Received 5 August 2020; | Revised 2020; | Accepted 2020

DOI: xxx/xxxx

ARTICLE TYPE

A new least square based reproducing kernel space method for solving regular and weakly singular 1D Volterra-Fredholm integral equations with smooth and nonsmooth solutions

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Summary

Based on the least square method, we proposed a new algorithm to obtain the solution of the second kind regular and weakly singular Volterra-Fredholm integral equations in reproducing kernel spaces. The stability and uniform convergence of the algorithm are investigated in details. Numerical experiments verify the theoretical findings. Meanwhile this method is also applicable to the nonlinear Volterra integral equations. Test problems which have non-smooth solutions are also considered and our proposed method is efficient as some recent Krylov subspace methods such as LSQR and LSMR.

KEYWORDS:
Volterra-Fredholm integral equation; multiscale basis; reproducing kernel spaces; least square method; convergence analysis; stability analysis

1 | INTRODUCTION

This article mainly discusses the following linear Volterra-Fredholm integral equation (VFIE) of the second kind:

$$u(x) - \int_a^x k_1(x,t)u(t)\mathrm{d}t - \int_a^b k_2(x,t)u(t)\mathrm{d}t = f(x),\ \ x \in [a,b] \tag{1.1}$$

with the solution u to be confirmed, where k_1, k_2, f are known functions, k_1, k_2 are smooth enough.

The VFIE of this type arised in the mathematical modeling of biological and chemical phenomena [1,2]. In [3], the authors studied some theoretical results of the equation (1.1), including the existence and uniqueness of the solution. Because equation (1.1) is usually difficult to be solved analytically, several numerical methods are used, here we refer to [4,5,6,7,8,9,11,17]. For more details, in [4] a variable transformation method is applied to solving the Volterra integral equations (VIE) of the second kind. Also, Du and Chen [5] proposed a high order reproducing kernel method to solve linear VFIE in the form of (1.1). Chen and his coworkers suggested approximate and exact schemes for solving some classes of (1.1) in [6] and [7], respectively. Regarding collocation methods, one can refer to [7] and [8], [9] and [10]. Taylor operational matrices are also implemented in [11].

The reproducing kernel method (RKM) has been widely used in solving some scientific models [12,13,14,15,16]. For more details, Li and Wu have implemented a new RKM for variable order fractional boundary value problems (BVPs) in [12]. Geng and Qian considered the optimal RKM for solving linear nonlocal BVPs in [13]. Xu and Lin implemented a simplified RKM for delay fractional ordinary differential equations (ODEs) in [14]. The aforementioned simplified RKM was then extended to solve 1D

[0]**Abbreviations:** ANA, anti-nuclear antibodies; APC, antigen-presenting cells; IRF, interferon regulatory factor

elliptic interface problems in [15]. Also, Abbasbandy and his coworkers used a variant of RKM to approximate the solutions of Brinkman-Forchheimer momentum equation in [16]. Moreover, a mixture of Newton method and simplified RKM was applied for computing the solutions of nonlinear fractional ODEs numerically in [17]. In this work, we construct a set of multi-scale standard orthogonal bases of reproducing kernel space $W_2^m[a,b]$ in section 2. Based on the least square method, we design an efficient numerical scheme in $W_2^m[a,b]$ for (1.1) in section 3. We proved that the condition number of the coefficient matrix of the system of linear equations is uniformly bounded. Therefore, the presented scheme is stable. The convergence of this method is also discussed. We find that the error of W_2^m-norm has the first-order convergence while the absolute error has $m+1$ order convergence as long as $u \in W_2^m[a,b]$. Some numerical results are provided to agree with the theoretical analysis in section 4. Moreover, in last Section, some interesting models including linear Volterra integro-differential equations (VIDE) of the first order [18] and the second order [19], pantograph type delay differential equations [20], nonlinear fractional integro-differential equations (FIDE) [21], noncompact Volterra integral equations [22,23], integral algebraic equations [24] and Volterra integro-differential algebraic equations [25] are introduced as future works that can be solved by the our proposed least square based reproducing kernel space method.

2 | PRELIMINARIES

Definition 1. The reproducing kernel space $W_2^m[a,b]$ is defined by

$$W_2^m[a,b] = \{u | u^{(m-1)} \text{is an absolute continuous function on } [a,b], u^{(m)} \in L^2[a,b]\}, \tag{2.1}$$

and equipped with the inner product and norm

$$\langle u, v\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)v^{(i)}(a) + \int_a^b u^{(m)}v^{(m)}\mathrm{d}x,\ \forall\, u,\ v \in W_2^m[a,b].$$

$$\|u\|^2_{W_2^m[a,b]} = \langle u, u\rangle_{W_2^m[a,b]},\ \forall\, u \in W_2^m[a,b].$$

Definition 2. [26] The kernel function $R_s(t)$ of the reproducing kernel space W satisfies

$$\langle u, R_s(t)\rangle = u(s), \forall u \in W. \tag{2.2}$$

For convenience, we abbreviate $\langle\cdot,\cdot\rangle_{W_2^m[a,b]}$ and $\|\cdot\|_{W_2^m[a,b]}$ as $\langle\cdot,\cdot\rangle_m$ and $\|\cdot\|_m$, respectively. We denote $R_x^m(y)$ as the reproducing kernel of W_2^m and use $(\cdot,\cdot)$ and $\|\cdot\|_0$ to represent the inner product and norm in $L^2[a,b]$.

Lemma 1. $\forall u \in W_2^m[a,b]$, satisfies the following where M is a constant

$$\|u\|_0 \le M\|u\|_m. \tag{2.3}$$

Proof. $\forall x \in [a,b]$,

$$u(x) = \langle u, R_x^m\rangle_m \le \|u\|_m\|R_x\|_m \le M_1\|u\|_m,\ \forall u \in W_2^m[a,b],$$

where $M_1 = \|R_x\|_m$. By the definition of $\|\cdot\|_0$, we deduce that $\|u\|_0 \le M\|u\|_m$. □

Next, we will give a standard orthogonal basis of $W_2^1[a,b]$. For $k = 1,2,3,\cdots$, $i = 0,1,\cdots,2^{k-1}-1$, let

$$\varphi_{k,i} = 2^{\frac{k-1}{2}}\sqrt{b-a}\begin{cases} \frac{x-a}{b-a} - \frac{i}{2^{k-1}},\ x \in [\frac{i}{2^{k-1}}(b-a)+a, \frac{i+1/2}{2^{k-1}}(b-a)+a], \\ \frac{i+1}{2^{k-1}} - \frac{x-a}{b-a},\ x \in [\frac{i+1/2}{2^{k-1}}(b-a)+a, \frac{i+1}{2^{k-1}}(b-a)+a], \\ \quad 0, \qquad others. \end{cases} \tag{2.4}$$

Lemma 2. $\left\{1, \frac{x-a}{\sqrt{b-a}}, \varphi_{1,0}, \varphi_{2,0}, \varphi_{2,1}, \cdots, \varphi_{k,0}, \varphi_{k,1}, \cdots, \varphi_{k,2^{k-1}-1}, \cdots\right\}$ is a standard orthogonal basis of $W_2^1[a,b]$.

Proof. By a direct computation, we derive

$$\langle 1, 1\rangle_1 = 1, \langle 1, \frac{x-a}{\sqrt{b-a}}\rangle_1 = 0, \langle \frac{x-a}{\sqrt{b-a}}, \frac{x-a}{\sqrt{b-a}}\rangle_1 = 1,$$

$$\langle 1, \varphi_{k,i}\rangle_1 = \langle \frac{x-a}{\sqrt{b-a}}, \varphi_{k,i}\rangle_1 = 0, k = 1,2,\cdots, i = 0,1,2,\cdots,2^{k-1}-1,$$

and

$$\langle \varphi_{k,i}, \varphi_{n,j} \rangle_1 = \begin{cases} 1, \ k = n, i = j, \\ 0, \ k \neq n, i \neq j. \end{cases}$$

Next, we will prove the basis presented above is complete in $W_2^1[a,b]$. Assume $u \in W_2^1[a,b]$ and it satisfies

$$\langle 1, u \rangle_1 = \langle \frac{x-a}{\sqrt{b-a}}, u \rangle_1 = 0, \langle \varphi_{k,i}, u \rangle_1 = 0, k = 1, 2, \cdots, i = 0, 1, 2, \cdots, 2^{k-1} - 1.$$

We need to prove $u \equiv 0$. By the definition of $\| \cdot \|_m$, it's easily to justify $\langle 1, u \rangle_1 = u(0) = 0, \langle u, \frac{x-a}{\sqrt{b-a}} \rangle_1 = \frac{1}{\sqrt{b-a}} u(1) = 0$, and $\langle u, \varphi_{k,i} \rangle_1 = u(\frac{i}{2^{k-1}}(b-a) + a) = 0, k = 1, 2, \cdots, i = 0, 1, 2, \cdots, 2^{k-1} - 1$. Due to the density of $\left\{ \frac{i}{2^{k-1}}(b-a) + a \right\}_{k,i}$ in [a,b] and the continuity of u, we obtain that $u \equiv 0$. □

Let $\mathcal{I}$ be an integral operator, that is, for any $u \in L^2[a,b]$, $\mathcal{I}u = \int_a^x u(t)\mathrm{d}t$. Then we have the following two lemmas.

Lemma 3. $\left\{ 1, x-a, \frac{1}{2\sqrt{b-a}}(x-a)^2, \mathcal{I}\varphi_{1,0}, \mathcal{I}\varphi_{2,0}, \mathcal{I}\varphi_{2,1}, \cdots, \mathcal{I}\varphi_{k,0}, \mathcal{I}\varphi_{k,1}, \cdots, \mathcal{I}\varphi_{k,2^{k-1}-1}, \cdots \right\}$ is a standard orthogonal basis of $W_2^2[a,b]$.

Proof. According to the definition of inner product and norm in $W_2^2[a,b]$ and simple calculation, the orthogonality of $\left\{ 1, x-a, \frac{1}{2\sqrt{b-a}}(x-a)^2, \mathcal{I}\varphi_{1,0}, \mathcal{I}\varphi_{2,0}, \mathcal{I}\varphi_{2,1}, \cdots, \mathcal{I}\varphi_{k,0}, \mathcal{I}\varphi_{k,1}, \cdots, \mathcal{I}\varphi_{k,2^{k-1}-1}, \cdots \right\}$ can be obtained. Similar to the proof of lemma 2.2, the conditions

$$\langle x-a, u \rangle_2 = \langle \frac{(x-a)^2}{2\sqrt{b-a}}, u \rangle_2 = 0, \langle \mathcal{I}\varphi_{k,i}, u \rangle_2 = 0, k = 1, 2, \cdots, i = 0, 1, 2, \cdots, 2^{k-1} - 1,$$

lead to $u' \equiv 0$. Thus $u \equiv C$. Besides, $\langle 1, u \rangle_2 = 0$ results in $u = 0$. Therefore, $u \equiv 0$. The completeness of the above basis is proved. □

Lemma 4. $\left\{ 1, x-a, \frac{1}{2}(x-a)^2, \frac{1}{6\sqrt{b-a}}(x-a)^3, \mathcal{I}^2\varphi_{1,0}, \mathcal{I}^2\varphi_{2,0}, \mathcal{I}^2\varphi_{2,1}, \cdots, \mathcal{I}^2\varphi_{k,0}, \mathcal{I}^2\varphi_{k,1}, \cdots, \mathcal{I}^2\varphi_{k,2^{k-1}-1}, \cdots \right\}$ is a standard orthogonal basis of $W_2^3[a,b]$.

The proof is similar to Lemma 2.3, we omit it here.

Exactly, a standard orthogonal basis of $W_2^m[a,b]$ can be obtained. For the sake of simplification, we denote the basis obtained above in $W_2^m[a,b]$ as $\{\psi_i\}_{i=0}^m \bigcup \{\phi_{k,0}, \phi_{k,1}, \cdots, \phi_{k,2^{k-1}-1}\}_{k=1}^{\infty}$.

According to Eq.(1.1), we define an operator $\mathcal{L} : W_2^m[a,b] \to W_2^m[a,b]$ as following:

$$\mathcal{L}u = u(x) - \int_a^x k_1(x,t)u(t)\mathrm{d}t - \int_a^b k_2(x,t)u(t)\mathrm{d}t, \ x \in [a,b].$$

Obviously, the operator $\mathcal{L}$ is linear and bounded.

Remark 2.1: Suppose that $k_1(x,t), k_2(x,t) \in C^m([a,b])^2$.

$$\|\mathcal{L}u_n^* - f\|_0^2 = \| \sum_{j=0}^m d_j^* \mathcal{L}\psi_j + \sum_{k=1}^n \sum_{i=0}^{2^{k-1}-1} c_{k,i}^* \mathcal{L}\phi_{k,i} - f \|_0^2 = \min \|\mathcal{L}u_n - f\|_0^2. \tag{2.5}$$

Definition 3. For any positive ϵ, v is termed as an ϵ-approximating solution of Eq.(1.1) in $W_2^m[a,b]$ if

$$\|\mathcal{L}v - f\|_m^2 \leq \epsilon^2.$$

Lemma 5. Assume that u is the exact solution of Eq.(1.1) in $W_2^m[a,b]$, then the Eq.(1.1) exists an ϵ-approximating solution for any positive ϵ.

Proof. As u belongs to $W_2^m[a,b]$, it follows that u can be expressed by

$$u(x) = \sum_{j=0}^m d_j^\star \psi_j + \sum_{k=1}^\infty \sum_{i=0}^{2^{k-1}-1} c_{k,i}^\star \phi_{k,i},$$

where $d_j^\star = \langle \psi_j, u \rangle_m, c_{k,i}^\star = \langle \phi_{k,i}, u \rangle_m$.

Thus, $\forall \epsilon > 0$, $\exists N \in N^*$ satisfies

$$\|u - \sum_{j=0}^{m} d_j^{\star}\psi_j - \sum_{k=1}^{n}\sum_{i=0}^{2^{k-1}-1} c_{k,i}^{\star}\phi_{k,i}\|_m^2 \leq \frac{1}{\|\mathcal{L}\|^2}\epsilon^2,\ \forall\, n \geq N.$$

Taking $u_n^{\star} = \sum_{j=0}^{m} d_j^{\star}\psi_j - \sum_{k=1}^{n}\sum_{i=0}^{2^{k-1}-1} c_{k,i}^{\star}\phi_{k,i}$, we have

$$\|\mathcal{L}u_n^{\star} - f\|_m^2 = \|\mathcal{L}u_n^{\star} - \mathcal{L}u\|_m^2 \leq \|\mathcal{L}\|^2\|u_n^{\star} - u\|_m^2 \leq \|\mathcal{L}\|^2\frac{1}{\|\mathcal{L}\|^2}\epsilon^2 = \epsilon^2,$$

which indicates that $u_n^{\star}$ is an ϵ-approximating solution of Eq.(1.1). □

3 | A LEAST SQUARE METHOD AND ITS CONVERGENCE ANALYSIS

We will raise a stable scheme for (1.1) and give the theoretical analysis of the scheme in this section.

Let

$$u_n = \sum_{j=0}^{m} d_j\psi_j + \sum_{k=1}^{n}\sum_{i=0}^{2^{k-1}-1} c_{k,i}\phi_{k,i}$$

be an approximating solution of Eq.(1.1) in $W_2^m[a,b]$. We need to determine the unknown coefficients. We will apply the least square method to determine the unknown coefficients. Let

$$J(d_0, d_1, \ldots, d_m, c_{1,0}, c_{2,0}, c_{2,1}, \cdots, c_{n,0}, c_{n,1}, \cdots, c_{n,2^{n-1}-1}) = \|\mathcal{L}u_n - f\|_m^2.$$

Algorithm: The least square method for problem (1.1) reads as: Finding $d_0^*, d_0^*, \ldots, d_m^*, c_{1,0}^*, c_{2,0}^*, c_{2,1}^*, \cdots, c_{n,0}^*, c_{n,1}^*, \cdots, c_{n,2^{n-1}-1}^*$ such that

$$\|\mathcal{L}u_n^* - f\|_m^2 = \|\sum_{j=0}^{m} d_j^*\mathcal{L}\psi_j + \sum_{k=1}^{n}\sum_{i=0}^{2^{k-1}-1} c_{k,i}^*\mathcal{L}\phi_{k,i} - f\|_m^2 = \min J. \tag{3.1}$$

It is trivial to derive that the algorithm (3.1) of searching the minimum value of J is equivalent to solving the following linear equations:

$$\begin{cases} \langle \mathcal{L}\psi_j, \mathcal{L}u_n\rangle_m &= \langle \mathcal{L}\psi_j, f\rangle_m,\ j = 1, 2, \cdots, m, \\ \langle \mathcal{L}\phi_{k,i}, \mathcal{L}u_n\rangle_m &= \langle \mathcal{L}\phi_{k,i}, f\rangle_m,\ k = 1, 2, \cdots, n, i = 1, 2, \cdots, 2^{k-1} - 1. \end{cases} \tag{3.2}$$

Therefore, the unknown coefficients $d_0^*, d_1^*, \ldots, d_m^*, c_{1,0}^*, c_{2,0}^*, c_{2,1}^*, \cdots, c_{n,0}^*, c_{n,1}^*, \cdots, c_{n,2^{n-1}-1}^*$ can be determined by (3.2).

Remark 3.1: In $W_2^m[a,b]$, the computation of $\|\cdot\|_0$ is simpler than that of $\|\cdot\|_m$. Thus we can simplify the scheme (3.1), that is, finding $d_0^*, d_1^*, \ldots, d_m^*, c_{1,0}^*, c_{2,0}^*, c_{2,1}^*, \cdots, c_{n,0}^*, c_{n,1}^*, \cdots, c_{n,2^{n-1}-1}^*$ such that

$$\|\mathcal{L}u_n^* - f\|_0^2 = \|\sum_{j=0}^{m} d_j^*\mathcal{L}\psi_j + \sum_{k=1}^{n}\sum_{i=0}^{2^{k-1}-1} c_{k,i}^*\mathcal{L}\phi_{k,i} - f\|_0^2 = \min\|\mathcal{L}u_n - f\|_0^2. \tag{3.3}$$

Remark 3.2: Selecting limited points $\{x_l\}_{l=1}^{N}$ in the interval [a, b], we can discretize the scheme (3.1), that is, finding $d_0^*, d_0^*, \ldots, d_m^*, c_{1,0}^*, c_{2,0}^*, c_{2,1}^*, \cdots, c_{n,0}^*, c_{n,1}^*, \cdots, c_{n,2^{n-1}-1}^*$ such that

$$\sum_{l=1}^{N}\left|\mathcal{L}u_n^*(x_l) - f(x_l)\right|^2 = \min\sum_{l=1}^{N}|\sum_{j=0}^{m} d_j\mathcal{L}\psi_j(x_l) + \sum_{k=1}^{n}\sum_{i=0}^{2^{k-1}-1} c_{k,i}\mathcal{L}\phi_{k,i}(x_l) - f(x_l)|^2. \tag{3.4}$$

Theorem 1. The approximating solutions u_n^* obtained by schemes (3.1) and (3.3) are ϵ-approximating solutions.

Proof. From the proof of the Lemma 2.5, it holds that there exists an ϵ-approximating solution $u_n^{\star}$ of Eq.(1.1) in $W_2^m[a,b]$, that is,

$$\|\mathcal{L}u_n^{\star} - f\|_m^2 \leq \epsilon^2,$$

As $\mathcal{L}u_n^*$ is the closest to f under $\|\cdot\|_m^2$-norm, we derive that

$$\|\mathcal{L}u_n^* - f\|_m^2 \leq \|\mathcal{L}u_n^{\star} - f\|_m^2 \leq \epsilon^2,$$

which implies that u_n^* obtained by the scheme (3.1) is an ϵ-approximating solution. Analogously, we deduce that u_n^* obtained by the scheme (3.3) is also an ϵ-approximating solution by Lemma 2.1. □

Theorem 2. Assume that $\mathcal{L}$ is a reversible operator from $W_2^m[a,b]$ to $W_2^m[a,b]$, then the scheme (3.1) is uniquely solvable and stable.

Proof. Denote the coefficient matrix of the system (3.2) by G. As G is a Gram matrix, we only need to show $\{\mathcal{L}\psi_i\}_{i=0}^m \bigcup \{\mathcal{L}\phi_{k,0}, \mathcal{L}\phi_{k,1}, \cdots, \mathcal{L}\phi_{k,2^{k-1}-1}\}_{k=1}^n$ are linearly independent. The combination of reversibility of the operator $\mathcal{L}$ and linear independence of $\{\psi_i\}_{i=0}^m \bigcup \{\phi_{k,0}, \phi_{k,1}, \cdots, \phi_{k,2^{k-1}-1}\}_{k=1}^n$ lead to the unique solvability of the scheme (3.1).

To show the stability, we consider the condition number of G. Let λ be an eigenvalue of G and $Y = (y_1, y_2, \cdots, y_L)$ be the related eigenvector with $\|Y\|_0 = 1$. Then it holds

$$Gy = \lambda y.$$

Therefore, we have $\lambda y_i = G(i,:)Y$. It follows that

$$\lambda = \lambda \sum_{i=1}^{N} y_i^2 = \|\mathcal{L}(\sum_{j=1}^{m} y_j\psi_j + \sum_{k=1}^{n}\sum_{i=0}^{2^{k-1}-1} y_{k,i}\phi_{k,j})\|_m^2 \leq \|\mathcal{L}\|^2 \|\sum_{j=1}^{m} y_j\psi_j + \sum_{k=1}^{n}\sum_{i=0}^{k-1} y_{k,i}\phi_{k,j}\|_m^2 = \|\mathcal{L}\|^2. \tag{3.5}$$

which indicates that λ is greater than zero. On the other hand, we obtain that

$$1 = \|Y\|_0^2 = \|\sum_{j=1}^{m} y_j\psi_j + \sum_{k=1}^{n}\sum_{i=0}^{k-1} y_{k,i}\phi_{k,j}\|_m^2 = \|\mathcal{L}^{-1}\mathcal{L}(\sum_{j=1}^{m} y_j\psi_j + \sum_{k=1}^{n}\sum_{i=0}^{k-1} y_{k,i}\phi_{k,j})\|_m^2 \leq \|\mathcal{L}^{-1}\|^2 \cdot \lambda. \tag{3.6}$$

Combined with (3.5) and (3.6), we have

$$\mathrm{cond}(G) = \frac{\lambda_{\max}}{\lambda_{\min}} \leq \|\mathcal{L}\|^2\|\mathcal{L}^{-1}\|^2,$$

that means the condition number of G is uniformly bounded. Thus, we have verified the stability of the scheme (3.1). □

Remark 3.3: Similar to the proof of Theorems 3.2, the unique solvability of the solution of the scheme (3.3) can be derived. Moreover, estimation (3.5) is valid, but estimation (3.6) does not hold.
Remark 3.4: The scheme (3.4) is a collocation method. However, we find it difficult to prove its stability.
Remark 3.5: Numerical results illustrate that schemes (3.3) and (3.4) are stable.

Theorem 3. Assume that the exact solution $u \in W_2^m[a,b]$, then $u_n^* \rightrightarrows u$. Furthermore,

$$\|u - u_n^*\|_m \leq 2^{-n}M.$$

Proof. By the reproducibility of R_x^m and the Cauchy-Schwarz inequality, it follows that

$$|u(x) - u_n^*(x)| = \langle u - u_n^*, R_x^m\rangle_m \leq \|u - u_n^*\|_m\|R_x^m\| \leq M_1\|u - u_n^*\|_m. \tag{3.7}$$

Let $u_n^\star$ be the ϵ-approximating solution defined in Lemma 2.5. Then $\|u - u_n^\star\|_m \to 0$ as $n \to \infty$. Since the operators $\mathcal{L}^{-1}$ and $\mathcal{L}$ are bounded, we derive that

$$\|u - u_n^*\|_m = \|\mathcal{L}^{-1}\mathcal{L}(u - u_n^*)\|_m \leq \|\mathcal{L}^{-1}\| \cdot \|\mathcal{L}\| \cdot \|u - u_n^*\|_m \leq \|\mathcal{L}^{-1}\| \cdot \|\mathcal{L}\| \cdot \|(u - u_n^\star)\|_m \to 0.$$

In fact, we have

$$\|u - u_n^\star\|_m^2 = \|\sum_{k=n+1}^{\infty}\sum_{i=0}^{2^{k-1}-1} c_{k,i}^\star\phi_{k,i}\|_m^2 = \sum_{k=n+1}^{\infty}\sum_{i=0}^{2^{k-1}-1} c_{k,i}^{\star 2} = \sum_{k=n+1}^{\infty}\sum_{i=0}^{2^{k-1}-1} (\int_a^b u^{(m)}\varphi'_{k,i}\mathrm{d}x)^2. \tag{3.8}$$

Because

$$u^{(m)}(x) = u^{(m)}\left(\frac{b-a}{2^{k-1}}i + a\right) + u^{(m+1)}(\xi)\left(x - \frac{b-a}{2^{k-1}}i - a\right),$$

we immediately obtain that

$$|c_{k,i}^\star| \leq \left|\int_{\frac{b-a}{2^{k-1}}i+a}^{\frac{b-a}{2^{k-1}}(i+1)+a} u^{(m)}\left(\frac{b-a}{2^{k-1}}i + a\right)\varphi'_{k,i}\right| + \left|\int_{\frac{b-a}{2^{k-1}}i+a}^{\frac{b-a}{2^{k-1}}(i+1)+a} u^{(m+1)}(\xi)\left(x - \frac{b-a}{2^{k-1}}i - a\right)\varphi'_{k,i}\right| = I_1 + I_2.$$

By a straightforward calculation, it holds I_1 equals to zero and

$$I_2 \leq \frac{2^{\frac{k-1}{2}}}{\sqrt{b-a}}|u^{(m+1)}|_C \int_{\frac{b-a}{2^{k-1}}i+a}^{\frac{b-a}{2^{k-1}}(i+1)+a} \left(x - \frac{b-a}{2^{k-1}}i - a\right)\mathrm{d}x \leq 2^{-\frac{3}{2}k}M_2.$$

Thus, we have

$$\|u-u_n^{\star}\|_m^2 \le \sum_{k=n+1}^{\infty}\sum_{i=0}^{2^{k-1}-1}(2^{-\frac{3}{2}k}M_2)^2 \le 4^{-n}M_3,$$

which implies that $\|u-u_n^*\|_m \le 2^{-n}M$. The proof is completed. □

Remark 3.6: Let $|u|_m^2 = \int_a^b u^{(m)}u^{(m)}\mathrm{d}x$. By the definition of $\|\cdot\|_m$, we have $|\cdot|_m$ is bounded by $\|\cdot\|_m$. Theorem 3.3 demonstrates that the error under $|\cdot|_m$-seminorm at least has first-order convergence.

4 | NUMERICAL EXAMPLES

The efficiency and robustness of the presented approach are verified via six numerical examples in this section. Numerical results obtained by schemes (3.1), (3.3) and (3.4) are similar for Example 4.1, 4.2 and 4.5, so we only illustrate results from the scheme (3.1). All of the codes associated to the proposed method are written in Mathematica software. In addition, we shall examine several numerical errors, denoted in the following notations:

$$\|e_n\|_\infty = \max_{x\in[a,b]}|u-u_n^*|,\ \ \|e_n\|_{W_2^m} = \|u-u_n^*\|_m,$$

and the convergence order is tested by the following:

$$\text{C.O.} = \log_2\frac{\|e_n\|}{\|e_{n+1}\|}.$$

Example 4.1: Let us consider the problem suggested in [5], wherein $a=0$, $b=1$, $k_1(t,x)=e^t\cos(x)$, $k_2(t,x)=e^t\sin(x)$ and $f(x)=e^x-\frac{1}{2}\cos(x)(e^{2x}-1)+\frac{1}{2}\sin(x)(e^2-1)$. $u(x)=e^x$ is the exact solution.

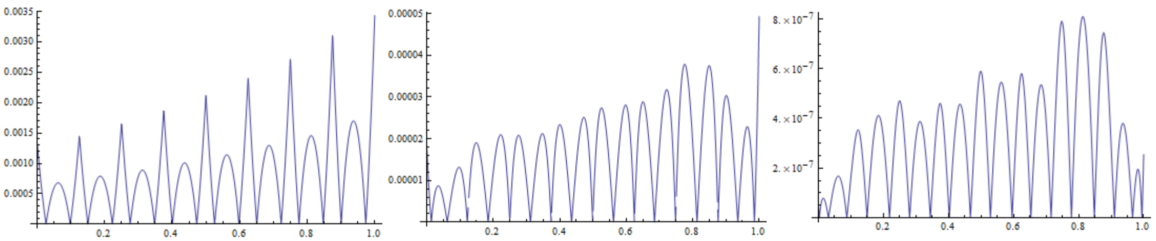

FIGURE 1 Absolute errors in W_2^1(left), W_2^2(middle) and W_2^3(right) for Example 4.1 with $n=3$.

TABLE 1 Convergence order for Example 4.1

	$W_2^1[0,1]$		$W_2^2[0,1]$		$W_2^3[0,1]$	
n	$\|e_n\|_{L^\infty}$	$\|e_n\|_{W_2^1}$	$\|e_n\|_{L^\infty}$	$\|e_n\|_{W_2^2}$	$\|e_n\|_{L^\infty}$	$\|e_n\|_{W_2^3}$
3	3.46×10^{-3}	6.45×10^{-2}	4.93×10^{-5}	6.47×10^{-2}	8.00×10^{-7}	6.47×10^{-2}
4	8.84×10^{-4}	3.23×10^{-2}	6.18×10^{-6}	3.23×10^{-2}	5.01×10^{-8}	3.24×10^{-2}
5	2.21×10^{-4}	1.62×10^{-2}	7.73×10^{-7}	1.63×10^{-2}	3.13×10^{-9}	1.62×10^{-2}
6	5.56×10^{-5}	8.01×10^{-3}	9.60×10^{-8}	8.02×10^{-3}	1.96×10^{-10}	8.01×10^{-3}
7	1.37×10^{-5}	4.01×10^{-3}	1.21×10^{-9}	4.01×10^{-3}	1.22×10^{-11}	4.01×10^{-3}
8	3.45×10^{-6}	2.00×10^{-3}	1.50×10^{-10}	2.01×10^{-3}	7.64×10^{-12}	2.00×10^{-3}
C.O.	2.0	1.0	3.0	1.0	4.0	1.0

Tables 1 and Figure 1 illustrate the performance of the proposed scheme (3.1) for the test problem 4.1 in $W_2^m[0,1](m=1,2,3)$. It can be seen from the Table 1 that the convergence order of W_2^m-norm error is 1, which is in agreement with our theoretical finding. As expected, the L^∞-norm error reaches to $\mathcal{O}(2^{-(m+1)n})$ in $W_2^m[0,1]$.

Example 4.2: Let us research the question from [11,17], wherein $a=0$, $b=1$, $k_1(t,x)=e^{t+x}$, $k_2(t,x)=-e^{t+h(x)}$ and $f(x)=e^{-x}-xe^x(h(x)-1)$. $u(x)=e^{-x}$ is the exact solution.

We take $k_2(t,x)=-e^{t+\ln(x+1)}$, the absolute error is given in Figure 2. In addition, the L^∞-norm error of our method is compared, the Taylor collocation method (TCM) in [10] and Taylor polynomial method (TPM) in [11] in Table 2. Our method approximates the exact solution more closely.

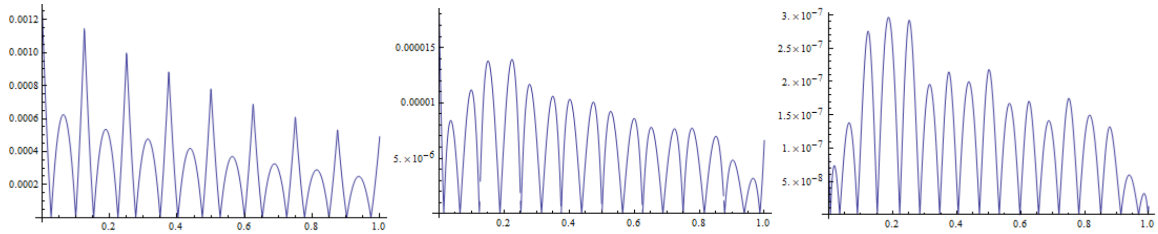

FIGURE 2 Absolute errors in W_2^1(left), W_2^2(middle) and W_2^3(right) for Example 4.2 with $n=3$.

TABLE 2 Comparison of the absolute errors for Example 4.2

	Our method		**TCM in [10]**		**TPM in [11]**	
n	$h(x)=x$	$h(x)=\ln(x+1)$	$h(x)=x$	$h(x)=\ln(x+1)$	$h(x)=x$	$h(x)=\ln(x+1)$
2	4.07×10^{-6}	4.03×10^{-6}	3.68×10^{-3}	3.27×10^{-3}	2.23×10^{-2}	3.59×10^{-2}
5	9.94×10^{-10}	9.90×10^{-10}	4.03×10^{-7}	4.30×10^{-7}	1.41×10^{-4}	3.05×10^{-4}
8	2.43×10^{-13}	2.41×10^{-13}	0.00×10^{-0}	5.96×10^{-8}	2.52×10^{-7}	5.61×10^{-7}
9	1.52×10^{-14}	1.50×10^{-14}	0.00×10^{-0}	8.84×10^{-8}	2.47×10^{-8}	1.41×10^{-7}

Example 4.3: Let us consider a VIE with weakly singular kernel, where $a=0$, $b=1$, $k_1(x,t)=\frac{1}{\sqrt{x-t}}$, $k_2(x,t)=0$ and $f(x)=\sin(\pi x)x+\sqrt{2}\left[-\cos(\pi x)S(\sqrt{2x})+\sin(\pi x)C(\sqrt{2x})\right]$, in which $S(x)=\int_0^x\cos(\frac{\pi t^2}{2})$ and $C(x)=\int_0^x\sin(\frac{\pi t^2}{2})$, $u(x)=\sin(\pi x)$. This example is found in [7].

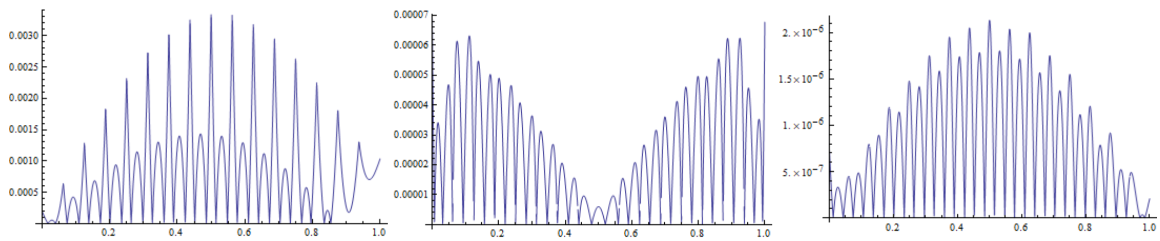

FIGURE 3 Absolute errors in W_2^1(left), W_2^2(middle) and $W_2^3[0,1]$(right) for Example 4.3 with $n=4$.

Due to the singularity of $k_1(s,x)$, we apply the scheme (3.3) to solve Problem 4.3. The surface plots of the absolute error with $n=4$ are demonstrated in Figure 3, which implies that our method is also applicable to this kind of problems. The convergence order of the L^∞-norm and W_2^m-norm errors are listed in Table 3, which shows that the singularity of $k_1(x,t)$ does not reduce the convergence order of the algorithm.

TABLE 3 Convergence order for Example 4.3

	$W_2^1[0,1]$		$W_2^2[0,1]$		$W_2^3[0,1]$	
n	$\|e_n\|_{L^\infty}$	$\|e_n\|_{W_2^1}$	$\|e_n\|_{L^\infty}$	$\|e_n\|_{W_2^2}$	$\|e_n\|_{L^\infty}$	$\|e_n\|_{W_2^3}$
3	1.58×10^{-2}	2.52×10^{-1}	5.57×10^{-4}	7.99×10^{-1}	3.45×10^{-5}	2.50×10^{-0}
4	3.34×10^{-3}	1.26×10^{-1}	6.79×10^{-5}	3.97×10^{-1}	2.13×10^{-6}	1.24×10^{-0}
5	7.92×10^{-4}	6.30×10^{-2}	8.35×10^{-6}	1.98×10^{-1}	1.32×10^{-7}	6.22×10^{-1}
6	1.96×10^{-4}	3.15×10^{-2}	1.06×10^{-6}	9.89×10^{-2}	8.25×10^{-9}	3.11×10^{-1}
7	4.85×10^{-5}	1.58×10^{-2}	1.32×10^{-7}	4.94×10^{-2}	5.16×10^{-10}	1.56×10^{-1}
8	1.21×10^{-5}	7.89×10^{-3}	1.66×10^{-8}	2.47×10^{-2}	3.22×10^{-11}	7.79×10^{-2}
C.O.	2.0	1.0	3.0	1.0	4.0	1.0

Example 4.4: Consider the VIE with the exact solution has singularity near a. In this example, $a = 0$, $b = 1$, $k_1(x,t) = x+t$, $k_2(x,t) = xt$ and $f(x) = -\frac{2}{9}x + x^{5/2} - \frac{32}{63}x^{9/2}$. The exact solution of this problem is $u(x) = x^{5/2}$.

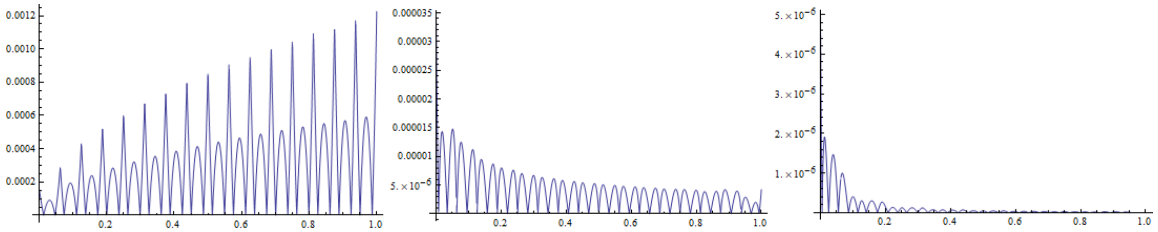

FIGURE 4 Absolute errors in W_2^1(left), W_2^2(middle) and $W_2^3[0,1]$(right) for Example 4.4 with $n = 4$.

TABLE 4 Convergence order for Example 4.4

	$W_2^1[0,1]$		$W_2^2[0,1]$		$W_2^3[0,1]$	
n	$\|e_n\|_{L^\infty}$	C.O	$\|e_n\|_{L^\infty}$	C.O	$\|e_n\|_{L^\infty}$	C.O
3	5.02×10^{-3}		1.93×10^{-4}	2.50	2.91×10^{-5}	
4	1.23×10^{-3}	2.03	3.42×10^{-5}	2.52	4.94×10^{-6}	2.56
5	3.05×10^{-4}	2.06	5.97×10^{-6}	2.95	8.38×10^{-7}	2.56
6	7.62×10^{-5}	2.01	7.70×10^{-7}	2.95	1.08×10^{-7}	2.95
7	1.91×10^{-5}	2.00	9.96×10^{-8}	2.95	1.40×10^{-8}	2.96
8	4.76×10^{-6}	2.00	1.29×10^{-8}	2.95	1.81×10^{-9}	2.95

The scheme (3.3) is applied to solve Problem 4.4. In Figure 4, we plot the absolute error. In Table 4, we list the convergence order of the L^∞-norm and W_2^m-norm errors. Since $u(x)$ is singular near $x = 0$, the convergence order of L^∞-norm error in W_2^2 is nearly to third and that in W_2^3 reduces to third.

Example 4.5: Consider the nonlinear VIE:

$$u(x) = \int_0^x \frac{u^4(t)}{\sqrt{x+t}}\mathrm{d}t + f(x),\ \ x \in [0,1].$$

where $f(x) = x^5 - 0.0340676x^{41/2}$ with the exact solution given by $u(x) = x^5$.

We apply QNM in [17] and the scheme (3.3) to solve Problem 4.5. We take $n = 4$, the absolute errors are displayed in Figure 5. The results in Table 5 illustrate the convergence order of the L^∞-norm and W_2^m-norm errors. Again the convergence order of absolute error is still 2, 3 and 4 in W_2^1, W_2^2 and W_2^3 is observed for this nonlinear problem.

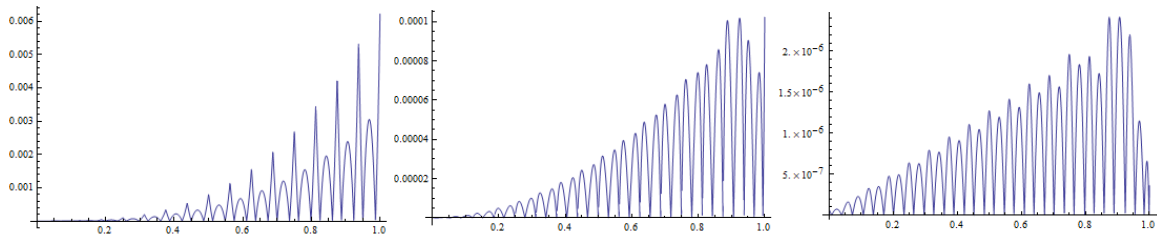

FIGURE 5 Absolute errors in W_2^1(left), W_2^2(middle) and $W_2^3[0,1]$(right) for Example 4.5 with $n = 4$.

TABLE 5 Convergence order for Example 4.5

	$W_2^1[0,1]$		$W_2^2[0,1]$		$W_2^3[0,1]$	
n	$\|e_n\|_{L^\infty}$	C.O	$\|e_n\|_{L^\infty}$	C.O	$\|e_n\|_{L^\infty}$	C.O
3	2.38×10^{-2}		9.38×10^{-4}		3.52×10^{-5}	
4	6.21×10^{-3}	1.93	1.03×10^{-4}	3.18	2.35×10^{-6}	3.88
5	1.58×10^{-3}	1.97	1.24×10^{-5}	3.05	1.58×10^{-7}	3.91
6	3.97×10^{-4}	1.99	1.53×10^{-6}	3.02	1.02×10^{-8}	3.96
7	9.93×10^{-5}	2.00	1.91×10^{-7}	3.00	6.43×10^{-9}	3.98
8	2.48×10^{-5}	2.00	2.39×10^{-8}	3.00	4.02×10^{-10}	4.00

Example 4.6:[27] Consider the VIE with nonsmooth solution:

$$u(x) = x^{\frac{1}{2}} - \frac{\pi^3}{24}x^{\frac{3}{2}} + \int_0^{\frac{\pi}{2}} (xt)^{\frac{3}{2}} u(t)\mathrm{d}t, \quad x \in [0, \frac{\pi}{2}],$$

the exact solution is $u(x) = x^{\frac{1}{2}}$.

We apply the scheme (3.3) to solve Problem 4.6. We take $n = 4, 5, 6$, the absolute errors in logarithm format are exhibited in Figure 6 . However, the rate of convergence of the approximate solutions are not the same of previous examples, but our proposed method is efficient for solving such this type of problems.

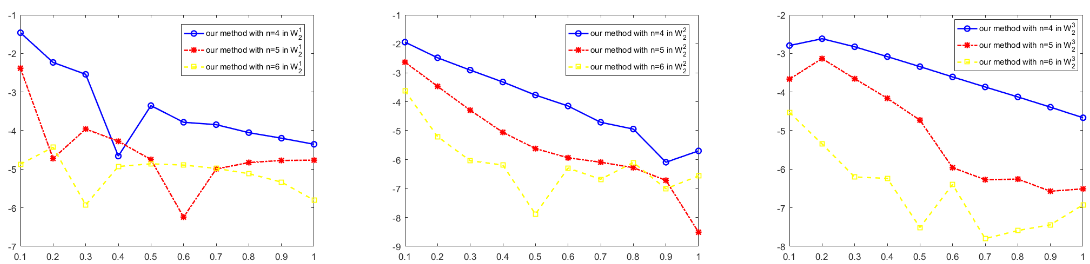

FIGURE 6 Absolute errors in W_2^1(left), W_2^2(middle) and $W_2^3[0,1]$(right) for Example 4.6 with $n = 4, 5, 6$ in loglog format

5 | CONCLUSIONS AND FUTURE WORKS

In this work, we introduce the least square method in reproducing kernel space $W_2^m[a,b]$ for the second type linear VFIE. We discuss the stability of the scheme and obtain that convergence order of W_2^m-norm error is 1. Numerical examples show that the approximating space we choose is smoother, the convergence order of absolute error is higher. What's more, the presented algorithm works well for weakly singular and nonlinear Volterra integral problems. As our future work, we will extend our proposed method for solving the first order linear VFIDE[18]:

$$\begin{cases} u'(x) - c(x)u(x) - \int_a^x k_1(x,t)u(t)dt - \int_a^b k_2(x,t)u(t)dt = f(x), \quad x \in [a,b], \\ u(a) = u_0, \end{cases}$$

and the second order linear VFIDE[19]:

$$\begin{cases} u''(x) - c_1(x)u(x) - c_2(x)u'(x) - \int_a^x k_1(x,t)u(t)dt - \int_a^b k_2(x,t)u(t)dt = f(x), \quad x \in [a,b], \\ u(a) = u_0, \quad u'(a) = u_0^{(1)}. \end{cases}$$

Also, our aim is for solving numerically linear VFIDE with vanishing pantograph delays[20]:

$$u(x) - \int_a^{q_1 x} k_1(x,t)u(t)\mathrm{d}t - \int_a^{q_2 x} k_2(x,t)u(t)\mathrm{d}t - \int_a^{b} k_3(x,t)u(t)\mathrm{d}t = f(x), \quad x \in [a,b].$$

Moreover, our suggsted scheme can be generalized for treating the following class of nonlinear FIDE approximately[21]

$${}_0^cD_x^\gamma u(x) - \int_a^x k_1(x,t,u(t))dt - \int_a^b k_2(x,t,u(t))dt = f(x,u(x)), \qquad x \in [a,b].$$

In addition, with some modifications in our novel and accurate method, noncompact Volterra-Fredholm integral equations can be considered in details both numerically and theoretically[22,23]:

$$u(x) - \int_a^x \frac{t^{\mu-1}}{x^\mu} k_1(x,t)u(t)dt - \int_a^b \frac{t^{\mu-1}}{x^\mu} k_2(x,t)u(t)dt = f(x), \qquad x \in [a,b].$$

The area of integral algebraic equations is another aim and future work to investigate our numerical technique for solving them numerically[24]:

$$\begin{cases} u(x) - \int_a^x k_{11}(x,t)u(t)dt - \int_a^x k_{12}(x,t)v(t)dt = f_1(x), \qquad x \in [a,b], \\ \quad 0 - \int_a^x k_{21}(x,t)u(t)dt - \int_a^x k_{22}(x,t)v(t)dt = f_2(x). \end{cases}$$

Finally, integro-differential algebraic equations can be a good candidate for testing the accuracy of the suggested least square based RKM[25]:

$$\begin{cases} u'(x) - c_{11}(x)u(x) - c_{12}(x)v(x) - \int_a^x k_{11}(x,t)u(t)dt - \int_a^x k_{12}(x,t)v(t)dt = f_1(x), \qquad x \in [a,b], \\ \quad 0 - \int_a^x k_{21}(x,t)u(t)dt - \int_a^x k_{22}(x,t)v(t)dt = f_2(x). \end{cases}$$

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How to cite this article: Minqiang Xu, Jing Niu, Emran Tohidi, Jinjiao Hou (2020), A new least square based reproducing kernel space method for solving regular and weakly singular 1D Volterra-Fredholm integral equations with smooth and nonsmooth solutions, *Math Meth Appl Sci*, *2020;00:1–6*.