Asymptotic behavior of random coupled Ginzburg-Landau equation driven by colored noise on unbounded domains

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Abstract

In this paper, random coupled Ginzburg-Landau equation driven by colored noise on unbounded domains is considered, in which nonlinear term satisfies local Lipschitz condition. It is shown that random attractor of such coupled Ginzburg-Landau equation is singleton set, and the components of solutions are very close when the coupling parameter becomes large enough.

Keywords: Random coupled Ginzburg-Landau equation, colored noise, random attractor, singleton set

1. Introduction

Synchronization phenomenon, which was discovered in many fields such as physics, biology, and social science [1, 2, 3], has been paid more attention due to its extensive applications in secure communications, optimization of nonlinear system performance [4, 5]. Synchronization of deterministic coupled dissipative systems has been investigated [6, 7, 8].

Since noise is omnipresent in real world, random perturbation is an important factor worthy of being considered in synchronization. The persistence and convergence rate of synchronization under additive noise were investigated in [9] and [10], respectively. Moreover, synchronization of coupled sine-Gordon wave

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model on bounded domain perturbed by additive white noise was investigated by the quasi-stability method [11].

For linear multiplicative noises, synchronization of Stratonovich stochastic differential equations was investigated in [12] by transforming it to random ordinary differential equations. Recently, synchronization for additive noise and linear multiplicative noise was investigated in [13] by the theory of Imkeller and Schmalfuss. However, the methods in the above references can not deal with synchronization for systems with nonlinear noise. Z. Li and J. Liu [14] proved the synchronization result for stochastic differential equations with general nonlinear multiplicative noise in the mean square sense.

It is worth mentioning that nonlinear terms in the above literature satisfy one-sided dissipative Lipschitz conditions or global Lipschitz conditions. When nonlinear term satisfies local Lipschitz condition, random attractor and synchronization were studied for stochastic reaction-diffusion system with additive space-time noise on a thin bounded domain [15].

Motivated by the above literature, in this paper, we will consider random coupled complex Ginzburg-Landau equation driven by colored noise on unbounded domains

$$\begin{cases} \dfrac{\partial u^{\varepsilon}}{\partial t} - (1+i\lambda)\Delta u^{\varepsilon} = -\rho_1 u^{\varepsilon} + f(u^{\varepsilon}) + \varepsilon(v^{\varepsilon} - u^{\varepsilon}) + u^{\varepsilon}\mathcal{G}_{\delta}\left(\theta_t\omega\right), \\ \dfrac{\partial v^{\varepsilon}}{\partial t} - (1+i\lambda)\Delta v^{\varepsilon} = -\rho_2 v^{\varepsilon} + f(v^{\varepsilon}) + \varepsilon(u^{\varepsilon} - v^{\varepsilon}) + v^{\varepsilon}\mathcal{G}_{\delta}\left(\theta_t\omega\right), \\ u^{\varepsilon}(\tau,x) = u_{\tau}(x),\ v^{\varepsilon}(\tau,x) = v_{\tau}(x), \end{cases} \tag{1.1}$$

where $u^{\varepsilon}(t,x)$, $v^{\varepsilon}(t,x)$ are unknown complex-value functions, $t \geq \tau$, $x \in \mathbb{R}$, i is the imaginary unit, λ, $\mu \in \mathbb{R}$, ρ_1, $\rho_2 > 0$, the nonlinear term $f(u) = -(1+i\mu) \mid u \mid^2 u$ is complex-valued function, $\varepsilon > 0$ is coupling parameter, and $\mathcal{G}_{\delta}(\theta_t\omega)$ is colored noise introduced in [16, 17] and is the unique stationary solution of stochastic differential equation $\mathrm{d}\mathcal{G}_{\delta} + \frac{1}{\delta}\mathcal{G}_{\delta}\mathrm{d}t = \frac{1}{\delta}\mathrm{d}W$.

It is worth noting that the nonlinear term f in (1.1) does not satisfy global Lipschitz conditions such as [11] and one-sided Lipschitz conditions such as [12, 13]. Moreover, different from the case of bounded domain in [11, 15], Sobolev embedding on unbounded domain is noncompact. In [19], the authors investigated random attractor for nonautonomous random Ginzburg-Landau equation on unbounded domain driven by nonlinear colored noise by the tail-estimates method and the properties of the colored noise. In this paper, we will further prove that the solutions of (1.1) converge pathwise to each other and random attractor set is singleton set in Section 3. Moreover, it will be also proved the solution $(u^\varepsilon, v^\varepsilon)$ of coupled system (1.1) satisfies $\lim\limits_{\varepsilon\to+\infty} \|u^\varepsilon(t) - v^\varepsilon(t)\|^2 = 0$ uniformly on any bounded time-interval. In addition, one can refer to [18] for random attractor of fractional Ginzburg-Landau equations on bounded domain driven by colored noise and [20] for random attractor of coupled fractional Ginzburg-Landau equation.

Throughout this paper, let $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product of $L^2(\mathbb{R})$, respectively. The Sobolev space $H^k(\mathbb{R})$ $(k \in \mathbb{N})$ consists of all $u \in L^2(\mathbb{R})$ whose weak derivatives up to order k belong to $L^2(\mathbb{R})$ as well, which is a separable Banach space with norm $\|u\|_{H^k(\mathbb{R})} := \left(\sum\limits_{|\alpha|\le k} \int_{\mathbb{R}} |D^\alpha u(x)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$. Denote $|\xi|^2_{H^k(\mathbb{R})} := \|u\|^2_{H^k(\mathbb{R})} + \|v\|^2_{H^k(\mathbb{R})}$, $|\xi|^k_{L^k(\mathbb{R})} := \|u\|^k_{L^k(\mathbb{R})} + \|v\|^k_{L^k(\mathbb{R})}$, where $\xi = (u,v)^T$.

2. Preliminaries

In this section, we recall some properties about colored noise, which are useful for proof of main results.

Lemma 2.1 ([21]). (1) *For every $\omega \in \Omega$, the mapping $t \mapsto \mathcal{G}_\delta(\theta_t\omega)$ is continuous, and for every $0 < \delta \le 1$,*

$$\lim_{t\to\pm\infty} \frac{|\mathcal{G}_\delta(\theta_t\omega)|}{t} = 0.$$

(2) *For every $\omega \in \Omega$,*

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \mathcal{G}_\delta(\theta_s\omega)\,\mathrm{d}s = 0 \quad \textit{uniformly for } 0 < \delta \le 1.$$

Lemma 2.2 ([22]). *Let $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $T > 0$. Then there exist $\delta_0 = \delta_0(\tau,\omega,T) > 0$ and $M = M(\tau,\omega,T) > 0$ such that for all $0 < \delta < \delta_0$ and*

$t \in [\tau, \tau+T]$,

$$\left|\int_0^t \mathcal{G}_\delta\left(\theta_s \omega\right) \mathrm{d} s\right| \leq M.$$

3. The asymptotic behavior of the coupled system

The system (1.1) can be rewritten as

$$\frac{\partial \xi^{\varepsilon}}{\partial t}-(1+i \lambda) \Delta \xi^{\varepsilon}=F(\xi^{\varepsilon})+\varepsilon B \xi^{\varepsilon}+\xi^{\varepsilon} \mathcal{G}_\delta(\theta_t \omega) \tag{3.1}$$

with initial datum $(u_\tau, v_\tau)^T$, where

$$\xi^{\varepsilon}=\begin{bmatrix} u^{\varepsilon} \\ v^{\varepsilon} \end{bmatrix}, \quad F(\xi^{\varepsilon})=\begin{bmatrix} -\rho_1 u^{\varepsilon}+f(u^{\varepsilon}) \\ -\rho_2 v^{\varepsilon}+f(v^{\varepsilon}) \end{bmatrix}, \quad B=\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Similar to [19], one can know that system (3.1) exists a unique solution $\xi^{\varepsilon} \in C\left([\tau, \infty); L^2(\mathbb{R}) \times L^2(\mathbb{R})\right) \bigcap L_{loc}^2\left([\tau, \infty); H^1(\mathbb{R}) \times H^1(\mathbb{R})\right)$, and $\xi^{\varepsilon} \in L_{loc}^4\left([\tau, \infty); L^4(\mathbb{R}) \times L^4(\mathbb{R})\right)$, and system (3.1) exists a unique random attractor. In what follows, we will first show random attractor is singleton set.

Theorem 3.1. *For any ρ_1, $\rho_2>0$, random attractor sets of coupled system (3.1) are singleton sets for any given $\varepsilon>0$.*

Proof. Let $\xi^{\varepsilon}=(u_1^{\varepsilon}, v_1^{\varepsilon})^T$ and $\eta^{\varepsilon}=(u_2^{\varepsilon}, v_2^{\varepsilon})^T$ be the solutions of (3.1) with initial data $\xi_\tau=(u_{1,\tau}, v_{1,\tau})^T$ and $\eta_\tau=(u_{2,\tau}, v_{2,\tau})^T$, respectively. Then we have

$$\begin{aligned}
& \frac{d}{dt}|\xi^{\varepsilon}-\eta^{\varepsilon}|^2+2|\nabla(\xi^{\varepsilon}-\eta^{\varepsilon})|^2 \\
= & 2 \operatorname{Re}\langle F(\xi^{\varepsilon})-F(\eta^{\varepsilon}), \xi^{\varepsilon}-\eta^{\varepsilon}\rangle+2 \operatorname{Re}\langle\varepsilon B(\xi^{\varepsilon}-\eta^{\varepsilon}), \xi^{\varepsilon}-\eta^{\varepsilon}\rangle \\
& +2 \mathcal{G}_\delta\left(\theta_t \omega\right)|\xi^{\varepsilon}-\eta^{\varepsilon}|^2 .
\end{aligned} \tag{3.2}$$

For the first term on the right-hand side of (3.2), it follows from the Hölder inequality and the Young inequality that

$$\begin{aligned}
& \operatorname{Re}\langle F(\xi^{\varepsilon})-F(\eta^{\varepsilon}), \xi^{\varepsilon}-\eta^{\varepsilon}\rangle \\
\leq & -\rho_1\|u_1^{\varepsilon}-u_2^{\varepsilon}\|^2-\rho_2\|v_1^{\varepsilon}-v_2^{\varepsilon}\|^2+\min\{\frac{\rho_1}{2}, \frac{\rho_2}{2}, 1\}\|u_1^{\varepsilon}-u_2^{\varepsilon}\|_{H^1(\mathbb{R})}^2 \\
& +\min\{\frac{\rho_1}{2}, \frac{\rho_2}{2}, 1\}\|v_1^{\varepsilon}-v_2^{\varepsilon}\|_{H^1(\mathbb{R})}^2 \\
& +\frac{3}{2^{\frac{4}{3}}}\left(1+2 \sqrt{2} \sqrt{1+\mu^2}\right)^{\frac{4}{3}}\left(\min\{\frac{\rho_1}{2}, \frac{\rho_2}{2}, 1\}\right)^{-\frac{1}{3}} \\
& [\left(\|u_1^{\varepsilon}\|_{L^4(\mathbb{R})}^{\frac{8}{3}}+\|u_2^{\varepsilon}\|_{L^4(\mathbb{R})}^{\frac{8}{3}}\right)\|u_1^{\varepsilon}-u_2^{\varepsilon}\|^2+\left(\|v_1^{\varepsilon}\|_{L^4(\mathbb{R})}^{\frac{8}{3}}+\|v_2^{\varepsilon}\|_{L^4(\mathbb{R})}^{\frac{8}{3}}\right)\|v_1^{\varepsilon}-v_2^{\varepsilon}\|^2]
\end{aligned}$$

$$\leq -\min\{\rho_1,\rho_2\}|\xi^\varepsilon-\eta^\varepsilon|^2+\min\{\frac{\rho_1}{2},\frac{\rho_2}{2},1\}|\xi^\varepsilon-\eta^\varepsilon|^2_{H^1(\mathbb{R})}+C(\xi^\varepsilon,\eta^\varepsilon)|\xi^\varepsilon-\eta^\varepsilon|^2,$$

where $C(\xi^\varepsilon,\eta^\varepsilon)=a\left[|\xi^\varepsilon|^{\frac{8}{3}}_{L^4(\mathbb{R})}+|\eta^\varepsilon|^{\frac{8}{3}}_{L^4(\mathbb{R})}\right]$,
$a=\frac{3}{2^{\frac{4}{3}}}\left(1+2\sqrt{2}\sqrt{1+\mu^2}\right)^{\frac{4}{3}}\left(\min\{\frac{\rho_1}{2},\frac{\rho_2}{2},1\}\right)^{-\frac{1}{3}}$.
Then together with (3.2), we have

$$\begin{aligned}&\frac{d}{dt}|\xi^\varepsilon-\eta^\varepsilon|^2\\ \leq& -\min\{\rho_1,\rho_2\}|\xi^\varepsilon-\eta^\varepsilon|^2+2C(\xi^\varepsilon,\eta^\varepsilon)|\xi^\varepsilon-\eta^\varepsilon|^2+2\mathcal{G}_\delta\left(\theta_t\omega\right)|\xi^\varepsilon-\eta^\varepsilon|^2.\end{aligned}$$

From the Gronwall inequality, we can obtain

$$|\xi^\varepsilon-\eta^\varepsilon|^2\leq e^{-\min\{\rho_1,\rho_2\}(t-\tau)+2\int_\tau^t\mathcal{G}_\delta(\theta_s\omega)ds+2\int_\tau^t C(\xi^\varepsilon,\eta^\varepsilon)ds}|\xi_\tau-\eta_\tau|^2. \tag{3.3}$$

Since

$$\begin{aligned}&\frac{d}{dt}|\xi^\varepsilon|^2+2|\nabla(\xi^\varepsilon)|^2\\ =&2\mathrm{Re}\langle F(\xi^\varepsilon),\xi^\varepsilon\rangle+2\mathrm{Re}\langle\varepsilon B\xi^\varepsilon,\xi^\varepsilon\rangle+2\mathcal{G}_\delta\left(\theta_t\omega\right)|\xi^\varepsilon|^2\\ \leq& -2\rho_1\|u_1^\varepsilon\|^2-2\rho_2\|v_1^\varepsilon\|^2-2\|u_1^\varepsilon\|^4_{L^4(\mathbb{R})}-2\|v_1^\varepsilon\|^4_{L^4(\mathbb{R})}+2\mathcal{G}_\delta\left(\theta_t\omega\right)|\xi^\varepsilon|^2\\ \leq& -2\min\{\rho_1,\rho_2\}|\xi^\varepsilon|^2-2\|u_1^\varepsilon\|^4_{L^4(\mathbb{R})}-2\|v_1^\varepsilon\|^4_{L^4(\mathbb{R})}+2\mathcal{G}_\delta\left(\theta_t\omega\right)|\xi^\varepsilon|^2,\end{aligned}$$

we have

$$\begin{aligned}&\frac{d}{dt}|\xi^\varepsilon|^2+2|\nabla(\xi^\varepsilon)|^2+2\|u_1^\varepsilon\|^4_{L^4(\mathbb{R})}+2\|v_1^\varepsilon\|^4_{L^4(\mathbb{R})}\\ \leq& -2\min\{\rho_1,\rho_2\}|\xi^\varepsilon|^2+2\mathcal{G}_\delta\left(\theta_t\omega\right)|\xi^\varepsilon|^2.\end{aligned} \tag{3.4}$$

By the Gronwall inequality, one can obtain

$$|\xi^\varepsilon|^2\leq e^{2\int_\tau^t[-\min\{\rho_1,\rho_2\}+\mathcal{G}_\delta(\theta_s\omega)]ds}|\xi_\tau|^2. \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\begin{aligned}&\int_\tau^t\|u_1^\varepsilon\|^4_{L^4(\mathbb{R})}ds+\int_\tau^t\|v_1^\varepsilon\|^4_{L^4(\mathbb{R})}ds\\ \leq&\frac{1}{2}|\xi_\tau|^2+\int_\tau^t\mathcal{G}_\delta\left(\theta_s\omega\right)|\xi^\varepsilon|^2ds\\ \leq&\frac{1}{2}|\xi_\tau|^2+\int_\tau^t|\mathcal{G}_\delta\left(\theta_s\omega\right)|e^{2\int_\tau^s[-\min\{\rho_1,\rho_2\}+\mathcal{G}_\delta(\theta_r\omega)]dr}|\xi_\tau|^2ds.\end{aligned} \tag{3.6}$$

By Lemma 2.1, there exists $T(\omega)$ such that for all $t>T(\omega)$, $\int_\tau^t\mathcal{G}_\delta\left(\theta_s\omega\right)ds\leq\frac{\min\{\rho_1,\rho_2\}}{4}(t-\tau)$ and $|\mathcal{G}_\delta\left(\theta_t\omega\right)|\leq t$. In addition, by Lemma 2.2, there exists

$M(\omega)$ such that $\int_\tau^{T(\omega)} |\mathcal{G}_\delta(\theta_t\omega)| dt \le M(\omega)$. Then together with (3.6), we obtain

$$\begin{aligned} &\int_\tau^t \|u_1^\varepsilon\|^4_{L^4(\mathbb{R})} ds + \int_\tau^t \|v_1^\varepsilon\|^4_{L^4(\mathbb{R})} ds \\ \le & M'(\omega)|\xi_\tau|^2 + |\xi_\tau|^2 \int_{T(\omega)}^t s e^{-\frac{3}{2}\min\{\rho_1,\rho_2\}(s-\tau)} ds, \end{aligned} \tag{3.7}$$

where $M'(\omega) = \int_\tau^{T(\omega)} |\mathcal{G}_\delta(\theta_s\omega)| e^{2\int_\tau^s [-\min\{\rho_1,\rho_2\}+\mathcal{G}_\delta(\theta_r\omega)]dr} ds + \frac{1}{2}$.

Similarly, we have

$$\begin{aligned} &\int_\tau^t \|u_2^\varepsilon\|^4_{L^4(\mathbb{R})} ds + \int_\tau^t \|v_2^\varepsilon\|^4_{L^4(\mathbb{R})} ds \\ \le & M'(\omega)|\eta_\tau|^2 + |\eta_\tau|^2 \int_{T(\omega)}^t s e^{-\frac{3}{2}\min\{\rho_1,\rho_2\}(s-\tau)} ds. \end{aligned} \tag{3.8}$$

From (3.7), (3.8) and the Hölder inequality, it follows that

$$\begin{aligned} &\int_\tau^t C(\xi^\varepsilon, \eta^\varepsilon) ds \\ \le & a\left[\left(\int_\tau^t |\xi^\varepsilon|^4_{L^4(\mathbb{R})} ds\right)^{\frac{2}{3}} + \left(\int_\tau^t |\eta^\varepsilon|^4_{L^4(\mathbb{R})} ds\right)^{\frac{2}{3}}\right](t-\tau)^{\frac{1}{3}} \\ \le & a\left[\left(M'(\omega)|\xi_\tau|^2 + |\xi_\tau|^2 \int_{T(\omega)}^{+\infty} s e^{-\frac{3}{2}\min\{\rho_1,\rho_2\}(s-\tau)} ds\right)^{\frac{2}{3}}\right. \\ &\left. + \left(M'(\omega)|\eta_\tau|^2 + |\eta_\tau|^2 \int_{T(\omega)}^{+\infty} s e^{-\frac{3}{2}\min\{\rho_1,\rho_2\}(s-\tau)} ds\right)^{\frac{2}{3}}\right](t-\tau)^{\frac{1}{3}} \\ := & C(\omega, \xi_\tau, \eta_\tau)(t-\tau)^{\frac{1}{3}}, \end{aligned} \tag{3.9}$$

which together with (3.3) implies that

$$|\xi^\varepsilon - \eta^\varepsilon|^2 \le e^{-\min\{\rho_1,\rho_2\}(t-\tau)+2\int_\tau^t \mathcal{G}_\delta(\theta_s\omega)ds + C(\omega,\xi_\tau,\eta_\tau)(t-\tau)^{\frac{1}{3}}} |\xi_\tau - \eta_\tau|^2.$$

Noticing that ρ_1, $\rho_2 > 0$, thus we can obtain

$$\lim_{t\to+\infty} |\xi^\varepsilon(t) - \eta^\varepsilon(t)|^2 = 0,$$

which implies that random attractor sets of coupled system (3.1) are singleton sets. $\square$

Remark 3.1. *Since* $(0,0)$ *is the solution of* (3.1)*, it follows by Theorem* 3.1 *that random attractor is actually singleton set* $\{(0,0)\}$*.*

Theorem 3.2. *The solution $(u^\varepsilon, v^\varepsilon)$ of coupled system (1.1) satisfies*

$$\lim_{\varepsilon\to+\infty} \|u^\varepsilon(t) - v^\varepsilon(t)\|^2 = 0$$

uniformly on any bounded time-interval $[T_1, T_2]$ of $\mathbb{R}$.

Proof. Let $\xi^\varepsilon = (u^\varepsilon, v^\varepsilon)^T$ be the solution of (1.1) with initial datum $\xi_\tau = (u_\tau, v_\tau)^T$, then we have

$$\begin{aligned}
&\frac{d}{dt}\|u^\varepsilon - v^\varepsilon\|^2 + 2\|\nabla(u^\varepsilon - v^\varepsilon)\|^2\\
\le& -2\rho_1\|u^\varepsilon - v^\varepsilon\|^2 + 2|\rho_2-\rho_1|\|u^\varepsilon - v^\varepsilon\|^2 + 2|\rho_2-\rho_1|\|v^\varepsilon\|^2\\
&-2\mathrm{Re}(1+i\mu)\langle |u^\varepsilon|^2u^\varepsilon - |v^\varepsilon|^2 v^\varepsilon, u^\varepsilon - u^\varepsilon\rangle - 4\varepsilon\|u^\varepsilon - v^\varepsilon\|^2 + 2\mathcal{G}_\delta(\theta_t\omega)\|u^\varepsilon - v^\varepsilon\|^2\\
\le& [-2\rho_1 + 2|\rho_2-\rho_1| - 4\varepsilon + 2\mathcal{G}_\delta(\theta_t\omega)]\|u^\varepsilon - v^\varepsilon\|^2 + \min\{\frac{\rho_1}{2}, 1\}\|u^\varepsilon - v^\varepsilon\|^2_{H^1(\mathbb{R})}\\
&+ C(u^\varepsilon, v^\varepsilon)\|u^\varepsilon - v^\varepsilon\|^2 + 2|\rho_2-\rho_1|\|v^\varepsilon\|^2,
\end{aligned}$$

where $C(u^\varepsilon, v^\varepsilon) = b\left(\|u^\varepsilon\|^{\frac{8}{3}}_{L^4(\mathbb{R})} + \|v^\varepsilon\|^{\frac{8}{3}}_{L^4(\mathbb{R})}\right)$,
$b = \frac{3}{2^{\frac{1}{3}}}\left(1 + 2\sqrt{2}\sqrt{1+\mu^2}\right)^{\frac{4}{3}}\left(\min\{\frac{\rho_1}{2}, 1\}\right)^{-\frac{1}{3}}$.
Therefore, we have

$$\begin{aligned}
\frac{d}{dt}\|u^\varepsilon - v^\varepsilon\|^2 \le& [-\rho_1 + 2|\rho_2-\rho_1| - 4\varepsilon + C(u^\varepsilon, v^\varepsilon) + 2\mathcal{G}_\delta(\theta_t\omega)]\|u^\varepsilon - v^\varepsilon\|^2\\
&+ 2|\rho_2-\rho_1|\|v^\varepsilon\|^2.
\end{aligned}$$

By the Gronwall inequality, one can obtain

$$\begin{aligned}
&\|u^\varepsilon(t) - v^\varepsilon(t)\|^2\\
\le& e^{\int_\tau^t[-\rho_1+2|\rho_2-\rho_1|-4\varepsilon+C(u^\varepsilon,v^\varepsilon)+2\mathcal{G}_\delta(\theta_s\omega)]ds}\|u_\tau - v_\tau\|^2\\
&+ 2|\rho_2-\rho_1|\int_\tau^t \|v^\varepsilon\|^2 e^{\int_s^t[-\rho_1+2|\rho_2-\rho_1|-4\varepsilon+C(u^\varepsilon,v^\varepsilon)+2\mathcal{G}_\delta(\theta_r\omega)]dr}ds.
\end{aligned} \tag{3.10}$$

Similar to (3.5) and (3.9) in the proof of Theorem 3.1, we have

$$\|v^\varepsilon(t)\|^2 \le |\xi^\varepsilon(t)|^2 \le e^{2\int_\tau^t[-\min\{\rho_1,\rho_2\}+\mathcal{G}_\delta(\theta_s\omega)]ds}|\xi_\tau|^2 \le C(T_1, T_2, \omega, \xi_\tau), \tag{3.11}$$

$$\int_\tau^t C(u^\varepsilon, v^\varepsilon)ds \le 2bM''^{\frac{2}{3}}(\omega)|\xi_\tau|^{\frac{4}{3}}(T_2-\tau)^{\frac{1}{3}} \tag{3.12}$$

for t on any bounded time-interval $[T_1, T_2]$, where

$$M''(\omega) = \int_\tau^{T_2} |\mathcal{G}_\delta(\theta_s\omega)| e^{2\int_\tau^s[-\min\{\rho_1,\rho_2\}+\mathcal{G}_\delta(\theta_r\omega)]dr}ds + \frac{1}{2}.$$

By (3.10)-(3.12) and Lemma 2.2, we obtain

$$
\begin{aligned}
&\|u^{\varepsilon}(t)-v^{\varepsilon}(t)\|^2\\
\leq& e^{2bM''^{\frac{2}{3}}(\omega)|\xi_\tau|^{\frac{4}{3}}(T_2-\tau)^{\frac{1}{3}}} e^{\int_\tau^t[-\rho_1+2|\rho_2-\rho_1|-4\varepsilon+2\mathcal{G}_\delta(\theta_s\omega)]ds}\|u_\tau-v_\tau\|^2\\
&+2|\rho_2-\rho_1|\int_\tau^t C(T_1,T_2,\omega,\xi_\tau)e^{2bM''^{\frac{2}{3}}(\omega)|\xi_\tau|^{\frac{4}{3}}(T_2-\tau)^{\frac{1}{3}}}\\
&\qquad\qquad \cdot e^{\int_s^t[-\rho_1+2|\rho_2-\rho_1|-4\varepsilon+2\mathcal{G}_\delta(\theta_r\omega)]dr}ds\\
\leq& M(T_1,T_2,\omega,\xi_\tau)e^{-4\varepsilon(T_2-\tau)}.
\end{aligned}
$$

Then we conclude that

$$
\|u^{\varepsilon}(t)-v^{\varepsilon}(t)\|^2\to 0,\qquad \varepsilon\to+\infty
$$

uniformly for t on any bounded time-interval $[T_1,T_2]$. □

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