Effect of the wave speeds on the decay rate of the thermoelastic structure in the whole line with interfacial slip

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Abstract

In this work we the study of the asymptotic behavior in the whole line of a thermoelastic structure with interfacial slip and second sound. We prove several polynomial decay estimates depending on the smoothness of initial data. The proof is based on the semigroup approach, the energy method by introducing a Lyapunov functional and Fourier transform.

Keywords: Thermoelastic structure, polynomial stability, interfacial slip, semigroup approach, energy method, Fourier transform.

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1 Introduction

In case of bounded domain, the well-posedness and stability of structure of two-layered beams with interfacial slip have been studied by many authors for the last twenty years. Let us mention here some of these works. In [7], it was derived the following model for two-layered beams with structural damping due to the interfacial slip:

$$\begin{cases} \rho\,\varphi_{tt} + G\,(\psi - \varphi_x)_x = 0 \\ I_\rho\,(3\,S_{tt} - \psi_{tt}) - G\,(\psi - \varphi_x) - D\,(3\,S_{xx} - \psi_{xx}) = 0 \\ 3\,I_\rho\,S_{tt} + 3\,G(\psi - \varphi_x) + 4\,\delta_0\,S + 4\,\gamma_0\,S_t - 3\,D\,S_{xx} = 0 \end{cases} \tag{1.1}$$

where $x \in (0,\,1)$, $t \geq 0$, $\varphi(x,\,t)$ denotes the transverse displacement, $\psi(x,\,t)$ represents the rotation angle and $S(x,\,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x, and ρ, G, I_ρ, D, δ_0 and γ_0 are the density of the beams, the shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness and adhesive damping of the beams, respectively. For papers that deal with interfacial slip we mention, for instance, [1, 7, 10, 15, 19].

In [5], Guesmia considered the system (1.1) in more general form: $3\,I_p$ and $3\,G$ in $(1.1)_3$ were replaced by positive constants ρ_3 and k_3, respectively. The subject of [5] was stabilizing the system by one control defined in term of an infinite memory or in term of a frictional damping, and acting only on one equation. The author proved that this control is capable alone to guarantee the strong and polynomial stability of the system; that is bringing it back to its equilibrium state with a decay rate of type t^{-d}, where d is a positive constant depending on the regularity of initial data. Moreover, it was also proved that, when the control is effective on the first equation, the system is not exponentially stable independently of the values of the parameters and, when the control is effective on the second or the third equation the exponential stability is equivalent to the equality between the three speeds of wave propagations.

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From now we consider $s(x,\, t) = 3\, S(x,\, t),\ \rho_1 = \rho,\ \rho_2 = I_\rho,\ k = G,\ b = D,\ \delta = \frac{1}{3}\,\delta_0$ and $\gamma = \frac{1}{3}\,\gamma_0$. Then we deduce from (1.1) the following system:

$$\begin{cases} \rho_1\, \varphi_{tt} + k\,(\psi - \varphi_x)_x = 0, \\ \rho_2\,(s-\psi)_{tt} - b\,(s-\psi)_{xx} - k\,(\psi - \varphi_x) = 0, \\ \rho_2\, s_{tt} - b\, s_{xx} + 3\,k\,(\psi - \varphi_x) + 4\,\delta\, s + 4\,\gamma\, s_t = 0. \end{cases} \tag{1.2}$$

Note that, (1.2) can be derived from the following more general model related to Bresse-type systems (known as the circular arch problem):

$$\begin{cases} \rho_1\, \varphi_{tt} - k_1\,(\varphi_x + \psi + \ell\, w)_x - \ell\, k_3\,(w_x - \ell\,\varphi) = 0, \\ \rho_2\, \psi_{tt} - k_2\, \psi_{xx} + k_1\,(\varphi_x + \psi + \ell\, w) = 0, \\ \rho_1\, w_{tt} - k_3\,(w_x - \ell\,\varphi)_x + \ell\, k_1\,(\varphi_x + \psi + \ell\, w) = 0, \end{cases} \tag{1.3}$$

where ℓ and k_j are positive constants. On the other hand, combining the last two equations in (1.2), we have

$$\begin{cases} \rho_1\, \varphi_{tt} + k\,(\psi - \varphi_x)_x = 0, \\ \rho_2\, \psi_{tt} - b\, \psi_{xx} + 4\,k\,(\psi - \varphi_x) + 4\,\delta\, s + 4\,\gamma\, s_t = 0. \end{cases} \tag{1.4}$$

For $s = 0$ in (1.4), we obtain the following system:

$$\begin{cases} \rho_1\, \varphi_{tt} + k\,(\psi - \varphi_x)_x = 0, \\ \rho_2\, \psi_{tt} - b\, \psi_{xx} + 4\,k\,(\psi - \varphi_x) = 0, \end{cases} \tag{1.5}$$

that is, a conservative system closely related with Timoshenko theory.

In [19], it was proved that the frictional damping created by the interfacial slip alone is not enough to stabilize the system (1.1) exponentially to its equilibrium state, then another dissipative mechanism is necessary to be introduced to stabilize this system. In this direction, Raposo [15] proved the exponential stability for the model of structure taking in account the frictional damping as bellow

$$\begin{cases} \rho_1\, \varphi_{tt} + k\,(\psi - \varphi_x)_x + \alpha\, \varphi_t = 0, \\ \rho_2\,(s-\psi)_{tt} - b\,(s-\psi)_{xx} - k\,(\psi - \varphi_x) + \beta\,(s-\psi)_t = 0, \\ \rho_2\, s_{tt} - b\, s_{xx} + 3\,k\,(\psi - \varphi_x) + 4\,\delta\, s + 4\,\gamma\, s_t = 0. \end{cases}$$

About thermoelastic Timoshenko's system, for the problem

$$\begin{cases} \rho_1\, \varphi_{tt} - \sigma\,(\varphi_x,\, \psi)_x + \mu\, \varphi_t = 0, \\ \rho_2\, \varphi_{tt} - b\, \psi_{xx} + k\,(\varphi_x + \psi) + \beta\, \theta_x = 0, \\ \rho_3\, \theta_t + \gamma\, q_x + \delta\, \psi_{tx} = 0, \\ \tau_0\, q_t + q + \kappa\, \theta_x = 0, \end{cases}$$

in the presence of the frictional damping and heat conduction modeled by Cattaneo's law, Messoaudi *et al.* [11] gave the exponential stability in linear and nonlinear cases.

In the absence of the frictional damping, Sare and Racke [18] showed that the coupling via Cattaneo's law; that is

$$\begin{cases} \rho_1\, \varphi_{tt} - \sigma\,(\varphi_x + \psi)_x = 0, \\ \rho_2\, \varphi_{tt} - b\, \psi_{xx} + k\,(\varphi_x + \psi) + \delta\, \theta_x = 0, \\ \rho_3\, \theta_t + \gamma\, q_x + \delta\, \psi_{x\,t} = 0, \\ \tau\, q_t + q + \kappa\, \theta_x = 0, \end{cases}$$

causes loss of the exponential decay usually obtained in the case of coupling via Fourier's law.

For Timoshenko systems of classical thermoelasticity, Rivera and Racke [13] considered the system

$$\begin{cases} \rho_1\,\varphi_{tt} - \sigma\,(\varphi_x,\,\psi)_x = 0, \\ \rho_2\,\varphi_{tt} - b\,\psi_{xx} + k\,(\varphi_x + \psi) + \gamma\,\theta_x = 0, \\ \rho_3\,\theta_t - k\,\theta_{xx} + \gamma\,\psi_{x\,t} = 0 \end{cases}$$

and proved several exponential decay results for the linearized system and non-exponential stability result for the case of different wave speeds.

For more information on thermoelastic Timoshenko system, see [4, 6] and references therein, where it was investigated the decay property in bounded domain for frictional damping, infinite memory, and Fourier or Cattaneo law of heat conduction. For thermoelastic Timoshenko system with second sound, we cite also [12, 14] and references therein.

The stability of Bresse type systems in unbounded domain has been also treated in the literature for the last ten years. In this direction, we mention the papers [2, 3, 17] (see also references therien), where some polynomial stability estimates for L^2-norm have been proved using frictinal dampings or heat conduction effects or memory controls. In some particular cases, the optimality of the decay rate was also proved. Similar results exist in the literature for Timoshenko type systems, see [8, 9, 16] and references therein.

In the best of our knowledge, the stability of structures with interfacial slip in unbounded domains has never been considered in the literature. For this reason, we are interested in studying the asymptotic behavior in the whole line $\mathbb{R}$ of a thermoelastic structure with interfacial slip, derived from (1.1). More precisely, we deal with the system below

$$\begin{cases} \rho_1\,\varphi_{tt} + k\,(\psi - \varphi_x)_x + \gamma_0\,\varphi_t = 0, \\ \rho_2\,(s-\psi)_{tt} - b\,(s-\psi)_{xx} - k\,(\psi - \varphi_x) + \beta\,(s-\psi)_t = 0, \\ \rho_2\,s_{tt} - b\,s_{xx} + 3\,k\,(\psi - \varphi_x) + 4\,\gamma\,s_t = 0 \end{cases} \tag{1.6}$$

with initial data

$$\begin{cases} (\varphi(x,\,0),\,\psi(x,\,0),\,s(x,\,0)) = (\varphi_0(x),\,s_0(x),\,\psi_0(x)), \\ (\varphi_t(x,\,0),\,\psi_t(x,\,0),\,s_t(x,\,0)) = (\varphi_1(x),\,s_1(x),\,\psi_1(x)). \end{cases} \tag{1.7}$$

In this system, $x \in \mathbb{R}$, $t \geq 0$, ρ_1, ρ_2, k and b are positive constants, and β, γ and γ_0 are nonnegative constants. We will prove the polynomial stability of the system (1.6)-(1.7) and present several decay estimates depending on the smoothness of initial data and the coefficients β, γ and γ_0.

This paper is organized as follows: in Section 2, we deal with the formulation of the problem in a first order system. In Section 3, we show the polynomial stability.

2 Formulation of the problem

Without loss of generality, we consider $\rho_1 = \rho_2 = 1$. For convenience, we introduce the variable Θ of the effective rotation angle given by

$$\Theta = s - \psi,$$

then (1.6)-(1.7) change to

$$\begin{cases} \varphi_{tt} + k\,(s - \Theta - \varphi_x)_x + \gamma_0\,\varphi_t = 0, \\ \Theta_{tt} - b\,\Theta_{xx} - k\,(s - \Theta - \varphi_x) + \beta\,\Theta_t = 0, \\ s_{tt} - b\,s_{xx} + 3\,k\,(s - \Theta - \varphi_x) + 4\,\gamma\,s_t = 0 \end{cases} \tag{2.1}$$

with initial data

$$\begin{cases} (\varphi(x,\,0),\,\Theta(x,\,0),\,s(x,\,0)) = (\varphi_0(x),\,\Theta_0(x),\,s_0(x)), \\ (\varphi_t(x,\,0),\,\Theta_t(x,\,0),\,s_t(x,\,0)) = (\varphi_1(x),\,\Theta_1(x),\,s_1(x)), \end{cases} \tag{2.2}$$

where $\Theta_0(x) = s_0(x) - \psi_0(x)$ and $\Theta_1(x) = s_1(x) - \psi_1(x)$. Taking the new variables

$$\begin{aligned} \varphi_t = u, \qquad \Theta_t = y, \qquad s_t = \eta, \\ s - \Theta - \varphi_x = v, \qquad \Theta_x = z, \qquad s_x = \phi, \end{aligned} \tag{2.3}$$

we get from (2.1) the following system:

$$\begin{cases} v_t + u_x + y - \eta = 0, \\ u_t + k\, v_x + \gamma_0\, u = 0, \\ z_t - y_x = 0, \\ y_t - b\, z_x - k\, v + \beta\, y = 0, \\ \phi_t - \eta_x = 0 \\ \eta_t - b\, \phi_x + 3\, k\, v + 4\, \gamma\, \eta = 0. \end{cases} \tag{2.4}$$

Let

$$\begin{cases} U = (v,\, u,\, z,\, y,\, \phi,\, \eta)^T, \\ U_0(x) = (v(x,\, 0),\, u(x,\, 0),\, z(x,\, 0),\, y(x,\, 0),\, \phi(x,\, 0),\, \eta(x,\, 0))^T. \end{cases}$$

The system (2.4) is equivalent to

$$\begin{cases} U_t(x,\, t) + \mathcal{A} U_x(x,\, t) + \mathcal{L} U(x,\, t) = 0, \\ U(x,\, 0) = U_0(x), \end{cases} \tag{2.5}$$

where

$$\mathcal{A} U_x = \begin{pmatrix} u_x \\ k\, v_x \\ -\, y_x \\ -\, b\, z_x \\ -\, \eta_x \\ -\, b\, \phi_x \end{pmatrix} \qquad \text{and} \qquad \mathcal{L} U = \begin{pmatrix} y - \eta \\ \gamma_0\, u \\ 0 \\ -\, k\, v + \beta\, y \\ 0 \\ 3\, k\, v + 4\, \gamma\, \eta \end{pmatrix}. \tag{2.6}$$

Using the Fourier transform (with respect to the space variable x) for (2.5), we obtain the following Cauchy problem of a first order system:

$$\begin{cases} \widehat{U}_t(\xi,\, t) + i\, \xi\, \mathcal{A} \widehat{U}(\xi,\, t) + \mathcal{L} \widehat{U}(\xi,\, t) = 0, \\ \widehat{U}(\xi,\, 0) = \widehat{U}_0(\xi), \end{cases} \tag{2.7}$$

or

$$\begin{cases} \widehat{v}_t + i\, \xi\, \widehat{u} + \widehat{y} - \widehat{\eta} = 0, \\ \widehat{u}_t + i\, \xi\, k\, \widehat{v} + \gamma_0\, \widehat{u} = 0, \\ \widehat{z}_t - i\, \xi\, \widehat{y} = 0, \\ \widehat{y}_t - i\, \xi\, b\, \widehat{z} - k\, \widehat{v} + \beta\, \widehat{y} = 0, \\ \widehat{\phi}_t - i\, \xi\, \widehat{\eta} = 0 \\ \widehat{\eta}_t - i\, \xi\, b\, \widehat{\phi} + 3\, k\, \widehat{v} + 4\, \gamma\, \widehat{\eta} = 0, \end{cases} \tag{2.8}$$

for $\xi \in \mathbb{R}$. The solution of (2.7) is given by $\widehat{U}(\xi,\, t) = e^{F(\xi)\, t}\, \widehat{U}_0(\xi)$, where

$$F(\xi) = -\, (i\, \xi\, \mathcal{A} + \mathcal{L}).$$

Let $\widehat{E}$ be the total energy associated to system (2.7) given by

$$\widehat{E}(\xi,\, t) \quad = \quad \frac{1}{2}\, |\widehat{U}(\xi,\, t)|^2$$

$$= \frac{1}{2}\left[3\,k\,|\widehat{v}|^2 + 3\,|\widehat{u}|^2 + 3\,b\,|\widehat{z}|^2 + 3\,|\widehat{y}|^2 + b\,|\widehat{\phi}|^2 + |\widehat{\eta}|^2\right]. \tag{2.9}$$

Then simple and direct computations give

$$\frac{d}{dt}\widehat{E}(\xi,\,t) = -\,3\,\gamma_0\,|\widehat{u}|^2 - 3\,\beta\,|\widehat{y}|^2 - 4\,\gamma\,|\widehat{\eta}|^2, \tag{2.10}$$

which shows the dissipative nature of system (2.7) if $(\gamma_0,\,\beta,\,\gamma) \neq (0,\,0,\,0)$; that is

$$\widehat{E}(\xi,\,t) \leq \widehat{E}(\xi,\,0), \quad \forall\, t \geq 0.$$

If $(\gamma_0,\,\beta,\,\gamma) = (0,\,0,\,0)$, then (2.7) is conservative; that is

$$\widehat{E}(\xi,\,t) = \widehat{E}(\xi,\,0), \quad \forall\, t \geq 0.$$

3 Polynomial stability

This section is dedicated, first, to the investigation of the asymptotic behavior of $\widehat{E}$ when at least two controlers are effective; that is, when at least two constants amoung $\gamma,\ \gamma_0$ and β are positive. We show that $\widehat{E}$ converges exponentially to zero with respect of time t. And then, we deduce some polynomial decay estimates on $\|\partial_x^k U\|_{L^2(\mathbb{R})}$, where $k \in \mathbb{N}$ and the decay rate depends on the smoothness of U_0.

In this section, C denotes a generic positive constant, and C_ε denotes a generic positive constant depending on some positive value ε. We distinguish four cases.

3.1 Case 1: $\gamma_0,\ \beta,\ \gamma > 0$.

We start by proving a crucial decay estimate on $|\widehat{U}(\xi,\,t)|$.

Lemma 3.1 *We assume that $\gamma_0,\ \beta,\ \gamma > 0$. Let $\widehat{U}$ be the solution of (2.7). Then there exist $c,\ \widetilde{c} > 0$ such that the following estimate holds:*

$$|\widehat{U}(\xi,\,t)|^2 \leq \widetilde{c}\; e^{-\,c\,f(\xi)\,t}\,|\widehat{U}_0(\xi)|^2, \quad \forall\, \xi \in \mathbb{R}, \quad \forall\, t \geq 0, \tag{3.1}$$

where

$$f(\xi) = \frac{\min\{\xi^2,\ \xi^4\}}{(1+\xi^2+\xi^4)}. \tag{3.2}$$

Proof. Multiplying $(2.8)_4$ by $i\,\xi\,\overline{\widehat{z}}$, we get

$$i\,\xi\,\widehat{y}_t\,\overline{\widehat{z}} + b\,\xi^2\,|\widehat{z}|^2 - i\,\xi\,k\,\widehat{v}\;\overline{\widehat{z}} + i\,\xi\,\beta\,\widehat{y}\;\overline{\widehat{z}} = 0. \tag{3.3}$$

Multiplying $(2.8)_3$ by $i\,\xi\,\overline{\widehat{y}}$, we find

$$i\,\xi\,\widehat{z}_t\,\overline{\widehat{y}} + \xi^2\,|\widehat{y}|^2 = 0. \tag{3.4}$$

Taking the real part of (3.3) and (3.4) and subtracting the resulting equations, we obtain

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{y}\;\overline{\widehat{z}}\right) + \xi^2\left(b\,|\widehat{z}|^2 - |\widehat{y}|^2\right) = k\,Re\left(i\,\xi\,\widehat{v}\;\overline{\widehat{z}}\right) - \beta\,Re\left(i\,\xi\,\widehat{y}\;\overline{\widehat{z}}\right). \tag{3.5}$$

Using the Young inequality, it follows from (3.5) that, for any $\varepsilon_0,\,\varepsilon_1 > 0$,

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{y}\;\overline{\widehat{z}}\right) \leq -\,(b-\varepsilon_0-\varepsilon_1)\,\xi^2\,|\widehat{z}|^2 + C_{\varepsilon_0}\,\left(1+\xi^2\right)|\widehat{y}|^2 + C_{\varepsilon_1}\,|\widehat{v}|^2. \tag{3.6}$$

On the other hand, multiplying $(2.8)_1$ by $i\,\xi\,\overline{\widehat{u}}$, we have

$$i\,\xi\,\widehat{v}_t\;\overline{\widehat{u}} - \xi^2\,|\widehat{u}|^2 + i\,\xi\,\widehat{y}\;\overline{\widehat{u}} - i\,\xi\,\widehat{\eta}\;\overline{\widehat{u}} = 0. \tag{3.7}$$

Multiplying $(2.8)_2$ by $-\,i\,\xi\,\overline{\widehat{v}}$, we find

$$-\,i\,\xi\,\widehat{u}_t\;\overline{\widehat{v}}+k\,\xi^2\,|\widehat{v}|^2-i\,\gamma_0\,\xi\,\widehat{u}\;\overline{\widehat{v}}=0. \tag{3.8}$$

Adding (3.7) and (3.8) and taking the real part, we obtain

$$\frac{d}{dt}Re\left(i\,\xi\,\widehat{v}\;\overline{\widehat{u}}\right)+\xi^2\left(k\,|\widehat{v}|^2-|\widehat{u}|^2\right)=-\,Re(i\,\xi\,\widehat{y}\;\overline{\widehat{u}})+\gamma_0\,Re(i\,\xi\,\widehat{u}\;\overline{\widehat{v}})+\,Re(i\,\xi\,\widehat{\eta}\;\overline{\widehat{u}}). \tag{3.9}$$

Using the Young inequality, it follows from (3.9) that, for any $\varepsilon_0>0$,

$$\frac{d}{dt}Re(i\,\xi\,\widehat{v}\;\overline{\widehat{u}})\le-\,(k-\varepsilon_0)\,\xi^2\,|\widehat{v}|^2+C_{\varepsilon_0}\,\left(1+\xi^2\right)\left(|\widehat{y}|^2+|\widehat{u}|^2+|\widehat{\eta}|^2\right). \tag{3.10}$$

Now, multiplying $(2.8)_5$ by $-\,i\,\xi\,\overline{\widehat{\eta}}$, we see that

$$-\,i\,\xi\,\widehat{\phi}_t\;\overline{\widehat{\eta}}-\xi^2\,|\widehat{\eta}|^2=0. \tag{3.11}$$

Multiplying $(2.8)_6$ by $i\,\xi\,\overline{\widehat{\phi}}$, it follows that

$$i\,\xi\,\widehat{\eta}_t\;\overline{\widehat{\phi}}+b\,\xi^2\,|\widehat{\phi}|^2+3\,i\,k\,\xi\,\widehat{v}\;\overline{\widehat{\phi}}+4\,i\,\gamma\,\xi\,\widehat{\eta}\;\overline{\widehat{\phi}}=0. \tag{3.12}$$

Adding (3.11) and (3.12), and taking the real part, we have

$$\frac{d}{dt}Re\left(-\,i\,\xi\,\widehat{\phi}\;\overline{\widehat{\eta}}\right)+\xi^2\left(b\,|\widehat{\phi}|^2-|\widehat{\eta}|^2\right)=-\,3\,k\,Re\left(i\,\xi\,\widehat{v}\;\overline{\widehat{\phi}}\right)-4\,\gamma\,Re\left(i\,\xi\,\widehat{\eta}\;\overline{\widehat{\phi}}\right). \tag{3.13}$$

Using the Young inequality, we conclude that, for any $\varepsilon_0,\,\varepsilon_1>0$,

$$\frac{d}{dt}Re(-\,i\,\xi\,\widehat{\phi}\;\overline{\widehat{\eta}})\le-\,(b-\varepsilon_0-\varepsilon_1)\,\xi^2\,|\widehat{\phi}|^2+C_{\varepsilon_0}\,\left(1+\xi^2\right)|\widehat{\eta}|^2+C_{\varepsilon_1}\,|\widehat{v}|^2. \tag{3.14}$$

Let us define the functional $\mathcal{F}$ as follows:

$$\mathcal{F}(\xi,\,t)=\xi^2 Re\left(i\,\xi\,\widehat{y}\;\overline{\widehat{z}}\right)+\lambda_1 Re\left(i\,\xi\,\widehat{y}\;\overline{\widehat{u}}\right)+\xi^2\,Re\left(-\,i\,\xi\,\widehat{\phi}\;\overline{\widehat{\eta}}\right), \tag{3.15}$$

and introduce the *Perturbed Energy* $\mathcal{L}$ as follows:

$$\mathcal{L}(\xi,\,t)=\lambda\,\widehat{E}(\xi,\,t)+\frac{1}{(1+\xi^2+\xi^4)}\,\mathcal{F}(\xi,\,t), \tag{3.16}$$

where λ and λ_1 are positive constants to be defined later. We put (3.6), (3.10), (3.14) into (3.15), and we deduce that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(\xi,\,t)\le&-\,(b-\varepsilon_0-\varepsilon_1)\,\xi^4\,|\widehat{z}|^2-(b-\varepsilon_0-\varepsilon_1)\,\xi^4\,|\widehat{\phi}|^2-[\lambda_1(k-\varepsilon_0)-C_{\varepsilon_1}]\,\xi^2\,|\widehat{v}|^2\\&+\;C_{\varepsilon_0,\;\lambda_1}\,\left(1+\xi^2+\xi^4\right)\left(|\widehat{\eta}|^2+|\widehat{y}|^2+|\widehat{u}|^2\right).\end{aligned} \tag{3.17}$$

We choose $0<\varepsilon_0<\min\{b,\;k\}$ and $0<\varepsilon_1<b-\varepsilon_0$. After, we choose λ_1 large enough such that $\lambda_1>\frac{C_{\varepsilon_1}}{k-\varepsilon_0}$. Hence, using the definition of $\widehat{E}$, (3.17) leads to, for some positive constant c_1,

$$\mathcal{F}(\xi,\,t)\le-\,c_1\,\min\{\xi^2,\;\xi^4\}\,\widehat{E}(\xi,\,t)+C\,\left(1+\xi^2+\xi^4\right)\left(|\widehat{\eta}|^2+|\widehat{y}|^2+|\widehat{u}|^2\right). \tag{3.18}$$

Then from (2.10), (3.16) and (3.18) we have, for $c_2=\min\{3\,\gamma_0,\;3\,\beta,\;4\,\gamma\}$,

$$\frac{d}{dt}\mathcal{L}(\xi,\,t)\le-\,c_1\,f(\xi)\widehat{E}(\xi,\,t)-(c_2\,\lambda-C)\left(|\widehat{\eta}|^2+|\widehat{y}|^2+|\widehat{u}|^2\right), \tag{3.19}$$

where f is defined in (3.2). Moreover, using the definition of $\widehat{E}$, $\mathcal{F}$ and $\mathcal{L}$, we get, for some $c_0 > 0$ (not depending on λ),

$$|\mathcal{L}(\xi,\, t) - \lambda\, \widehat{E}(\xi,\, t)| \leq \frac{c_0\,(|\xi| + |\xi|^3)}{(1+\xi^2+\xi^4)}\, \widehat{E}(\xi,\, t) \leq c_0\, \widehat{E}(\xi,\, t). \tag{3.20}$$

Therefore, for λ large enough so that $\lambda > \max\left\{\frac{C}{c_2},\, c_0\right\}$, we deduce from (3.19) and (3.20) that

$$\frac{d}{dt}\mathcal{L}(\xi,\, t) + c_1\, f(\xi)\, \widehat{E}(\xi,\, t) \leq 0 \tag{3.21}$$

and

$$c_3\, \widehat{E}(\xi,\, t) \leq \mathcal{L}(\xi,\, t) \leq c_4\, \widehat{E}(\xi,\, t), \tag{3.22}$$

where $c_3 = \lambda - c_0$ and $c_4 = \lambda + c_0$. Consequently, a combination of (3.21) and the second inequality in (3.22) leads to, for some positive constant c,

$$\frac{d}{dt}\mathcal{L}(\xi,\, t) + c\, f(\xi)\, \mathcal{L}(\xi,\, t) \leq 0. \tag{3.23}$$

Finally, by integration (3.23) and using the first inequality in (3.22), the Lemma 3.1 follows with $\widetilde{c} = \frac{c_4}{c_3}$.

Theorem 3.2 *We assume that $\gamma_0,\, \beta,\, \gamma > 0$. Let $N,\, \ell \in \mathbb{N}^*$ such that $\ell \leq N$,*

$$U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$$

and U be the solution of (2.5). Then, for any $j = 0,\, \ldots,\, N-\ell$, there exists $\widehat{c} > 0$ such that, for any $t \geq 0$,

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq \widehat{c}\,(1+t)^{-1/8-j/4}\, \|U_0\|_{L^1(\mathbb{R})} + \widehat{c}\,(1+t)^{-\ell/2}\, \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}. \tag{3.24}$$

Proof. From (3.2) we have (low and high frequences)

$$f(\xi) \geq \begin{cases} \frac{1}{3}\,\xi^4 & \text{if} \quad |\xi| \leq 1, \\ \frac{1}{3}\,\xi^{-2} & \text{if} \quad |\xi| > 1. \end{cases} \tag{3.25}$$

Applying Plancherel's theorem and lemma 3.1, we have

$$\begin{aligned} \|\partial_x^j U\|^2_{L^2(\mathbb{R})} &= \|\widehat{\partial_x^j U}(x,\, t)\|^2_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \xi^{2\,j}\, |\widehat{U}(\xi,\, t)|^2\, d\xi \\ &\leq \widetilde{c} \int_{\mathbb{R}} \xi^{2\,j}\, e^{-\,c\, f(\xi)\, t}\, |\widehat{U}_0(\xi)|^2\, d\xi \\ &= \widetilde{c} \int_{|\xi|\leq 1} \xi^{2\,j}\, e^{-\,c\, f(\xi)\, t}\, |\widehat{U}_0(\xi)|^2\, d\xi + \widetilde{c} \int_{|\xi|>1} \xi^{2\,j}\, e^{-\,c\, f(\xi)\, t}\, |\widehat{U}_0(\xi)|^2\, d\xi \\ &:= I_1 + I_2. \end{aligned} \tag{3.26}$$

To complete our proof, we prove the following lemma:

Lemma 3.3 *Let $\sigma > -1$ and $p,\ \bar{c} > 0$. Then there exists $C_{\sigma,\ p,\ \bar{c}} > 0$ such that*

$$\int_0^1 \xi^\sigma\, e^{-\,\bar{c}\, t\, \xi^p}\, d\xi \leq C_{\sigma,p,\bar{c}}\,(1+t)^{-\,(\sigma+1)/p}, \quad \forall\, t \geq 0. \tag{3.27}$$

Proof. In fact, for $0 \le t \le 1$, (3.27) is evident, for any $C_{\sigma,p,\bar c} \ge \frac{2^{(\sigma+1)/p}}{\sigma+1}$. For $t > 1$, we have

$$\int_0^1 \xi^\sigma \, e^{-\,\bar c\, t\, \xi^p} \; d\xi = \int_0^1 \xi^{\sigma+1-p}\, e^{-\,\bar c\, t\, \xi^p}\, \xi^{p-1}\; d\xi = \int_0^1 (\xi^p)^{(\sigma+1-p)/p}\, e^{-\,\bar c\, t\, \xi^p}\, \xi^{p-1}\; d\xi.$$

Taking $\tau = \bar c\, t\, \xi^p$. Then

$$\xi^p = \frac{\tau}{\bar c\, t} \qquad \text{and} \qquad \xi^{p-1}\; d\xi = \frac{1}{p\,\bar c\, t}\; d\tau.$$

Replacing in the above integral, we find

$$\begin{aligned} \int_0^1 (\xi^p)^{(\sigma+1-p)/p}\, e^{-\,\bar c\, t\, \xi^p}\, \xi^{p-1}\; d\xi &= \int_0^{\bar c\, t} \left(\frac{\tau}{\bar c\, t}\right)^{(\sigma+1-p)/p} e^{-\,\tau}\, \frac{1}{p\,\bar c\, t}\; d\tau \\ \le \quad \frac{1}{p\,(\bar c\, t)^{(\sigma+1)/p}} \int_0^{+\infty} \tau^{(\sigma+1-p)/p}\, e^{-\,\tau}\; d\tau &\le \frac{2^{(\sigma+1)/p}}{p\,(\bar c)^{(\sigma+1)/p}}\; C_{\sigma,p}\, (t+1)^{-(\sigma+1)/p}, \end{aligned}$$

where

$$C_{\sigma,p} = \int_0^{+\infty} \tau^{(\sigma+1-p)/p}\, e^{-\,\tau}\; d\tau$$

which is a convergent integral, for any $\sigma \ge 0$ and $p > 0$. This completes the proof of (3.27), where

$$C_{\sigma,p,\bar c} = \max\left\{ \frac{2^{(\sigma+1)/p}}{\sigma+1},\; \frac{2^{(\sigma+1)/p}}{p\,(\bar c)^{(\sigma+1)/p}} C_{\sigma,p} \right\}.$$

Now, using (3.25) and (3.27) (with $\sigma = 2\,j$ and $p = 4$), it follows, for low frequency region,

$$I_1 \le C\, \|\widehat U_0\|^2_{L^\infty(\mathbb{R})} \int_{|\xi|\le 1} \xi^{2\,j}\, e^{-\,\frac{c}{3}\, t\, \xi^4}\; d\xi \le C\,(1+t)^{-\,\frac14 (1+2\,j)}\, \|U_0\|^2_{L^1(\mathbb{R})}. \tag{3.28}$$

In the high frequency region, we have

$$\begin{aligned} I_2 \quad &\le \quad C \int_{|\xi|>1} |\xi|^{2\,j}\, e^{-\frac13\, t\, \xi^{-2}}\, |\widehat U(\xi,\,0)|^2\; d\xi \\ &\le \quad C \sup_{|\xi|>1} \left\{ |\xi|^{-2\,\ell}\, e^{-\frac13\, t\, |\xi|^{-2}} \right\} \int_{|\xi|>1} |\xi|^{2\,(j+\ell)}\, |\widehat U_0(\xi)|^2\; d\xi, \end{aligned}$$

then

$$I_2 \le C\,(1+t)^{-\,\ell}\; \|\, \partial_x^{j+\ell} U_0\,\|^2_{L^2(\mathbb{R})}, \tag{3.29}$$

since, by classical and direct arguments, we see that there exists $C > 0$ such that

$$\sup_{|\xi|\ge 1} |\xi|^{-2\ell}\, e^{-\,\frac13\, t\, |\xi|^{-\,2}} \le C\,(1+t)^{-\,\ell}, \quad \forall\, t \ge 0. \tag{3.30}$$

So, by combining (3.26), (3.28) and (3.29), we get (3.24).

3.2 Case 2: $\gamma_0,\ \beta > 0$ and $\gamma = 0$.

We start by proving the following lemma:

Lemma 3.4 *We assume that $\gamma = 0$ and $\gamma_0,\ \beta > 0$. Let $\widehat U$ be the solution of (2.7). Then there exist c, $\widetilde c > 0$ such that we have the following estimate:*

$$|\widehat U(\xi,\, t)|^2 \le \widetilde c\, e^{-\,c\, f(\xi)\, t}\, |\widehat U_0(\xi)|^2, \quad \forall\, \xi \in \mathbb{R}, \quad \forall\, t \ge 0, \tag{3.31}$$

where

$$f(\xi) = \frac{\xi^6}{(1+\xi^2+\xi^4+\xi^6+\xi^8)}. \tag{3.32}$$

Proof. Multiplying (3.5) by ξ^4 and using the Young inequality, it follows that, for any $\varepsilon_0,\, \varepsilon_1 > 0$,

$$\frac{d}{dt} Re(i\,\xi^5\,\widehat{y}\,\overline{\widehat{z}}) \leq -\,(b-\varepsilon_0-\varepsilon_1)\,\xi^6\,|\widehat{z}|^2 + C_{\varepsilon_0}\left(\xi^6+\xi^8\right)|\widehat{y}|^2 + C_{\varepsilon_1}\,\xi^4\,|\widehat{v}|^2. \tag{3.33}$$

Multiplying (3.9), first by ξ^2, second by ξ^4, and third by ξ^6, and using the Young inequality in each time, it follows that, for any $\varepsilon_0 > 0$,

$$\frac{d}{dt} Re(i\,\xi^3\,\widehat{v}\,\overline{\widehat{u}}) \leq -\,(k-\varepsilon_0)\,\xi^4\,|\widehat{v}|^2 + \varepsilon_0\,\xi^6\,|\widehat{\eta}|^2 + C_{\varepsilon_0}\left(1+\xi^2+\xi^4\right)\left(|\widehat{y}|^2+|\widehat{u}|^2\right), \tag{3.34}$$

$$\frac{d}{dt} Re(i\,\xi^5\,\widehat{v}\,\overline{\widehat{u}}) \leq -\,(k-\varepsilon_0)\,\xi^6\,|\widehat{v}|^2 + \varepsilon_0\,\xi^6\,|\widehat{\eta}|^2 + C_{\varepsilon_0}\left(\xi^2+\xi^4+\xi^6\right)\left(|\widehat{y}|^2+|\widehat{u}|^2\right) \tag{3.35}$$

and

$$\frac{d}{dt} Re(i\,\xi^7\,\widehat{v}\,\overline{\widehat{u}}) \leq -\,(k-\varepsilon_0)\,\xi^8\,|\widehat{v}|^2 + \varepsilon_0\,\xi^6\,|\widehat{\eta}|^2 + C_{\varepsilon_0}\left(\xi^4+\xi^6+\xi^8\right)\left(|\widehat{y}|^2+|\widehat{u}|^2\right). \tag{3.36}$$

On the other hand, multiplying (3.13) by ξ^4, noting that $\gamma = 0$ and using the Young inequality, it follows that, for any $\varepsilon_1 > 0$,

$$\frac{d}{dt} Re(-\,i\,\xi^5\,\widehat{\phi}\,\overline{\widehat{\eta}}) \leq -\,(b-\varepsilon_1)\,\xi^6\,|\widehat{\phi}|^2 + \xi^6\,|\widehat{\eta}|^2 + C_{\varepsilon_1}\,\xi^4\,|\widehat{v}|^2. \tag{3.37}$$

Now, multiplying $(2.8)_1$ by $-\,\xi^6\,\overline{\widehat{\eta}}$ and $(2.8)_6$ by $-\,\xi^6\,\overline{\widehat{v}}$, noting that $\gamma = 0$, adding the obtained formulas and taking the real part, we see that

$$\frac{d}{dt} Re\left(-\,\xi^6\,\widehat{v}\,\overline{\widehat{\eta}}\right) = \xi^6\left(3\,k\,|\widehat{v}|^2 - |\widehat{\eta}|^2\right) + \xi^6 Re\left(i\,\xi\,\widehat{u}\,\overline{\widehat{\eta}} - i\,b\,\xi\,\widehat{\phi}\,\overline{\widehat{v}} + \widehat{y}\,\overline{\widehat{\eta}}\right). \tag{3.38}$$

Using the Young inequality, we conclude from (3.38) that, for any $\varepsilon_0,\, \varepsilon_1 > 0$,

$$\begin{aligned}\frac{d}{dt} Re\left(-\,\xi^6\,\widehat{v}\,\overline{\widehat{\eta}}\right) \quad\leq\quad & -\,(1-\varepsilon_0)\,\xi^6\,|\widehat{\eta}|^2 + \varepsilon_1\,\xi^6\,|\widehat{\phi}|^2 \\ & +\, C_{\varepsilon_1}\,\left(\xi^6+\xi^8\right)|\widehat{v}|^2 + C_{\varepsilon_0}\,\left(\xi^6+\xi^8\right)\left(|\widehat{y}|^2+|\widehat{u}|^2\right).\end{aligned} \tag{3.39}$$

Let us define the functional $\mathcal{F}$ and the *Perturbed Energy* $\mathcal{L}$ as follows:

$$\mathcal{F}(\xi,\, t) = Re\left(i\,\xi^5\left(\widehat{y}\,\overline{\widehat{z}} + \widehat{\eta}\,\overline{\widehat{\phi}}\right)\right) + \lambda_1 Re\left(i\left(\xi^3+\xi^5+\xi^7\right)\widehat{v}\,\overline{\widehat{u}}\right) + \lambda_2\,Re\left(-\xi^6\,\widehat{v}\,\overline{\widehat{\eta}}\right) \tag{3.40}$$

and

$$\mathcal{L}(\xi,\, t) = \lambda\,\widehat{E}(\xi,\, t) + \frac{1}{\left(1+\xi^2+\xi^4+\xi^6+\xi^8\right)}\,\mathcal{F}(\xi,\, t), \tag{3.41}$$

where λ, λ_1 and λ_2 are positive constants to be defined later. Replacing (3.33) and (3.37) together with (3.34), (3.35), (3.36) and (3.39) into (3.40) and (3.41) we deduce that

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}(\xi,\, t) \\ &\leq\; -\,(b-\varepsilon_0-\varepsilon_1)\,\xi^6\,|\widehat{z}|^2 - (b-\varepsilon_1-\varepsilon_1\,\lambda_2)\,\xi^6\,|\widehat{\phi}|^2 - \left((1-\varepsilon_0)\,\lambda_2 - 3\,\varepsilon_0\,\lambda_1\right)\xi^6\,|\widehat{\eta}|^2 \\ &\quad\; -\left[\left[\lambda_1\,(k-\varepsilon_0) - C_{\varepsilon_1}\right]\xi^4 + \left[\lambda_1\,(k-\varepsilon_0) - \lambda_2\,C_{\varepsilon_1}\right]\left(\xi^6+\xi^8\right)\right]|\widehat{v}|^2 \\ &\quad\; +\, C_{\varepsilon_0,\,\lambda_1,\,\lambda_2}\left(1+\xi^2+\xi^4+\xi^6+\xi^8\right)\left(|\widehat{y}|^2+|\widehat{u}|^2\right).\end{aligned} \tag{3.42}$$

We choose $0 < \varepsilon_1 < \min\{\frac{b}{2},\, k\}$, $1 < \lambda_2 < \frac{b-\varepsilon_1}{\varepsilon_1}$ and $\lambda_1 > \max\left\{\frac{C_{\varepsilon_1}}{k},\, \frac{\lambda_2\, C_{\varepsilon_1}}{k}\right\}$. After, we choose ε_0 small enough such that

$$0 < \varepsilon_0 < \min\left\{\frac{\lambda_2 - 1}{\lambda_2 + 3\,\lambda_1},\; b-\varepsilon_1,\; \frac{\lambda_1\,k - C_{\varepsilon_1}}{\lambda_1},\; \frac{\lambda_1\,k - \lambda_2\,C_{\varepsilon_1}}{k}\right\}.$$

Hence, using the definition of $\widehat{E}$, (3.42) implies that, for some positive constant c_1,

$$\frac{d}{dt}\mathcal{F}(\xi,\,t) \leq -\,c_1\,\xi^6\,\widehat{E}(\xi,\,t) + C\,\left(1+\xi^2+\xi^4+\xi^6+\xi^8\right)\left(|\widehat{y}|^2+|\widehat{u}|^2\right). \tag{3.43}$$

Then from (2.10) (with $\gamma = 0$), (3.41) and (3.43) we have, for $c_2 = \min\{3\,\gamma_0,\ 3\,\beta\}$,

$$\frac{d}{dt}\mathcal{L}(\xi,\,t) \leq -\,c_1\,f(\xi)\widehat{E}(\xi,\,t) - (c_2\,\lambda - C)\left(|\widehat{y}|^2+|\widehat{u}|^2\right), \tag{3.44}$$

where f is defined in (3.32). Moreover, using the definition of $\widehat{E}$, $\mathcal{F}$ and $\mathcal{L}$, we get, for some $c_0 > 0$ (not depending on λ),

$$|\mathcal{L}(\xi,\,t) - \lambda\,\widehat{E}(\xi,\,t)| \leq \frac{c_0\,(|\xi|^3+|\xi|^5+|\xi|^6+|\xi|^7)}{(1+\xi^2+\xi^4+\xi^6+\xi^8)}\,\widehat{E}(\xi,\,t) \leq 4\,c_0\,\widehat{E}(\xi,\,t). \tag{3.45}$$

Therefore, for λ large enough so that $\lambda > \max\left\{\frac{C}{c_2},\ 4\,c_0\right\}$, we deduce from (3.59) and (3.45) that (3.21) and (3.22) are satisfied with $c_3 = \lambda - 4\,c_0$, $c_4 = \lambda + 4\,c_0$ and f is defined (3.32). Consequently, (3.21) and the second inequality in (3.22) imply (3.23). So, an integration of (3.23) and use of the first inequality in (3.22) lead to (3.31), which ends the proof of Lemma 3.4.

Theorem 3.5 *We assume that $\gamma = 0$ and $\gamma_0,\ \beta\ > 0$. Let $N,\ \ell \in \mathbb{N}^*$ such that $\ell \leq N$,*

$$U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$$

and U be the solution of (2.5). Then, for any $j = 0,\,\ldots,\,N-\ell$, there exists $\widehat{c} > 0$ such that, for any $t \geq 0$,

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq \widehat{c}\,(1+t)^{-1/12-j/6}\,\|U_0\|_{L^1(\mathbb{R})} + \widehat{c}\,(1+t)^{-\ell/2}\,\|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}. \tag{3.46}$$

Proof. From (3.32) we have (low and high frequences)

$$f(\xi) \geq \begin{cases} \frac{1}{5}\,\xi^6 & \text{if} \quad |\xi| \leq 1, \\ \\ \frac{1}{5}\,\xi^{-2} & \text{if} \quad |\xi| > 1. \end{cases} \tag{3.47}$$

As for (3.26), applying Plancherel's theorem and using (3.31) and (3.47), we get

$$\|\partial_x^j U\|_{L^2(\mathbb{R})}^2 \leq \widetilde{c}\int_{|\xi|\leq 1} \xi^{2\,j}\,e^{-\,\frac{c}{5}\,t\,\xi^6}\,|\widehat{U}_0(\xi)|^2\,d\xi + \widetilde{c}\int_{|\xi|>1} \xi^{2\,j}\,e^{-\,\frac{c}{5}\,t\,\xi^{-2}}\,|\widehat{U}_0(\xi)|^2\,d\xi. \tag{3.48}$$

Then, using (3.27) (with $\sigma = 2\,j$ and $p = 6$) and proceeding as for (3.28) and (3.29), we get

$$\|\partial_x^j U\|_{L^2(\mathbb{R})}^2 \leq C\,(1+t)^{-\,\frac{1}{6}\,(1+2\,j)}\,\|U_0\|_{L^1(\mathbb{R})}^2 + C\,(1+t)^{-\,\ell}\,\|\,\partial_x^{j+\ell} U_0\,\|_{L^2(\mathbb{R})}^2;$$

which implies (3.46).

3.3 Case 3: $\gamma_0,\ \gamma > 0$ and $\beta = 0$.

As in the previous two subsections, we start by proving the following lemma:

Lemma 3.6 *The result of Lemma 3.4 holds true also when $\beta = 0$ and $\gamma_0,\ \gamma\ > 0$.*

Proof. Multiplying (3.5) by ξ^4, noting that $\beta = 0$ and using the Young inequality, it follows that, for any $\varepsilon_1 > 0$,

$$\frac{d}{dt} Re\left(i\,\xi^5\,\widehat{y}\,\overline{\widehat{z}}\right) \le -\,(b-\varepsilon_1)\,\xi^6\,|\widehat{z}|^2 + \xi^6\,|\widehat{y}|^2 + C_{\varepsilon_1}\,\xi^4|\widehat{v}|^2. \tag{3.49}$$

Multiplying (3.9), first by ξ^2, second by ξ^4, and third by ξ^6, and using the Young inequality in each time, it follows that, for any $\varepsilon_0 > 0$,

$$\frac{d}{dt} Re\left(i\,\xi^3\,\widehat{v}\,\overline{\widehat{u}}\right) \le -\,(k-\varepsilon_0)\,\xi^4\,|\widehat{v}|^2 + \varepsilon_0\,\xi^6\,|\widehat{y}|^2 + C_{\varepsilon_0}\left(1+\xi^2+\xi^4\right)\left(|\widehat{\eta}|^2 + |\widehat{u}|^2\right), \tag{3.50}$$

$$\frac{d}{dt} Re\left(i\,\xi^5\,\widehat{v}\,\overline{\widehat{u}}\right) \le -\,(k-\varepsilon_0)\,\xi^6\,|\widehat{v}|^2 + \varepsilon_0\,\xi^6\,|\widehat{y}|^2 + C_{\varepsilon_0}\left(\xi^4+\xi^6\right)\left(|\widehat{\eta}|^2 + |\widehat{u}|^2\right) \tag{3.51}$$

and

$$\frac{d}{dt} Re\left(i\,\xi^7\,\widehat{v}\,\overline{\widehat{u}}\right) \le -\,(k-\varepsilon_0)\,\xi^8\,|\widehat{v}|^2 + \varepsilon_0\,\xi^6\,|\widehat{y}|^2 + C_{\varepsilon_0}\left(\xi^6+\xi^8\right)\left(|\widehat{\eta}|^2 + |\widehat{u}|^2\right). \tag{3.52}$$

On the other hand, multiplying (3.13) by ξ^4 and using the Young inequality, it follows that, for any $\varepsilon_0,\ \varepsilon_1 > 0$,

$$\frac{d}{dt} Re\left(-\,i\,\xi^5\,\widehat{\phi}\,\overline{\widehat{\eta}}\right) \le -\,(b-\varepsilon_0-\varepsilon_1)\,\xi^6\,|\widehat{\phi}|^2 + C_{\varepsilon_1}\,\xi^4\,|\widehat{v}|^2 + C_{\varepsilon_0}\left(\xi^4+\xi^6\right)|\widehat{\eta}|^2. \tag{3.53}$$

Now, multiplying $(2.8)_1$ by $\xi^6\,\overline{\widehat{y}}$ and $(2.8)_4$ by $\xi^6\,\overline{\widehat{v}}$, noting that $\beta = 0$, adding the obtained formulas and taking the real part, we see that

$$\frac{d}{dt} Re\left(\xi^6\,\widehat{v}\,\overline{\widehat{y}}\right) = \xi^6\left(k\,|\widehat{v}|^2 - |\widehat{y}|^2\right) + \xi^6\,Re\left(-\,i\,\xi\,\widehat{u}\,\overline{\widehat{y}} + i\,b\,\xi\,\widehat{z}\,\overline{\widehat{v}} + \widehat{\eta}\,\overline{\widehat{y}}\right). \tag{3.54}$$

Using the Young inequality, we conclude from (3.54) that, for any $\varepsilon_0,\ \varepsilon_1 > 0$,

$$\begin{aligned}\frac{d}{dt} Re\left(\xi^6\,\widehat{v}\,\overline{\widehat{y}}\right) \quad \le \quad & -\,(1-\varepsilon_0)\,\xi^6\,|\widehat{y}|^2 + \varepsilon_1\,\xi^6\,|\widehat{z}|^2 \\ & +\, C_{\varepsilon_1}\,\left(\xi^6+\xi^8\right)|\widehat{v}|^2 + C_{\varepsilon_0}\,\left(\xi^6+\xi^8\right)\left(|\widehat{\eta}|^2 + |\widehat{u}|^2\right).\end{aligned} \tag{3.55}$$

Let us define the functional $\mathcal{F}$ by

$$\mathcal{F}(\xi,\,t) = Re\left(-\,i\,\xi^5\,\widehat{\phi}\,\overline{\widehat{\eta}} + \xi^6\,\widehat{v}\,\overline{\widehat{y}}\right) + \lambda_1\,Re\left(i\,(\xi^3+\xi^5+\xi^7)\,\widehat{v}\,\overline{\widehat{u}}\right) + \lambda_2\,Re\left(i\,\xi^5\,\widehat{y}\,\overline{\widehat{z}}\right) \tag{3.56}$$

and the *Perturbed Energy* $\mathcal{L}$ by (3.41), where $\lambda,\ \lambda_1$ and λ_2 are positive constants to be fixed later. Replacing (3.53) and (3.55) together with (3.50), (3.51), (3.52) and (3.33) into (3.56), we deduce that

$$\begin{aligned}&\frac{d}{dt}\mathcal{F}(\xi,\,t) \le -\,(\lambda_2\,(b-\varepsilon_1) - \varepsilon_1)\,\xi^6\,|\widehat{z}|^2 \\ \le \quad & -\,(b-\varepsilon_0-\varepsilon_1)\,\xi^6\,|\widehat{\phi}|^2 - (1-(\varepsilon_0 + 3\,\varepsilon_0\,\lambda_1 + \lambda_2))\,\xi^6\,|\widehat{y}|^2 \\ & -\left[\left[\lambda_1\,(k-\varepsilon_0) - \lambda_2\,C_{\varepsilon_1}\right]\xi^4 + \left[\lambda_1\,(k-\varepsilon_0) - C_{\varepsilon_1}\right](\xi^6+\xi^8)\right]\,|\widehat{v}|^2 \\ & +\, C_{\varepsilon_0,\ \lambda_1}\left(1+\xi^2+\xi^4+\xi^6+\xi^8\right)\left(|\widehat{\eta}|^2 + |\widehat{u}|^2\right).\end{aligned} \tag{3.57}$$

We choose $0 < \varepsilon_1 < \frac{b}{2},\ \frac{\varepsilon_1}{b-\varepsilon_1} < \lambda_2 < 1$ and $\lambda_1 > \max\left\{\frac{C_{\varepsilon_1}}{k},\ \frac{\lambda_2\,C_{\varepsilon_1}}{k}\right\}$. After, we choose ε_0 small enough such that

$$0 < \varepsilon_0 < \min\left\{\frac{1-\lambda_2}{3\,\lambda_1+1},\ b-\varepsilon_1,\ \lambda_1\,k - C_{\varepsilon_1},\ \lambda_1\,k - \lambda_2\,C_{\varepsilon_1}\right\}.$$

Hence, using the definition of $\widehat{E}$, (3.57) implies that, for some positive constant c_1,

$$\frac{d}{dt}\mathcal{F}(\xi,\,t) \leq -\,c_1\,\xi^6\,\widehat{E}(\xi,\,t) + C\,\left(1+\xi^2+\xi^4+\xi^6+\xi^8\right)\left(|\widehat{\eta}|^2+|\widehat{u}|^2\right). \tag{3.58}$$

Then from (2.10) (with $\beta = 0$), (3.41) and (3.58) we find, for $c_2 = \min\{3\,\gamma_0,\ 4\,\gamma\}$,

$$\frac{d}{dt}\mathcal{L}(\xi,\,t) \leq -\,c_1\,f(\xi)\,\widehat{E}(\xi,\,t) - (c_2\,\lambda - C)\left(|\widehat{\eta}|^2+|\widehat{u}|^2\right), \tag{3.59}$$

where f is defined in (3.32). Moreover, using the definition of $\widehat{E}$, $\mathcal{F}$ and $\mathcal{L}$, we get (3.45). So the proof of (3.31) can be ended as in the proof of Lemma 3.4.

Theorem 3.7 *The result of Theorem 3.5 holds true also when $\beta = 0$ and $\gamma_0,\ \gamma > 0$.*

Proof. The proof is identical to the one of Theorem 3.5.

3.4 Case 4: $\beta,\ \gamma > 0$ and $\gamma_0 = 0$.

As in the previous three subsections, we start by proving the following lemma:

Lemma 3.8 *The result of Lemma 3.1 holds true also when $\gamma_0 = 0$ and $\beta,\ \gamma > 0$.*

Proof. Multiplying (3.5) by ξ^2 and using the Young inequality, it follows that, for any $\varepsilon_0 > 0$,

$$\frac{d}{dt}Re\left(i\,\xi^3\,\widehat{y}\,\overline{\widehat{z}}\right) \leq -\,(b-\varepsilon_0)\,\xi^4\,|\widehat{z}|^2 + k\,\xi^2\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{z}}\right) + C_{\varepsilon_0}\,(\xi^2+\xi^4)\,|\widehat{y}|^2. \tag{3.60}$$

Multiplying (3.9) by -1, noting that $\gamma_0 = 0$ and using the Young inequality, it follows that, for any $\varepsilon_0 > 0$,

$$\frac{d}{dt}Re\left(-\,i\,\xi\,\widehat{v}\,\overline{\widehat{u}}\right) \leq -\,(1-\varepsilon_0)\,\xi^2\,|\widehat{u}|^2 + (k+\varepsilon_0)\,\xi^2\,|\widehat{v}|^2 + C_{\varepsilon_0}\left(|\widehat{\eta}|^2+|\widehat{y}|^2\right). \tag{3.61}$$

On the other hand, multiplying (3.13) by ξ^2 and using the Young inequality, it follows that, for any $\varepsilon_0,\ \varepsilon_1 > 0$,

$$\frac{d}{dt}Re\left(-\,i\,\xi^3\,\widehat{\phi}\,\overline{\widehat{\eta}}\right) \leq -\left(b-\varepsilon_0-\frac{3\,k}{2}\,\varepsilon_1\right)\xi^4\,|\widehat{\phi}|^2 + \frac{3\,k}{2\,\varepsilon_1}\,\xi^2\,|\widehat{v}|^2 + C_{\varepsilon_0}\left(\xi^2+\xi^4\right)|\widehat{\eta}|^2. \tag{3.62}$$

Now, multiplying $(2.8)_1$ by $-\,\xi^2\,\overline{\widehat{y}}$ and $(2.8)_4$ by $-\,\xi^2\,\overline{\widehat{v}}$, adding the obtained formulas and taking the real part, we see that

$$\begin{aligned}&\frac{d}{dt}Re\left(-\,\xi^2\,\widehat{y}\,\overline{\widehat{v}}\right)\\ &= \ b\,\xi^2\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{z}}\right) + \xi^2\left(-\,k\,|\widehat{v}|^2+|\widehat{y}|^2\right) + \xi^2\,Re\left(i\,\xi\,\widehat{u}\,\overline{\widehat{y}} + \beta\,\widehat{y}\,\overline{\widehat{v}} - \widehat{\eta}\,\overline{\widehat{y}}\right).\end{aligned} \tag{3.63}$$

Using the Young inequality, we conclude from (3.63) that, for any $\varepsilon_0 > 0$,

$$\begin{aligned}\frac{d}{dt}Re\left(-\,\xi^2\,\widehat{y}\,\overline{\widehat{v}}\right) \ \leq \ & b\,\xi^2\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{z}}\right) - (k-\varepsilon_0)\,\xi^2\,|\widehat{v}|^2\\ & +\,\varepsilon_0\,\xi^2\,|\widehat{u}|^2 + C_{\varepsilon_0}\,\left(\xi^2+\xi^4\right)\left(|\widehat{\eta}|^2+|\widehat{y}|^2\right).\end{aligned} \tag{3.64}$$

Multiplying $(2.8)_2$ by $\xi^2\,\overline{\widehat{z}}$ and $(2.8)_3$ by $\xi^2\,\overline{\widehat{u}}$, noting that $\gamma_0 = 0$, adding the obtained formulas and taking the real part, we see that

$$\frac{d}{dt}Re\left(\xi^2\,\widehat{u}\,\overline{\widehat{z}}\right) = -\,k\,\xi^2\,Re\left(i\,\xi\,\widehat{v}\,\overline{\widehat{z}}\right) + \xi^2\,Re\left(i\,\xi\,\widehat{y}\,\overline{\widehat{u}}\right). \tag{3.65}$$

Using the Young inequality, we conclude from (3.65) that, for any $\varepsilon_0 > 0$,

$$\frac{d}{dt} Re \left(\xi^2 \, \widehat{u} \, \overline{\widehat{z}}\right) \leq -\, k \, \xi^2 \, Re \left(i \, \xi \, \widehat{v} \, \overline{\widehat{z}}\right) + \varepsilon_0 \, \xi^2 \, |\widehat{u}|^2 + C_{\varepsilon_0} \, \xi^4 \, |\widehat{y}|^2. \tag{3.66}$$

Let us define the functional $\mathcal{F}$ by

$$\begin{aligned} \mathcal{F}(\xi,\, t) \;\; &= \;\; Re \left(-\, i \, \xi \, \widehat{v} \, \overline{\widehat{u}} - i \, \xi^3 \, \widehat{\phi} \, \overline{\widehat{\eta}}\right) + Re \left(i \, \xi^3 \, \widehat{y} \, \overline{\widehat{z}}\right) \\ &\quad + \lambda_1 \, Re \left(-\, \xi^2 \, \widehat{y} \, \overline{\widehat{v}}\right) + \left(1 + \frac{b}{k} \, \lambda_1\right) Re \left(\xi^2 \, \widehat{u} \, \overline{\widehat{z}}\right) \end{aligned} \tag{3.67}$$

and the *Perturbed Energy* $\mathcal{L}$ by (3.16), where λ and λ_1 are positive constants to be choosen later. Replacing (3.61), (3.62) together with (3.60), (3.64) and (3.66) into (3.67), we deduce that

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(\xi,\, t) \;\; &\leq \;\; -\left[1 - \varepsilon_0 \left(2 + \lambda_1 + \frac{b}{k} \, \lambda_1\right)\right] \xi^2 \, |\widehat{u}|^2 - \left(b - \varepsilon_0 - \frac{3\, k}{2} \, \varepsilon_1\right) \xi^4 \, |\widehat{\phi}|^2 \\ &\quad - (b - \varepsilon_0) \, \xi^4 \, |\widehat{z}|^2 - \left[k \left(\lambda_1 - 1 - \frac{3}{2\, \varepsilon_1}\right) - (1 + \lambda_1) \, \varepsilon_0\right] \xi^2 \, |\widehat{v}|^2 \\ &\quad + C_{\varepsilon_0,\ \lambda_1} \left(1 + \xi^2 + \xi^4\right) \left(|\widehat{\eta}|^2 + |\widehat{y}|^2\right). \end{aligned} \tag{3.68}$$

We choose $0 < \varepsilon_1 < \frac{2\, b}{3\, k}$, $\lambda_1 > 1 + \frac{3}{2\, \varepsilon_1}$ and

$$0 < \varepsilon_0 < \min \left\{ b - \frac{3\, k}{2} \, \varepsilon_1, \; \frac{1}{2 + \lambda_1 + \frac{b}{k} \, \lambda_1}, \; \frac{k \left(\lambda_1 - 1 - \frac{3}{2\, \varepsilon_1}\right)}{1 + \lambda_1} \right\}.$$

Hence, using the definition of $\widehat{E}$, (3.68) implies that, for some positive constant c_1,

$$\frac{d}{dt} \mathcal{F}(\xi,\, t) \leq -\, c_1 \, \min\{\xi^2,\ \xi^4\} \widehat{E}(\xi,\, t) + C \, \left(1 + \xi^2 + \xi^4\right) \left(|\widehat{\eta}|^2 + |\widehat{y}|^2\right). \tag{3.69}$$

Then from (2.10) (with $\gamma_0 = 0$), (3.16) and (3.69) we find, for $c_2 = \min\{3\, \beta,\ 4\, \gamma\}$,

$$\frac{d}{dt} \mathcal{L}(\xi,\, t) \leq -\, c_1 \, f(\xi) \, \widehat{E}(\xi,\, t) - (c_2 \, \lambda - C) \left(|\widehat{\eta}|^2 + |\widehat{y}|^2\right),$$

where f is defined in (3.2). So the end of proof is identical to the one of Lemma 3.1.

Theorem 3.9 *The result of Theorem 3.2 holds true also when* $\gamma_0 = 0$ *and* $\beta,\ \gamma > 0$.

Proof. The proof is identical to the one of Theorem 3.2.

Comments. 1. The function f tends to 0 when ξ goes to infinity. This means that the dissipation is very weak in the high frequency region, which leads to the regularity loss in the estimate on $\|\partial_x^j U\|_{L^2(\mathbb{R})}$.
2. It would be interesting to study the cases where only one damping is considered; that is [$\beta = \gamma = 0$ and $\gamma_0 > 0$] or [$\gamma_0 = \gamma = 0$ and $\beta > 0$] or [$\beta = \gamma_0 = 0$ and $\gamma > 0$].

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