The Probability of Events for Parabolic Equations

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August 4, 2020

Abstract In this short paper, we focus on the blowup phenomenon of stochastic parabolic equations. We first discuss the probability of the event that the solutions keep positive. Then, the blowup phenomenon in the whole space is considered. The probability of the event that the solutions blow up in finite time is given. Lastly, we obtain the probability of the event that blowup time of stochastic parabolic equations larger than or less than the deterministic case.

Keywords: Blowup; Stochastic heat equation; Impact of noise.

MSC (2010): 35K20, 60H15, 60H40.

1 Introduction

For a deterministic partial differential equation, when we add a noise on it, we first want to know how to change about the solution, that is, the effect of noise. More precisely, if the solutions of deterministic parabolic equations keep positive, what is the probability that the solutions keep positive for the stochastic case ? In this paper, we will give a partially positive answer. Similarly, for the blowup phenomenon, we want to know the probability of the event that the solutions blow up in finite time.

We firstly recall some known results of stochastic partial differential equations (SPDEs). In this paper, we only focus on the stochastic parabolic equations. It is known that the existence and uniqueness of global solutions to SPDEs can be established under appropriate conditions (see [2]). For the finite time blowup phenomenon of stochastic parabolic equations, we first consider the case on a bounded domain. Consider the following equation

$$\begin{cases} du(x,t) = (\Delta u + f(u))dt + \sigma(u)dB_t, & t > 0, \quad x \in D, \\ u(x,t)|_{t=0} = u_0(x) \geq 0, & \qquad\quad\ x \in D, \\ u(x,t) = 0, & t > 0, \quad x \in \partial D, \end{cases} \tag{1.1}$$

where B_t is a one-dimensional Brownian motion. Da Prato and Zabczyk [19] studied the existence of global solutions of (1.1) with constant σ. Manthey and Zausinger [16] considered (1.1), with σ satisfying a global Lipschitz condition. Dozzi and López-Mimbela [6] studied (1.1) with $\sigma(u) = u$ and proved that if $f(u) \geq u^{1+\alpha}$ ($\alpha > 0$) and the initial data is large enough, the solution will blow up in finite time with a positive probability, and that if $f(u) \leq u^{1+\beta}$ (β is a certain positive constant) and the initial data is small enough, the solution will exist globally almost surely (also see [18]). When σ does not satisfy the global Lipschitz condition, Chow [3, 4] obtained the finite time blowup phenomenon. Lv and Duan [13] described the competition between the nonlinear term and the noise term for equation (1.1). Bao and Yuan [1], and Li et al.[12] obtained the existence of local solutions of (1.1) with jump process and Lévy process, respectively, also see [21]. For more details in blowup phenomenon of stochastic parabolic equations, see [5, 14, 17] for details.

We remark that the method used to prove the finite time blowup of solutions on a bounded domain is the stochastic Kaplan's first eigenvalue method. In order to make sure the inner product (u, ϕ) positive, the authors firstly proved the solutions of (1.1) keep positive under some assumptions, see [1, 3, 4, 12, 13]. The method used to prove the positivity of solutions is that the negative part is zero. The main difficulty is to choose suitable test functions. In the present paper, we will introduce another method to prove the positivity of solutions. We also remark that, in our paper [15] a new method (stochastic concavity method) is introduced to prove the solutions blow up in finite time. The advantage of this method is that we need not the positivity of solutions.

In former papers [7, 15], the blowup phenomenon in the whole space is considered in the form of $\mathbb{E}u^2(x,t)$. That is to say, the moment of the solutions will blow up in finite time. From the point of probability theory, we want to know the probability of event that the solutions blow up in finite time. In present paper, we study the parabolic equations with linear multiplicative noise and give the probability of the event.

On the other hand, we remark that the existence of finite time blowup solution was obtained by Dozzi and López-Mimbela [6]. But the estimate of blowup time is no result. This is our second aim. We will estimate the probability that blowup time of stochastic parabolic equations large than or less than the deterministic case.

The advantage of linear multiplicative noise is that we can change stochastic parabolic equations into random parabolic equations. And then we can use the comparison principle and the results of deterministic case to get the results of the stochastic case.

Throughout this paper, we write C as a general positive constant and C_i, $i = 1, 2, \cdots$ as concrete positive constants.

2 The impact of additive noises

In this section, we consider the impact of additive noise on parabolic equations. Our aim is to find the probability of the event that the solutions keep positive or belong to some interval or are less (larger) than the solutions of the corresponding deterministic case.

We first consider a simple case:

$$\begin{cases} du(x,t) = \Delta u dt + \sigma dB_t, & t>0, \quad x\in\mathbb{R}^d, \\ u(x,t)|_{t=0} = u_0(x) \gneqq 0, & \qquad\quad\ x\in\mathbb{R}^d, \end{cases} \tag{2.1}$$

where $\sigma > 0$, and B_t is a one-dimensional Brownian motion. A mild solution to (2.1) in sense of Walsh [20] is any u, which is adapted to the filtration generated by the white noise and satisfies the following evolution equation

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s)\sigma dy dB_s, \tag{2.2}$$

where $K(x,t)$ is the heat kernel of Laplacian operator:

$$K(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4t}\right)$$

satisfying

$$\left(\frac{\partial}{\partial t} - \Delta\right) K(x,t) = 0 \quad \text{for} \quad (x,t) \neq (0,0).$$

Due to the properties of heat kernel K, we have

$$u(x,t) = \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy + \sigma B_t,$$

which implies that

$$\mathbb{P}(u(x,t)>0) = \mathbb{P}\Big(\frac{B_t}{\sqrt{t}} > \frac{A_t(x)}{\sigma\sqrt{t}}\Big) = 1 - \Phi\Big(\frac{A_t(x)}{\sigma\sqrt{t}}\Big),$$

where $A_t(x) = -\int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy$. Similarly, we have $\mathbb{P}(u(x,t)\le 0) = \Phi\left(\frac{A_t(x)}{\sigma\sqrt{t}}\right)$. Here $\Phi(x)$ is the distribution function of standard normal random variable.

Similarly, for $a,b\in\mathbb{R}$ and $a<b$, we have

$$\begin{aligned} \mathbb{P}(a < u(x,t) \le b) &= \mathbb{P}\left(\frac{a+A_t(x)}{\sigma\sqrt{t}} < \frac{B_t}{\sqrt{t}} \le \frac{b+A_t(x)}{\sigma\sqrt{t}}\right) \\ &= \Phi\left(\frac{b+A_t(x)}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a+A_t(x)}{\sigma\sqrt{t}}\right). \end{aligned}$$

Therefore, we have the following results. Here $C-$ means that the constants is a little lower than C, i.e., $C- > C-\varepsilon$ for some $0<\varepsilon\ll 1$.

Theorem 2.1 *Assume that the initial data $u_0 \geq 0$ is a bounded continuous function. Then the solution of (2.1) will keep positive with the probability $1 - \Phi\left(\frac{A_t(x)}{\sigma\sqrt{t}}\right)$ for any fixed point (x,t). For real numbers $a < b$, we have*

$$\mathbb{P}(a < u(x,t) \leq b) = \Phi\left(\frac{b + A_t(x)}{\sigma\sqrt{t}}\right) - \Phi\left(\frac{a + A_t(x)}{\sigma\sqrt{t}}\right).$$

Moreover, letting $\sigma \to 0$, the probability converges to 1 exponentially, and we get the exact rate. That is to say, the probability

$$\mathbb{P}(u(x,t) > 0) \to 1, \quad \text{as } \sigma \to 0$$

with the exponential rate β, where $\beta > 0$ is any fixed constant. More precisely, we have the following estimate

$$1 - \mathbb{P}(u(x,t) > 0) = O(e^{-\frac{A_t^2(x)}{2\sigma^2 t}}).$$

When $a + A_t(x)$ and $b + A_t(x)$ have the same sign, the event $\{a < u(x,t) \leq b\}$ will become impossible event as $\sigma \to 0$.

Proof. From the above discussion, we only note that

$$1 - \mathbb{P}(u(x,t) > 0) = \Phi\left(\frac{A_t(x)}{\sigma\sqrt{t}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{A_t(x)}{\sigma\sqrt{t}}} e^{-\frac{y^2}{2}} dy,$$

and for any fixed positive constant δ

$$e^{-\frac{A_t^2(x)}{2\sigma^2 t} - } \int_{-\infty}^{\frac{A_t(x)}{\sigma\sqrt{t}}} e^{-\frac{y^2}{2}} dy \to 0.$$

The proof is complete. $\square$

Remark 2.1 *1. The assumption that the initial data $u_0 \geq 0$ can be deleted, that is to say, for additive noises, the events that solutions are positive and negative are possible. In other words, the event that solutions always keep positive is not a certain event. Meanwhile, we note that if $u_0 \geq 0$, then $\mathbb{P}(u(x,t) \geq 0) \geq \frac{1}{2}$.*

2. If $a < u_0 \leq b$, then $a + A_t(x) < 0$ and $b + A_t(x) \geq 0$, then as $\sigma \to 0$, the solutions will belong to $(a, b]$. Furthermore, we have the exact convergence rate.

3. It is easy to see that Theorem 2.1 also holds if the operator Δ is replaced by $-(-\Delta)^\alpha$ with $\alpha \in (0,1)$. More generally, Δ can be replaced by $-(-\Delta)^\alpha + V(\cdot) \cdot \nabla$ if the operator $-(-\Delta)^\alpha + V(\cdot) \cdot \nabla$ has a heat kernel.

4. Theorem 2.1 is similar to large deviation principle, but there is a big difference from the classical theory. We give the description about the event, i.e., how to become to the certain event.

Now, we compare the solutions of stochastic parabolic with the corresponding deterministic case. For simplicity, we consider

$$\begin{cases} du(x,t) = (\Delta u + ku)dt + \sigma(x,t)dB_t, & t > 0, \quad x \in \mathbb{R}^d, \\ u(x,0) = u_0(x) \gneqq 0, & \qquad\quad\; x \in \mathbb{R}^d, \end{cases} \tag{2.3}$$

and

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = \Delta v + kv, & t > 0, \quad x \in \mathbb{R}^d, \\ v(x,0) = u_0(x) \gneqq 0, & \quad\quad\quad x \in \mathbb{R}^d, \end{cases} \tag{2.4}$$

where $k, \sigma > 0$.

Theorem 2.2 *Assume that the initial data u_0 is a bounded continuous function. Denote the event as*

$$\mathcal{A}_t(x) = \{\omega \in \Omega : u(x,t,\omega) \le v(x,t)\},$$

then $\mathbb{P}(\mathcal{A}_t(x)) = \frac{1}{2}$.

Proof. Let $w = u - v$, then w satisfies that

$$\begin{cases} dw(x,t) = (\Delta w + kw)dt + \sigma(x,t)dB_t, & t > 0, \quad x \in \mathbb{R}^d, \\ w(x,0) = 0, & \quad\quad\quad x \in \mathbb{R}^d. \end{cases}$$

Denote $\tilde{w} = e^{-kt}w$, then we have

$$\begin{cases} d\tilde{w}(x,t) = \Delta \tilde{w} dt + e^{-kt}\sigma(x,t)dB_t, & t > 0, \quad x \in \mathbb{R}^d, \\ \tilde{w}(x,0) = 0, & \quad\quad\quad x \in \mathbb{R}^d. \end{cases} \tag{2.5}$$

Then the solution of (2.5) can be expressed as

$$\tilde{w}(x,t) = \int_0^t \left(\int_{\mathbb{R}^d} K(x-y,t-s)e^{-ks}\sigma(y,s)dy \right) dB_s.$$

Let $f(x,s,t) = \int_{\mathbb{R}^d} K(x-y,t-s)e^{-ks}\sigma(y,s)dy$, then $\tilde{w}(x,t) = \int_0^t f(x,s,t)dB_s$. For any fixed $x \in \mathbb{R}^d$ and $t > 0$, we have $\int_0^t f(x,s,t)dB_s$ is a Gaussian process, whose expectation is 0. And we also remark that

$$\mathbb{P}(\mathcal{A}_t(x)) = \mathbb{P}(w \le 0) = \mathbb{P}(\tilde{w} \le 0) = \frac{1}{2}.$$

The proof is complete. $\square$

We only focus on the linear parabolic equation in Theorems 2.1 and 2.2. Actually, we can also consider the nonlinear parabolic equation

$$\begin{cases} du_t = (\Delta u + ku^p)dt + \sigma dB_t, & t > 0, \quad x \in \mathbb{R}^d, \\ u(x,0) = u_0(x) \gneqq 0, & \quad\quad\quad x \in \mathbb{R}^d, \end{cases} \tag{2.6}$$

where $k \in \mathbb{R}$. For the Cauchy problem (2.6), it is impossible that the solutions keep positive almost surely. However, we can use Jensen's inequality to deal with some special case. Since the proof is easy, we only give the result and omit the proof details here.

Proposition 2.1 *Assume that p is an even positive number and $k > 0$, then it holds that*

$$\mathbb{P}(u(x,t) \geq 0) \geq 1 - \Phi\left(\frac{A_t(x)}{\sigma\sqrt{t}}\right).$$

Assume that p is an even positive number and $k < 0$, then it holds that

$$\mathbb{P}(u(x,t) \leq 0) \geq \Phi\left(\frac{A_t(x)}{\sigma\sqrt{t}}\right).$$

For $0 < p \leq 1$, we have that $\mathbb{E}|u|$ is a lower solution of the following equation

$$\begin{cases} \frac{\partial}{\partial t}v = \Delta v + |k|v^p, \quad t > 0, & x \in \mathbb{R}^d, \\ v(x,0) = u_0(x) \gneqq 0, & x \in \mathbb{R}^d, \end{cases}$$

where u is a solution to (2.6). Consequently, when $0 < p \leq 1$, the solution of (2.6) exists globally almost surely.

3 The impact of linear multiplicative noises

In this section, we consider the impact of linear multiplicative noises on parabolic equations. Our aim is to get the probability of the event that the solutions keep positive or the solutions are less (larger) than those of corresponding deterministic case, and so on.

Firstly, we consider a multiplicative noise.

$$\begin{cases} du(x,t) = \Delta u dt + f(u)dt + \sigma u dB_t, \quad t > 0, & x \in \mathbb{R}^d, \\ u(x,0) = u_0(x) \gneqq 0, & x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

where $\sigma > 0$, and B_t is a one-dimensional Brownian motion. By using the Itô formula, it is easy to see that if we let $v(x,t) = e^{-\sigma B_t}u(x,t)$, then $v(x,t)$ satisfies that

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = \Delta v(x,t) - \frac{\sigma^2}{2}v(x,t) + e^{-\sigma B_t}f(e^{\sigma B_t}v), \quad t > 0, & x \in \mathbb{R}^d, \\ v(x,0) = u_0(x) \gneqq 0, & x \in \mathbb{R}^d. \end{cases}$$

Denote $w(x,t) = e^{\frac{\sigma^2}{2}t}v$, then we have

$$\begin{cases} \frac{\partial}{\partial t}w(x,t) = \Delta w(x,t) + e^{\frac{\sigma^2}{2}t - \sigma B_t}f(e^{\sigma B_t - \frac{\sigma^2}{2}t}w), \quad t > 0, & x \in \mathbb{R}^d, \\ w(x,0) = u_0(x) \gneqq 0, & x \in \mathbb{R}^d. \end{cases} \tag{3.2}$$

Therefore, using the comparison principle, we have the following result.

Theorem 3.1 *Assume that the nonlinear term f is locally Lipschitz continuous, then the Cauchy problem (3.1) with a nonnegative initial datum admits a local solution defined by (2.2). Moreover, the solution remains positive: $u(x,t) \geq 0$, a.s. for almost every $x \in \mathbb{R}^d$ and for all $t \in [0,T]$, where T is the lifetime.*

Remark 3.1 *Comparing Theorem 3.1 with [15, Theorem 4.1], we find the proof here is simple and the result is exact same as the deterministic case. The reason is that the problem (3.2) is random and thus we can use the comparison principle. Moreover, the result of Theorem 3.1 is better than that of [15, Theorem 4.1]. More precisely, in [15, Theorem 4.1], the assumption about f is that $f(u) \geq 0$ for $u \leq 0$, which is stronger than that in this paper.*

Theorem 3.2 *Assume all conditions in Theorem 3.1 hold and u is the solution of (3.1) with $f(u) = u^p$. Then for $p > 1$, $\mathbb{E}u$ is a super solution of the following equation*

$$\begin{cases} \frac{\partial}{\partial t} v = \Delta v + v^p, \quad t > 0, & x \in \mathbb{R}^d, \\ v(x, 0) = u_0(x) \gneqq 0, & x \in \mathbb{R}^d. \end{cases} \tag{3.3}$$

Consequently, when $p > 1$, $\mathbb{E}u$ will blow up in finite time if the initial data belongs to $\mathcal{U}_\infty$, where

$$\mathcal{U}_\infty = \left\{ v \mid v \in BC(\mathbb{R}^d, \mathbb{R}_+), v(x) \geq ce^{-k|x|^2}, \ k > 0, c \gg 1 \right\},$$

and BC is the set consisted of bounded and uniformly continuous functions.

Proof. It follows from Theorem 3.1 that $u \geq 0$ almost surely. The mild solution of (3.1) can be expressed as

$$\begin{aligned} u(x,t) \quad = \quad & \int_{\mathbb{R}^d} K(x-y,t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} K(x-y, t-s) u^p(y,s) dy ds \\ & + \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s) u(y,s) dy dB_s. \end{aligned}$$

Taking expectation in the above equality, we have

$$\mathbb{E}u(x,t) \geq \int_{\mathbb{R}^d} K(x-y,t) u_0(y) dy + \int_0^t \int_{\mathbb{R}^d} K(x-y,t-s)(\mathbb{E}u)^p(y,s) dy ds,$$

which implies that $\mathbb{E}u$ is a super solution of equation (3.3). From [9] (also see [8, 11]), the solution of (3.3) will blow up in finite time, then $\mathbb{E}u$ will blow up as well. $\Box$

Theorem 3.2 shows that $\mathbb{E}u$ will be easier to blow up in finite time than the solution of (3.3), but do not give the blowup probability. Now, we study this interesting problem. Let u be a mild solution to (3.1) in the sense of Walsh [20] (given by the heat kernel). It follows Theorem 3.1 that the solutions of (3.1) will keep positive if the initial data are nonnegative. Furthermore, we want to know the probability of event that the solutions of (3.1) blow up in finite time. It suffices to consider the equation (3.2). For simplicity, we only consider the case that $f(u) = u^p$. Following [10], if $1 < p \leq 1 + 2/d$, then any nontrivial, nonnegative solution solutions of (3.2) with $\sigma = 0$ blows up in finite time. When $\sigma \neq 0$, we have

Theorem 3.3 *(i) Assume that $1 < p \leq 1 + \frac{2}{pd}$. The probability that the solution of (3.2) blows up in finite time is lower bounded by $\Phi\left(\frac{\ln(\frac{1}{\varepsilon}) - \frac{(p-1)\sigma^2}{2}}{|\sigma|(p-1)} \right)$, where $0 < \varepsilon \ll 1$ is a fixed any small constant.*

(ii) Assume that $1+\frac{2}{pd}<p<1+\frac{2}{d}$*. The probability that the solution of (3.2) blows up in finite time is lower bounded by* $\Phi\left(\frac{\ln(\frac{1}{\epsilon})-\frac{2(p-1)\sigma^2T^\star}{2}}{|\sigma|(p-1)\sqrt{2T^\star}}\right)$*, where* $0<\epsilon\ll 1$ *is a fixed any small constant and* $T^\star$ *satisfies (3.10).*

Proof. It follows from Theorem 3.1 that $w(x,t)=e^{\frac{\sigma^2}{2}t}v\geq 0$ almost surely. By using the properties of heat kernel, we get

$$\begin{aligned} w(x,t) &= \int_{\mathbb{R}^d} K(x-y,t)u_0(y)dy+\int_0^t\int_{\mathbb{R}^d} K(x-y,t-s)e^{-(p-1)\frac{\sigma^2}{2}s+(p-1)\sigma B_s}w^p(y,s)dyds \\ &= : I_1(x,t)+I_2(x,t). \end{aligned}$$

We assume that the solution remains finite for all finite t almost surely and want to derive a contradiction. We may assume without loss of generality that $u_0(x)\geq C_1>0$ for $|x|<1$ by the assumption. A direct computation shows that

$$I_1(x,t) \geq \frac{C_1}{(4\pi t)^{\frac{d}{2}}}\int_{B_1(0)}\exp\left(-\frac{|x|^2+|y|^2}{4t}\right)dy\geq \frac{C}{(4\pi t)^{\frac{d}{2}}}\exp\left(-\frac{|x|^2}{4t}\right),$$

for $t\geq 1$ and $C>0$.

It is easy to see that

$$I_2(x,t) \geq C_0\int_0^t e^{-(p-1)\frac{\sigma^2}{2}s+(p-1)\sigma B_s}\left(\int_{\mathbb{R}^d} K(x-y,t-s)w(y,s)dy\right)^p ds.$$

Let

$$G(t)=\int_{\mathbb{R}^d} K(x,t)w(x,t)dx.$$

Then for $t\geq 1$,

$$\begin{aligned} G(t) &= \int_{\mathbb{R}^d} I_1(x,t)K(x,t)dx+\int_{\mathbb{R}^d} I_2(x,t)K(x,t)dx \\ &\geq \frac{C_2}{t^{\frac{d}{2}}}+C_0\int_0^t e^{-(p-1)\frac{\sigma^2}{2}s+(p-1)\sigma B_s} \\ &\quad\times\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,t)K(x-y,t-s)dxw(y,s)dy\right)^p ds. \end{aligned} \tag{3.4}$$

It follows from the estimates of [10, pp 42] that

$$\int_{\mathbb{R}^d} K(x,t)K(x-y,t-s)dx\geq C_3K(y,s)\frac{s^{\frac{d}{2}}}{t^{\frac{d}{2}}}.$$

Hence, (3.4) becomes

$$G(t)\geq \frac{C_2}{t^{\frac{d}{2}}}+C_3\int_0^t e^{-(p-1)\frac{\sigma^2}{2}s+(p-1)\sigma B_s}\left(\frac{s^{\frac{d}{2}}}{t^{\frac{d}{2}}}\right)^p G^p(s)ds,$$

where C_3 is a positive constant. We can rewrite the above inequality as

$$t^{\frac{pd}{2}}G(t) \geq C_2 t^{\frac{d(p-1)}{2}} + C_3 \int_0^t s^{\frac{pd}{2}} e^{-(p-1)\frac{\sigma^2}{2}s+(p-1)\sigma B_s} G^p(s)ds =: g(t). \tag{3.5}$$

Then for $t \geq 1$, we have

$$g(t) \geq C_2 t^{\frac{d(p-1)}{2}}$$

and

$$g'(t) \geq t^{\frac{pd}{2}} e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t} G^p(t) \geq t^{\frac{(p-p^2)d}{2}} e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t} g^p(t),$$

which implies

$$\begin{aligned} \frac{C_2^{1-p}}{p-1} t^{-\frac{d(p-1)^2}{2}} &\geq \frac{1}{p-1} g^{1-p}(t) \\ &\geq C_3 \int_t^T s^{\frac{(p-p^2)d}{2}} e^{-(p-1)\frac{\sigma^2}{2}s+(p-1)\sigma B_s} ds \quad \text{for } T > t \geq 1. \end{aligned} \tag{3.6}$$

(i) Assume that $1 < p \leq 1 + 2/(pd)$, i.e., $(p^2 - p)d/2 \leq 1$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the filtration generated by $\{B_t\}_{t\geq 0}$. Due to $\{e^{-\frac{\sigma^2}{2}t+\sigma B_t}\}_{t\geq 0}$ is martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, taking conditional expectation and using Jensen's inequality, we have

$$\begin{aligned} \frac{C_2^{1-p}}{p-1} t^{-\frac{d(p-1)^2}{2}} &\geq C_3 \int_t^T s^{\frac{(p-p^2)d}{2}} \mathbb{E}\left[e^{-(p-1)\frac{\sigma^2}{2}s+(p-1)\sigma B_s} | \mathcal{F}_t\right] ds \\ &\geq C_3 \int_t^T s^{\frac{(p-p^2)d}{2}} \left(\mathbb{E}\left[e^{-\frac{\sigma^2}{2}s+\sigma B_s} | \mathcal{F}_t\right]\right)^{p-1} ds \\ &\geq C_3 \int_t^T s^{\frac{(p-p^2)d}{2}} ds e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t}. \end{aligned}$$

Observing that

$$\int_t^T s^{\frac{(p-p^2)d}{2}} ds = \begin{cases} \frac{2}{(p-p^2)d+2}\left(T^{\frac{(p-p^2)d+2}{2}} - t^{\frac{(p-p^2)d+2}{2}}\right), & (p^2-p)d/2 < 1, \\ \ln T - \ln t, & (p^2-p)d/2 = 1, \end{cases}$$

we gain

$$\begin{aligned} &\frac{C_2^{1-p}}{p-1} t^{-\frac{d(p-1)^2}{2}} \\ &\geq \begin{cases} \frac{2C_3}{(p-p^2)d+2}\left(T^{\frac{(p-p^2)d+2}{2}} - t^{\frac{(p-p^2)d+2}{2}}\right) e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t}, & (p^2-p)d/2 < 1, \\ C_3(\ln T - \ln t) e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t}, & (p^2-p)d/2 = 1. \end{cases} \end{aligned} \tag{3.7}$$

Hence letting $T \to \infty$, we know that the probability of the inequality (3.7) does not hold is equivalent to the probability of the event that $\{\omega \in \Omega;\ e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t(\omega)} > \varepsilon\}$, where $\varepsilon > 0$ is any fixed. By using the fact that B_t is a Gaussian process and for any fixed $t > 0$, B_t obeys the Gaussian normal distribution $N(0, t)$, we have

$$\mathbb{P}\left(e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t} > \varepsilon\right) = \Phi\left(\frac{\ln(\frac{1}{\varepsilon}) - \frac{(p-1)\sigma^2 t}{2}}{|\sigma|(p-1)\sqrt{t}}\right).$$

It follows from the above discussion that we can take $t = 1$ in above equality.

(ii) Let us discuss the case: $1 + 2/(pd) < p < 1 + 2/d$. In this case, noting that $(p - p^2)d + 2 < 0$ and $d(p-1)^2/2 > -1 + d(p-1)^2/2$, we obtain that

$$\frac{C_2^{1-p}}{p-1} t^{-\frac{d(p-1)^2}{2}} \geq \frac{2C_3}{(p^2-p)d-2}\left(t^{\frac{(p-p^2)d+2}{2}} - T^{\frac{(p-p^2)d+2}{2}}\right) e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t}. \tag{3.8}$$

Letting $T \to \infty$, we get

$$\frac{C_2^{1-p}}{p-1} t^{-\frac{d(p-1)^2}{2}} \geq \frac{2C_3}{(p^2-p)d-2} t^{\frac{(p-p^2)d+2}{2}} e^{-(p-1)\frac{\sigma^2}{2}t+(p-1)\sigma B_t}. \tag{3.9}$$

For any fixed $\epsilon > 0$, let $T^\star$ satisfy

$$\frac{C_2^{1-p}}{p-1} (T^\star)^{-\frac{d(p-1)^2}{2}} < \frac{2C_3\epsilon}{(p^2-p)d-2} (T^\star)^{\frac{(p-p^2)d+2}{2}}. \tag{3.10}$$

Then the inequality (3.8) does not hold with the probability $\mathbb{P}(A_{T^\star})$, where

$$A_{T^\star} = \{\omega \in \Omega;\ e^{-(p-1)\frac{\sigma^2}{2}T^\star+(p-1)\sigma B_{T^\star}(\omega)} > \epsilon\}.$$

Similar to the former case, we have

$$\mathbb{P}(A_{T^\star}) = \Phi\left(\frac{\ln(\frac{1}{\epsilon}) - \frac{(p-1)\sigma^2 T^\star}{2}}{|\sigma|(p-1)\sqrt{T^\star}}\right).$$

The proof is complete. □

Remark 3.2 *The difference between the two cases* $1 < p < 1 + 2/(pd)$ *and* $1 + 2/(pd) < p < 1 + 2/d$ *is that: in the case* $(p - p^2)d/2 + 1 \geq 0$*, the constant* $\varepsilon > 0$ *does not depend on the time; however, in the second case* $(p - p^2)d/2 + 1 < 0$*, the constant* $\epsilon > 0$ *depends on the time and must satisfy the inequality (3.10). Consequently, in the case* $(p - p^2)d/2 + 1 > 0$*, the probability of the event that the solutions blow up in finite time is closed to* 1*. But in the other case, the probability has a certain distance with respect to* 1*.*

The linear multiplicative noise can be regarded as a perturbation and the profile of the solution will keep together with the deterministic case. Thus we should be care of the probability of the event that the solutions has the same properties as the deterministic case. But if the noise is nonlinear and multiplicative, the structure for the original equation will be changed, we can not deduce the same properties as the deterministic case in general.

Lastly, we prove the probability of the event that blowup time of stochastic parabolic equations large than or less than the deterministic case. Let $D \subset \mathbb{R}^d$. Consider the following stochastic parabolic equation

$$\begin{cases} du(x,t) = (\Delta u + G(u))dt + \kappa u dB_t, & t > 0,\ x \in D, \\ u(x,0) = u_0(x), & x \in D, \\ u(x,t) = 0, & t > 0,\ x \in \partial D, \end{cases} \tag{3.11}$$

where $G : \mathbb{R} \to \mathbb{R}_+$ is locally Lipschitz and satisfies

$$G(u) \geq C u^{1+\beta} \quad \text{for all } u > 0,$$

and C, β, κ are positive numbers, $\{B_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and $u_0 :\ D \to \mathbb{R}_+$ is of class C^2 and not identically zero. Dozzi and López-Mimbela [6] obtained the probability that the solution of (3.11) blows up in finite time is lower bounded by $\int_{\frac{1}{\beta}u(\phi,0)^{-\beta}}^{\infty} h(y)dy$ with

$$h(y) = \frac{(\kappa^2\beta^2 y/2)^{(2\lambda_1+\kappa^2)/\kappa^2\beta}}{y\Gamma((2\lambda_1+\kappa^2)/(\kappa^2\beta))} \exp\left(-\frac{2}{\kappa^2\beta^2 y}\right), \quad u(\phi,0) = \int_D u_0(x)\phi(x)dx,$$

where $\lambda_1 > 0$ is the first eigenvalue of the Laplacian on D, and ϕ is the corresponding eigenfunction normalized so that $\|\phi\|_{L^1} = 1$. It is not hard to prove that $v(x,t) = e^{-\kappa B_t}u(x,t)$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = \Delta v(x,t) - \frac{\kappa^2}{2}v(x,t) + e^{-\kappa B_t}G(e^{\kappa B_t}v(x,t)), & t > 0,\ x \in D, \\ v(x,0) = u_0(x), & x \in D, \\ v(x,t) = 0, & t > 0,\ x \in \partial D. \end{cases} \tag{3.12}$$

Similar to the proof of Theorem 3.1, we can prove the solutions of (3.12) will keep positive. Following the method of Theorem 3.3, one can give a different probability from [6] of the event that the solutions blow up in finite time. In paper [6], the blowup time is obtained, that is,

$$\tau := \inf\left\{ t \geq 0 \,\middle|\, \int_0^t e^{-(\lambda_1 + \frac{\kappa^2}{2})\beta s + \kappa\beta B_s} ds \geq \frac{1}{\beta}u(\phi,0) \right\}.$$

It is easy to see that when $\kappa = 0$, τ becomes the blowup time of deterministic case. Assume T^* satisfies

$$\int_0^{T^*} e^{-\lambda_1\beta s} ds = \frac{1}{\beta}u(\phi,0).$$

Now we want to prove $\mathbb{P}(\tau > T^*)$. It follows from the definition of T^*, we have

$$\begin{aligned} & \mathbb{P}(\tau > T^*) \\ = \ & \mathbb{P}\left(\tau > T^*, \int_0^{T^*} e^{-\lambda_1\beta s}(1 - e^{-\frac{\kappa^2\beta s}{2} + \kappa\beta B_s})ds = \int_{T^*}^{\tau} e^{-(\lambda_1 + \frac{\kappa^2}{2})\beta s + \kappa\beta B_s} ds \right) \\ = \ & \mathbb{P}\left(\int_0^{T^*} e^{-\lambda_1\beta s}(1 - e^{-\frac{\kappa^2\beta s}{2} + \kappa\beta B_s})ds > 0 \right) \\ = \ & \mathbb{P}\left(\int_0^{T^*} e^{-(\lambda_1 + \frac{\kappa^2}{2})\beta s + \kappa\beta B_s} ds < \frac{1}{\lambda_1\beta}(1 - e^{-\lambda_1\beta T^*}) \right). \end{aligned}$$

It follows from the results of [22] that the random variable $\int_0^{T^*} e^{-(\lambda_1+\kappa^2/2)\beta s+\kappa\beta B_s} ds$ has a probability law and we denote by P^{T^*}. Then we have

$$\mathbb{P}(\tau > T^*) = P^{T^*}(\frac{1}{\lambda_1\beta}(1 - e^{-\lambda_1\beta T^*})).$$

Combining the above discussion, we have the following result.

Theorem 3.4 *The probability that the blowup time of (3.11) is larger than the deterministic case (i.e., (3.11) with $\kappa = 0$) is $P^{T^*}(\frac{1}{\lambda_1\beta}(1 - e^{-\lambda_1\beta T^*}))$.*

Acknowledgment This work was supported in part by NSF of China grants 11771123 and 11501577. The authors thanks Prof. Feng-yu Wang for discussing this manuscript.

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