ and (3.11) that

$$v_j \rightharpoonup v \in X, \quad \limsup_{j\to\infty}\langle \mathcal{L}(v_j) - \mathcal{L}(v), v_j - v\rangle \leq 0, \quad \mathcal{L} \text{ is a mapping of type } (S_+),$$

which imply that $v_j \to v$ in X, thanks to Proposition 2.3. Consequently, $\mathcal{I}_\mu$ satisfies the Cerami condition. □

Lemma 3.2. *Suppose that $(M_1) - (M_3)$, $(G_1) - (G_2)$, (G_5), and $(H_1) - (H_2)$ hold. Then, for any $\mu > 0$ the functional $\mathcal{I}_\mu$ fulfills the Cerami condition at the level $c \in \mathbb{R}$.*

Proof. Take $\{v_j\}$ be a Cerami sequence in X fulfilling (3.1). Then, (3.2) holds. Thanks to Lemma (3.1), we only need to show that $\{v_j\}$ is bounded in X. For this, discussing by contradiction, it is supposed that $\|v_j\| > 1$ and $\|v_j\| \to \infty$ as $j \to \infty$, define $z_j = v_j/\|v_j\|$. Then, up to a subsequence, still denoted by z_j, we get

$$z_j \rightharpoonup z \text{ in } X, \qquad z_j \to z \text{ in } L_a^{q(\cdot)}(\mathbb{R}^N), \qquad z_j(x) \to z(x) \text{ a.e. in } \mathbb{R}^N. \tag{3.14}$$

for $1 < q(x) < p_s^*(x)$ thanks to Lemma 2.6. Take $\Omega_1 = \{x \in \mathbb{R}^N : z(x) \neq 0\}$. According to the same argument as in Lemma 3.1, we know that $|\Omega_1| = 0$, thereby, $z(x) = 0$ for almost all $x \in \mathbb{R}^N$. Because $I_\mu(tv_j)$ is continuous in $t \in [0,1]$, for each $j \in \mathbb{N}$, there exists $t_j \in [0,1]$ such that $I_\mu(t_jv_j) := \max_{t\in[0,1]} I_\mu(tv_j)$. Suppose that there exists a positive sequence $\{\zeta_n\}$ of real numbers such that $\lim_{n\to\infty}\zeta_n = \infty$ and $\zeta_n > 1$ for all n. Clearly, we can get $\|\zeta_n z_j\| = \zeta_n > 1$ for all n and j. Let n be fixed. On the basis of the continuity of the Nemytskii operator, we have that $G(x, \zeta_n z_j) \to 0$ in $L^1(\mathbb{R}^N)$ due to $z_j \to 0$ in $L_a^{q(\cdot)}(\mathbb{R}^N)$ as $j \to \infty$. Therefore,

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} G(x, \zeta_n z_j) = 0. \tag{3.15}$$

According to $\|v_j\| \to \infty$ as $j \to \infty$, we have $\|v_j\| = \zeta_n > 1$ for enough large j. Hence, from (3.15) and (M_2), we obtain

$$\begin{aligned}
\mathcal{I}_\mu(t_jv_j) &\geq \mathcal{I}_\mu\left(\frac{\zeta_n}{\|v_j\|}v_j\right) = \mathcal{I}_\mu(\zeta_n z_j) \\
&= \widetilde{M}\Big(\delta_{p(\cdot)}(\zeta_n z_j)\Big) + \int_{\mathbb{R}^N}\frac{1}{\overline{p}(x)}|\zeta_n z_j|^{\overline{p}(x)}dx - \mu\int_{\mathbb{R}^N} G(x, \zeta_n z_j)dx \\
&\geq m\delta_{p(\cdot)}(\zeta_n z_j) + \int_{\mathbb{R}^N}\frac{1}{\overline{p}(x)}|\zeta_n z_j|^{\overline{p}(x)}dx - \mu\int_{\mathbb{R}^N} G(x, \zeta_n z_j)dx \\
&\geq \frac{m}{p^+}\iint_{\mathbb{R}^{2N}}\frac{|\zeta_n z_j(x) - \zeta_n z_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy + \frac{1}{p^+}\int_{\mathbb{R}^N}|\zeta_n z_j|^{\overline{p}(x)}dx - \mu\int_{\mathbb{R}^N} G(x, \zeta_n z_j)dx \\
&\geq \frac{\min\{1,m\}}{p^+}\|\zeta_n z_j\|^{p^-} - \mu\int_{\mathbb{R}^N} G(x, \zeta_n z_j)dx \\
&\geq \frac{\min\{1,m\}}{2p^+}\zeta_n^{p^-}
\end{aligned}$$

as $j \to \infty$. In what follows, taking $j \to \infty$ and $n \to \infty$, we have

$$\lim_{j\to\infty} \mathcal{I}_\mu(t_j v_j) = \infty. \tag{3.16}$$

It follows from (3.2) and $\mathcal{I}_\mu(0) = 0$ that $t_j \in (0,1)$ and $\langle \mathcal{I}_\mu'(t_j v_j), t_j v_j\rangle = 0$. So, from (M_3) and (G_5), for all sufficiently large j, we obtain

$$\begin{aligned}
\frac{1}{\lambda}\mathcal{I}_\mu(t_j v_j) &= \frac{1}{\lambda}\mathcal{I}_\mu(t_j v_j) - \frac{1}{p^+\lambda}\langle \mathcal{I}_\mu'(t_j v_j), t_j v_j\rangle + o(1)\\
&= \frac{1}{\lambda}\widetilde{M}\Big(\delta_{p(\cdot)}(t_j v_j)\Big) - \frac{1}{p^+\lambda}M\Big(\delta_{p(\cdot)}(t_j v_j)\Big)\iint_{\mathbb{R}^{2N}} \frac{|t_j v_j(x) - t_j v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy\\
&\quad - \mu\int_{\mathbb{R}^N}\left(\frac{1}{\lambda}G(x,t_j v_j) - \frac{1}{p^+\lambda}g(x,t_j v_j)t_j v_j\right)dx + \frac{1}{\lambda}\int_{\mathbb{R}^N}\frac{1}{\overline{p}(x)}|t_j v_j|^{\overline{p}(x)}dx\\
&\quad - \frac{1}{p^+\lambda}\int_{\mathbb{R}^N}|t_j v_j|^{\overline{p}(x)}dx + o(1)\\
&\le \frac{1}{\lambda}\widetilde{M}\Big(\delta_{p(\cdot)}(t_j v_j)\Big) - \frac{1}{p^+\lambda}M\Big(\delta_{p(\cdot)}(t_j v_j)\Big)\iint_{\mathbb{R}^{2N}} \frac{|t_j v_j(x) - t_j v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy\\
&\quad - \frac{\mu}{p^+\lambda}\int_{\mathbb{R}^N}\mathcal{G}(x,t_j v_j)dx + \frac{1}{\lambda}\int_{\mathbb{R}^N}\frac{1}{\overline{p}(x)}|t_j v_j|^{\overline{p}(x)}dx - \frac{1}{p^+\lambda}\int_{\mathbb{R}^N}|t_j v_j|^{\overline{p}(x)}dx + o(1)\\
&\le \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) - \frac{1}{p^+}M\Big(\delta_{p(\cdot)}(v_j)\Big)\iint_{\mathbb{R}^{2N}} \frac{|v_j(x) - v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy\\
&\quad - \frac{\mu}{p^+}\int_{\mathbb{R}^N}\mathcal{G}(x,v_j)dx + \int_{\mathbb{R}^N}\frac{1}{\overline{p}(x)}|v_j|^{\overline{p}(x)}dx - \frac{1}{p^+}\int_{\mathbb{R}^N}|v_j|^{\overline{p}(x)}dx + o(1)\\
&\le \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) - \frac{1}{p^+}M\Big(\delta_{p(\cdot)}(v_j)\Big)\iint_{\mathbb{R}^{2N}} \frac{|v_j(x) - v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy\\
&\quad - \mu\int_{\mathbb{R}^N}\left(G(x,v_j) - \frac{1}{p^+}g(x,v_j)v_j\right)dx + \int_{\mathbb{R}^N}\frac{1}{\overline{p}(x)}|v_j|^{\overline{p}(x)}dx - \frac{1}{p^+}\int_{\mathbb{R}^N}|v_j|^{\overline{p}(x)}dx + o(1)\\
&= \frac{1}{\lambda}\mathcal{I}_\mu(v_j) - \frac{1}{p^+}\langle \mathcal{I}_\mu'(v_j), v_j\rangle + o(1)\\
&\le C_1
\end{aligned}$$

as $j \to \infty$, which contradicts (3.16). The proof is completed. □

4 Proof of Theorems 1.1-1.2

The proof of Theorems 1.1-1.2 are based on the application of an abstract critical point result for an energy functional fulfilling the Cerami condition, which can be found in [19]. For the reader's convenience, we now state the abstract critical point theorem.

Theorem 4.1. *Let X be a real Banach space and consider two locally Lispschitz continuous functionals $\mathcal{A}, \mathcal{B} : X \to \mathbb{R}$. Assume that $\mathcal{A}$ is bounded from below and $\mathcal{A}(0) = \mathcal{B}(0) = 0$. Set $\eta > 0$ be fixed, and it is supposed that for each*

$$\mu \in \Gamma_0 := \left(0, \frac{\eta}{\sup_{v\in\mathcal{A}^{-1}((-\infty,\eta))}\mathcal{B}(v)}\right),$$

the functional $\mathcal{I}_\mu = \mathcal{A} - \mu\mathcal{B}$ fulfills the Cerami condition for any $\mu \in \Gamma_0$ and is unbounded from below. Then, for any $\mu \in \Gamma_0$, the functional $\mathcal{I}_\mu$ has two distinct critical points.

Proof of Theorem 1.1. It is clear that $\mathcal{A}$ is bounded from below and $\mathcal{A}(0) = \mathcal{B}(0) = 0$. According to (G_3), for any $C_3 > 0$, there exists a constant $C_4 > 0$ such that

$$F(x,t) \geq C_3|t|^{\theta p^+} \tag{4.1}$$

for $|t| > C_4$ and for almost all $x \in \mathbb{R}^N$. Let $v \in X\backslash\{0\}$. Then, for enough large $t > 1$, by (4.1) we have

$$\begin{aligned}\mathcal{I}_\mu(tv) &= \widetilde{M}\Big(\delta_{p(\cdot)}(tv)\Big) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)}|tv|^{\overline{p}(x)}dx - \mu\int_{\mathbb{R}^N} G(x,tv)dx \\ &\leq \widetilde{M}(1)\left(\delta_{p(\cdot)}(tv)\right)^\theta + \frac{1}{p^-}\int_{\mathbb{R}^N} |tv|^{\overline{p}(x)}dx - \mu\int_{\mathbb{R}^N} G(x,tv)dx \\ &\leq |t|^{\theta p^+}\left(\frac{\widetilde{M}(1)}{(p^-)^\theta}\left(\delta_{p(\cdot)}(v)\right)^\theta + \frac{1}{p^-}\int_{\mathbb{R}^N}|v|^{\overline{p}(x)}dx - \mu C_3\int_{\mathbb{R}^N}|v|^{\theta p^+}\right)dx,\end{aligned} \tag{4.2}$$

where θ is given in (M_1), thanks to $\widetilde{M}(\iota) \leq \widetilde{M}(1)\iota^\theta$ for $\iota \geq 1$. If C_3 is large enough, then we infer that $\mathcal{I}_\mu(tv) \to -\infty$ as $t \to \infty$. Therefore, $\mathcal{I}_\mu$ is unbounded from below. It follows from (G_1) and Lemma 2.7 that

$$\begin{aligned}\mathcal{B}(v) = \int_{\mathbb{R}^N} G(x,v)dx &\leq \int_{\mathbb{R}^N}\frac{a(x)}{q(x)}|v(x)|^{q(x)}dx \\ &\leq \frac{1}{q^-}\varrho_{q,a}(v) \leq \frac{1}{q^-}C_{q,a}\|v\|^{\overline{q}} \leq \frac{1}{q^-}C_{q,a}\max\left\{\|v\|^{q^+}, \|v\|^{q^-}\right\},\end{aligned} \tag{4.3}$$

where $C_{q,a}$ is given in Lemma 2.7. Take $\eta = 1$. Then for each $v \in \mathcal{A}^{-1}((-\infty,1))$, we have

$$\begin{aligned}\|v\| &\leq \max\left\{\left(\frac{p^+}{\min\{1,m\}}\mathcal{A}(v)\right)^{\frac{1}{p^+}}, \left(\frac{p^+}{\min\{1,m\}}\mathcal{A}(v)\right)^{\frac{1}{p^-}}\right\} \\ &\leq \max\left\{\left(\frac{p^+}{\min\{1,m\}}\right)^{\frac{1}{p^+}}, \left(\frac{\theta p^+}{\min\{1,m\}}\right)^{\frac{1}{p^-}}\right\} = \left(\frac{p^+}{\min\{1,m\}}\right)^{\frac{1}{p^-}}.\end{aligned}$$

Denote

$$\mu^* = \left(\frac{1}{q^-}C_{q,a}\left(\frac{p^+}{\min\{1,m\}}\right)^{\frac{q^+}{p^-}}\right)^{-1}.$$

Considering (4.3), we get that

$$\sup_{v\in\mathcal{A}^{-1}((-\infty,1))}\mathcal{B}(v) \leq \frac{1}{q^-}C_{q,a}\left(\frac{p^+}{\min\{1,m\}}\right)^{\frac{q^+}{p^-}} = \frac{1}{\mu^*} < \frac{1}{\mu}.$$

By Lemma 3.1, we know that the functional $\mathcal{I}_\mu$ fulfills Cerami condition for any $\mu > 0$. Thus, all conditions of Theorem 4.1 are satisfied. So, for any $\mu \in (0,\mu^*) \subset \Gamma_0$, problem (1.1) admits two distinct weak solutions in X. This concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. By Lemma 3.2, obviously, this theorem holds. □

5 Proof of Theorems 1.3-1.4 and Theorems 1.5-1.6.

Next, we will use the following classic mountain pass theorem to prove our second main result Theorem 1.3.

Theorem 5.1. ([37]) Let X be a real Banach space and $\mathcal{I}_\mu \in C^1(X, \mathbb{R})$ with $\mathcal{I}_\mu(0) = 0$. Assume that $\mathcal{I}_\mu$ satisfies the Cerami condition and

(i) there exist $\alpha, \kappa > 0$ such that $\mathcal{I}_\mu(v) \geq \alpha$ for all $v \in X_0$, $\|v\| = \kappa$;

(ii) there exists $\omega \in X_0$ satisfying $\|\omega\| > \kappa$ such that $\mathcal{I}_\mu(\omega) < 0$.

Define

$$\Gamma = \{\gamma \in C^1([0,1], X_0) : \gamma(0) = 0, \gamma(1) = \omega\}.$$

Then

$$c_\mu = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}_\mu(\gamma(t)) \geq \alpha$$

is a critical value of $\mathcal{I}_\mu$.

Proof of Theorem 1.3. Obviously, according to an argument similar to (4.2) of Theorem 1.1. We know that (ii) in the mountain pass geometry is satisfied for any $\mu > 0$. So it is sufficient to prove the geometry (i). Indeed, let $v \in X$ and $\mu > 0$ be such that

$$\|v\| = \kappa \in \left(0, \min\left\{1, 1/C_{q,a}, \left(\frac{q^- \min\{1,m\}}{2\mu C_{q,a} p^+}\right)^{q^- - p^+}\right\}\right)$$

with $C_{q,a}$ given in Lemma 2.7. From (M_2), (G_1), Proposition 2.2 and Lemma 2.7, we have

$$\begin{aligned}
\mathcal{I}_\mu(v_j) &\geq \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)} |v_j|^{\overline{p}(x)} dx - \mu \int_{\mathbb{R}^N} \frac{a(x)}{q(x)} |v(x)|^{q(x)} dx \\
&\geq m\delta_{p(\cdot)}(v_j) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)} |v_j|^{\overline{p}(x)} dx - \mu \frac{1}{q^-} \varrho_{q,a}(v) \\
&\geq \frac{m}{p^+} \iint_{\mathbb{R}^{2N}} \frac{|v_j(x) - v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy + \frac{1}{p^+} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)} dx - \mu \frac{1}{q^-} C_{q,a} \|v\|^{\overline{q}} \\
&\geq \frac{\min\{1,m\}}{p^+} \|v_j\|^{p^+} - \mu \frac{1}{q^-} C_{q,a} \|v\|^{q^-} \\
&= \frac{\min\{1,m\}}{2p^+} \kappa^{p^+} = \alpha > 0.
\end{aligned} \tag{5.1}$$

Here we used the fact $q^- > p^+$. Therefore, the condition (i) is fulfilled. Again by Lemma 3.1, we know that the functional $\mathcal{I}_\mu$ fulfills Cerami condition for any $\mu > 0$. Moreover, observe that $\mathcal{I}_\mu(0) = 0$, from (G_1) we see that v is a strict local minimum for $\mathcal{I}_\mu(v)$. So all conditions of Theorem 5.1 are satisfied. Consequently, problem (1.1) has a nontrivial weak solution for any $\mu > 0$. This ends the proof. □

Proof of Theorem 1.4. Considering Lemma 3.2, it is clear that the proof of this theorem holds. □

At the end of the paper, in order to prove Theorem 1.5, we use the critical point theory in symmetric form, i.e., the fountain theorem.

To do this, since X is a reflexive and separable real Banach space, there exist $\{w_i\} \subset X$ and $\{w_i^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{w_i : i \in \mathbb{N}^+\}}, \qquad X^* = \overline{\operatorname{span}\{w_i^* : i \in \mathbb{N}^+\}}$$

and

$$X^i = \operatorname{span}\{w_i\}, \qquad Y_j = \bigoplus_{i=1}^{j} X^i, \qquad Z_j = \overline{\bigoplus_{i=j}^{\infty} X^i}, \quad j = 1, 2, \ldots.$$

Then, we can introduce the following version of the Fountain Theorem.

Theorem 5.2. ([37]) Consider an even functional $\mathcal{I}_\mu \in C^1(X, \mathbb{R})$. Assume that for every $j \in \mathbb{N}$, there exist $\rho_j > \gamma_j > 0$ such that

(I_1) $a_j := \max\limits_{v \in Y_j, \|v\|_X = \rho_j} \mathcal{I}_\mu(v) \leq 0$;

(I_2) $b_j := \inf\limits_{v \in Z_j, \|v\|_X = \gamma_j} \mathcal{I}_\mu(v) \to \infty, \ j \to \infty$;

(I_3) $\mathcal{I}_\mu$ fulfills the Cerami condition for every $c > 0$.

Then $\mathcal{I}_\mu$ has an unbounded sequence of critical values.

Lemma 5.1. *Suppose that $(H_1) - (H_2)$ hold. Let $q \in C_+(\overline{\Omega})$, with $1 < q(x) < p_s^*(x)$ for any $x \in \mathbb{R}^N$, and denote*

$$\xi_j := \sup \left\{ \int_{\mathbb{R}^N} a(x)|v(x)|^{q(x)} dx : \quad v \in Z_j, \quad \|v\|_X \leq 1 \right\}, \tag{5.2}$$

where a fulfills (A_1). Then, $\xi_j \to 0$ as $j \to \infty$.

Proof. It follows from the similar argument in [34, Lemma 2.4] that we can get the result of Lemma 5.1. □

Proof of Theorem 1.5. We have noted that the functional $\mathcal{I}_\mu$ fulfills the Cerami condition by Lemma 3.1 and $\mathcal{I}_\mu(v) = \mathcal{I}_\mu(-v)$ by (G_4). In what follows, we only show that $\mathcal{I}_\mu$ satisfies the condition (I_2) of Theorem 5.2 because the condition (I_1) is obviously true which can be obtained a discussion similar to (4.2) of Theorem 1.1 for any $v \in Y_j$ with $\|v\| = 1$. Let $\mu > 0$. According to (G_1), (5.2) and Proposition 2.2, for any $v \in Z_j$ with $\|v\|_X > 1$, we get

$$\begin{aligned}
\mathcal{I}_\mu(v_j) &\geq \widetilde{M}\Big(\delta_{p(\cdot)}(v_j)\Big) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)}|v_j|^{\overline{p}(x)} dx - \mu \int_{\mathbb{R}^N} \frac{a(x)}{q(x)}|v(x)|^{q(x)} dx \\
&\geq m\delta_{p(\cdot)}(v_j) + \int_{\mathbb{R}^N} \frac{1}{\overline{p}(x)}|v_j|^{\overline{p}(x)} dx - \mu \frac{1}{q^-}\varrho_{q,a}(v) \\
&\geq \frac{m}{p^+} \iint_{\mathbb{R}^{2N}} \frac{|v_j(x) - v_j(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy + \frac{1}{p^+} \int_{\mathbb{R}^N} |v_j|^{\overline{p}(x)} dx - \mu \frac{1}{q^-}\xi_j \|v\|^{\overline{q}} \\
&\geq \frac{\min\{1, m\}}{p^+}\|v_j\|^{p^+} - \mu \frac{1}{q^-}\xi_j \|v\|^{q^+} \\
&= \|v_j\|^{p^+} \left(\frac{\min\{1, m\}}{p^+} - \mu \frac{1}{q^-}\xi_j \|v\|^{q^+ - p^+} \right).
\end{aligned} \tag{5.3}$$

Define

$$\gamma_j := \left(\frac{q^- \min\{1, m\}}{2p^+ \mu \xi_j} \right)^{\frac{1}{q^+ - p^-}},$$

then since $\gamma_j \to \infty$ as $j \to \infty$ by Lemma 5.1 and the fact that $q^+ > p^-$ by (G_1), we can assume that $\gamma_j > 1$ for j even larger. Hence, by (5.3) applied for any $v \in Z_j$ with $\|v\|_X = \gamma_j$, we get

$$\mathcal{I}_\mu(v) \geq \frac{\min\{1, m\}}{2p^+} \gamma_j^{p^+} \to \infty$$

as $j \to \infty$, by Lemma 5.1. Hence, the condition (I_2) of Theorem 5.2 is fulfilled. The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.6. It follows from Lemma 3.2 that the proof of this theorem is complete. □

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