Existence of solution to a Dirichlet elliptic problem on the Sierpiński gasket

Arun Kumar Badajena and Rasmita Kar*

Department of Mathematics, National Institute of Technology Rourkela,

Odisha 769008, India

e-mails: ons.arun93@gmail.com, rasmitak6@gmail.com

Abstract. We study the existence of a weak (strong) solution of the nonlinear elliptic problem

$$-\Delta u - \lambda u g_1 + h(u) g_2 = f \quad \text{in } V \setminus V_0$$
$$u = 0 \quad \text{on } V_0,$$

where V is the Sierpiński gasket in $\mathbb{R}^{N-1} (N \geq 2)$, V_0 is its boundary (consisting of its N corners) and λ is a real parameter. Here, $f, g_1, g_2 : V \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are functions satisfying suitable hypotheses.

Keywords: Sierpiński gasket, Nonlinear elliptic equations, Fractal domains, Demicontinuous operators.
Mathematics Subject Classification 2010: 35J60; 28A80; 47H99.

1. Introduction

We study the existence of weak solutions for the following class of elliptic problem

$$\begin{aligned} -\Delta u - \lambda g_1 u + h(u) g_2 = f \quad &\text{in } V \setminus V_0 \\ u = 0 \quad &\text{on } V_0, \end{aligned} \tag{1.1}$$

where V denotes the Sierpiński gasket in $\mathbb{R}^{N-1} (N \geq 2)$, V_0 is its boundary (consisting of its N corners). Δ denotes the Laplacian operator on V, $\lambda \in \mathbb{R}$ and $f, g_1, g_2 : V \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$ are functions satisfying the following hypotheses:

(H_1) Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded (i.e., $|h(t)| \leq A, t \in \mathbb{R}$, $A > 0$) and continuous function;

(H_2) Assume $g_1 \in L^\infty(V)$, $g_2 \in L^2(V)$ and $f \in L^2(V)$.

Recently, there has been a considerable interest in the study of nonlinear partial differential equations on fractal domains and in particular on the Sierpiński gasket. Many physical problems on fractal regions such as reaction-diffusion problems, elastic properties of fractal media and flow through fractal regions are modeled by nonlinear equations. Now, a natural question is whether the classical existence results (we refer to [1, 24, 28]) in the standard framework of the Laplacian also hold in the corresponding fractal framework. To answer this we have to overcome several difficulties that arise due to the geometrical structure of fractal domains. The one main difficulty is that how to define differential operators, like the Laplacian operator, on the fractal domains

*Corresponding author

for there is no concept of a generalized derivative of functions defined on the fractal domains. However, a Laplacian is defined on a few special fractals, we refer to [2, 3, 20, 21] and a Hilbert space structure is introduced in [15]. This enables us to investigate the existence of solutions for equations of type (1.1) defined on fractal domains.

The study of nonlinear elliptic equations on the Sierpinski gasket was essentially initiated by Falconer and Hu in the paper [15]. Since then many authors have contributed to the literature in this direction. In [15], Falconer and Hu, considered the problem

$$\begin{aligned} \Delta u + a(x)u &= f(x,u) \;\; x \in V \setminus V_0 \\ u|_{V_0} &= 0, \end{aligned} \tag{1.2}$$

where V denotes the Sierpinki gasket with boundary V_0 and $a \in L^1(V)$ satisfies suitable conditon. The nonlinearity $f(x,u)$ satisfies the condition

(f) there exists constants $\nu > 2$ and $r \geq 0$ such that for $|t| \geq r$

$$tf(x,t) < \nu F(x,t) < 0 \tag{1.3}$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$.

A typical example of the function f is $f(x,t) = -t|t|^{p-1}, p > 1$. The authors formulated the problem in a suitable function space over Sierpinski gasket and used the Mountain Pass Theorem [1] to prove the existence of a solution. In [5], Molica Bisci et.al. considered a similar problem

$$\begin{aligned} \Delta u + \alpha(x)u &= \lambda f(x,u) \;\; x \in V \setminus V_0 \\ u|_{V_0} &= 0, \end{aligned}$$

where λ is a positive real parameter and proved the existence of at least two solutions for small values of λ. The authors used a recent result in variational principle due to Ricceri [25] to prove the existence of solutions. In [8], Breckner et. al. studied the existence of infinitely many solutions of the problem

$$\begin{aligned} \Delta u(x) + \alpha(x)u(x) &= g(x)f\left(u(x)\right) \;\; x \in V \setminus V_0 \\ u|_{V_0} &= 0. \end{aligned}$$

The authors proved the existence of infinitely many solutions by extending a method introduced by Faraci and Kristály [16] in the framework of Sobolev spaces to the case of function spaces on fractal domains. For more results on existence and multiplicity of solutions on the Sierpinski gasket we refer to the papers [4, 6-11, 13, 14] and [18, 19, 26, 27] as well as the references therein. The main tool used in these papers to prove the existence of nontrivial solutions are basically Mountain Pass theorems, saddle-point theorems or certain minimization procedures.

If the condition (f) does not hold then the functional associated to the problem (1.2) does not satisfy the Mountain Pass structure and so we can not use the Mountain Pass theorem to prove the existence of solutions. In this paper, we show an application of demicontinuous operators to the nonlinear elliptic problems in the fractal setting. In particular, the main tool we used to establish the existence of a solution is a result due to P. Hess [12] on linear demicontinuous operators. We note that the class of functions satisfying (H_1) is different than that of functions considered in [15]. For example, the function $f(x,u) = 1/(1+u^2)$ is bounded but does not satisfy (f). The study is inspired by a problem in bounded domain given in the book by Zeidler [30].

This paper is organized as follows; Section 2 deals with preliminaries and weak formulation of the problem. Section 3 concerns with the main result namely the existence of a weak solution of

(1.1). Finally, Section 4 deals with an extension to a class of continuous functions h that are not necessarily bounded.

2. Preliminaries

We recall the definition of the Sierpiński gasket in $\mathbb{R}^{N-1}$ $(N \geq 2)$ and the Hilbert space $H_0^1(V)$ as introduced in [15] (also we refer to [3,4,5–10]). Let $p_1, p_2, ..., p_N \in \mathbb{R}^{N-1}$ be such that $|p_i - p_j| = 1$ for $i \neq j$. Define, the map $F_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$F_i(x) = \frac{1}{2}x + \frac{1}{2}p_i.$$

Let $\mathcal{F} := \{F_1, F_2, ..., F_N\}$ and for any subset A of $\mathbb{R}^{N-1}$, define the map $L : \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ by

$$L(A) = \bigcup_{i=1}^{N} F_i(A).$$

Then, by [13, Theorem 9.1] there exists a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the attractor of the family $\mathcal{F}$, such that $L(V) = V$. The set V with boundary $V_0 = \{p_1, p_2, ..., p_N\}$ is called the Sierpiński gasket in $\mathbb{R}^{N-1}$.

Let $C(V)$ denotes the space of real-valued continuous functions on V and $C_0(V) = \{u \in C(V) : u|_{V_0} = 0\}$ both equipped with the usual supremum norm $\|\cdot\|_\infty$. For $m \in \mathbb{N}$ let $V_* := \bigcup_{m \geq 0} V_m$, where $V_m = L(V_{m-1})$. Now, for any function $u : V_* \to \mathbb{R}$ and $x, y \in V_m$ define

$$W_m(u) := \left(\frac{N+2}{N}\right)^m \sum_{|x-y|=2^{-m}} (u(x) - u(y))^2.$$

It turns out that, $W_m(u) \leq W_{m+1}(u)$ (we refer to [21]) and hence, the function $W(u)$ defined as

$$W(u) = \lim_{m \to \infty} W_m(u).$$

is well defined. Let $H_0^1(V)$ be the space of functions given by

$$H_0^1(V) := \{u \in C_0(V) \ : \ W(u) < \infty\}$$

with the norm

$$\|u\|_{H_0^1(V)} = \sqrt{W(u)}.$$

Let $H^{-1}(V)$ be the closure of $L^2(V)$ with respect to the pre-norm

$$\|u\|_{-1} = \sup_{v \in H_0^1(V)} |\langle u, v\rangle|,$$

where

$$\langle u, v\rangle = \int_V uv d\mu$$

for $u \in L^2(V)$, $v \in H_0^1(V)$ and μ denotes restriction of the normalized $\log N / \log 2-$dimensional Hausdorff measure on $\mathbb{R}^{N-1}$ such that $\mu(V) = 1$ (we refer to [8]). Note that with this structure $H^{-1}(V)$ is a Hilbert space. Here the space $H^{-1}(V)$ denotes the dual of $H_0^1(V)$.

Let the space $H_0^1(V)$ be given with the inner product

$$\mathcal{W}(u, v) = \lim_{m \to \infty} \left(\frac{N+2}{N}\right)^m \sum_{x,y \in V_m, |x-y|=2^{-m}} (u(x) - u(y))(v(x) - v(y)).$$

By the Cauchy–Schwarz inequality, $\mathcal{W}(u, v)$ as defined above exists and is finite. The space $H_0^1(V)$ with the inner product $\mathcal{W}(u, v)$ is a separable Hilbert space (we refer to [9]). Also we have

$$|\mathcal{W}(u, v)| \leq \|u\|_{H_0^1(V)} \|v\|_{H_0^1(V)}, \text{for all } u, v \in H_0^1(V) \tag{2.4}$$

Since $H^{-1}(V)$ is a Hilbert space, for each $u \in H_0^1(V)$, the relation

$$-\mathcal{W}(u,v) = \langle \Delta u, v\rangle \quad \text{for all } v \in H_0^1(V)$$

uniquely defines a function $\Delta u \in H^{-1}(V)$. The operator Δ is denoted as the weak Laplacian of u on V. Now, we can define the weak solution for the problem (1.1).

Definition 2.1. *We say that a function $u \in H_0^1(V)$ is a weak solution of* (1.1) *if it satisfies*

$$\mathcal{W}(u,v) - \lambda \int_V g_1(x)u(x)v(x)\, d\mu + \int_V h(u(x))g_2(x)v(x)\, d\mu = \int_V f(x)v(x)\, d\mu \tag{2.5}$$

for all $v \in H_0^1(V)$.

For further details on Laplacian operator on Sierpiński gasket, we refer to the paper [20].

We note that, if the functions f, g_1, g_2 and h are continuous then, the weak solutions of the equation (1.1) are also strong solutions which, is the following.

Lemma 2.2. *Assume that $u \in H_0^1(V)$ is a weak solution to the problem* (1.1). *If the functions $f, g_1, g_2 \in C(V)$ and $h \in C(\mathbb{R})$ then, u is a strong solution to* (1.1).

Proof. The proof is similar to [15, Lemma 2.16], hence omitted. □

At each step, a generic constant is denoted by C or c to avoid too many suffixes. We recall the embedding properties of $H_0^1(V)$ into the space $C_0(V)$ and to the space $L^2(V,\mu)$ (we refer to [15]), for sake of completeness.

Lemma 2.3. *The embedding $j : H_0^1(V) \hookrightarrow C_0(V)$ is compact and*

$$|u(x)| \leq (2N+3)\|u\|_{H_0^1(V)} \quad \text{for any } x \in V. \tag{2.6}$$

Also we have that the embedding $j : H_0^1(V) \hookrightarrow L^2(V)$ is compact and

$$\|u\|_2 \leq C\|u\|_{H_0^1(V)}, \tag{2.7}$$

where $\|u\|_2 = \left(\int_V |u(x)|^2 d\mu\right)^{\frac{1}{2}}$.

Let Y^* denotes the dual of the real Banach space Y. Let $\|.\|$ and $\|.\|_{Y^*}$ denote the norm on Banach space Y and dual space Y^* respectively. For $x \in Y$ and $f \in Y^*$, let $(f|x)$ denotes the evaluation of linear functional f at x.

Definition 2.4. Let B, $N : Y \to Y^*$ be operators on the real separable reflexive Banach space Y. Then,

(i) $B + N$ is asymptotically linear if B is linear and

$$\frac{\|Nu\|}{\|u\|} \to 0, \text{ as } \|u\| \to \infty.$$

(ii) B satisfies condition (S) if

$$u_n \rightharpoonup u \text{ and } \lim_{n\to\infty}(Bu_n - Bu|u_n - u) = 0, \text{ implies } u_n \to u. \tag{2.8}$$

We say that B is a (S)-operator if B satisfies condition (S) .

The following is on a real Gårding form G, (compare with [29, page 364]):

Definition 2.5. Let X and Z be Hilbert spaces over $\mathbb{R}$ with the continuous embedding $X \subseteq Z$. Then, $G : X \times X \to \mathbb{R}$ is called a *Gårding form* iff G is bilinear and bounded, and there is a constant $c > 0$ and a real constant C such that

$$G(u,u) \geq c\,\|u\|_X^2 - C\,\|u\|_Z^2, \quad \text{for all } u \in X. \tag{2.9}$$

The equation (2.9) is called Gårding inequality. If $C = 0$ then, G is called a *strict Gårding form*. The Gårding form G is called *regular* iff the embedding $X \subseteq Z$ is compact.

In Section 3, we need the following result.

Proposition 2.6. *Let $B, N : Y \to Y^*$ be operators on the real separable reflexive Banach space Y. Assume:*

(i) the operator $B : Y \to Y^*$ is linear and continuous;

(ii) the operator $N : Y \to Y^*$ is demicontinuous and bounded;

(iii) $B + N$ is asymptotically linear;

(iv) for each $T \in Y^*$ and for each $t \in [0,1]$, the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S) in Y.

If $Bu = 0$ implies $u = 0$ then, for each $T \in Y^*$, the equation $Bu + Nu = T$ has a solution in Y.

For a detailed proof of the above theorem, we refer to [12] or to [30, Theorem 29.C].

We define the functionals $B_1, B_2 : H_0^1(V) \times H_0^1(V) \to \mathbb{R}$ by

$$B_1(u,\varphi) = \mathcal{W}(u,\varphi) - \lambda \int_V u(x) g_1(x) \varphi(x)\, \mathrm{d}\mu$$

$$B_2(u,\varphi) = \int_V h(u(x)) g_2(x) \varphi(x)\, \mathrm{d}\mu.$$

Also define $T : H_0^1(V) \to \mathbb{R}$ by

$$T(\varphi) = \int_V f(x)\varphi(x)\, \mathrm{d}\mu.$$

A function $u \in H_0^1(V)$ is a solution of (1.1) if

$$B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi), \quad \forall \varphi \in H_0^1(V).$$

By applying Cauchy-Schwarz inequality, we note that

$$|B_1(u,\varphi)| \leq |\mathcal{W}(u,\varphi)| + \lambda \int_V |g_1(x)||u(x)||\varphi(x)|\, \mathrm{d}\mu \tag{2.10}$$

$$\begin{aligned} &= \|u\|_{H_0^1(V)} \|\varphi\|_{H_0^1(V)} + |\lambda| \|g_1\|_\infty \|u\|_2 \|\varphi\|_2 \\ &\leq (1 + C|\lambda| \|g_1\|_\infty) \|u\|_{H_0^1(V)} \|\varphi\|_{H_0^1(V)}, \end{aligned} \tag{2.11}$$

where C is a constant arising out of the inequality (2.7) in Lemma 2.3.

By hypotheses (H_1),(H_2) and Holder's inequality, we have

$$\begin{aligned} |B_2(u,\varphi)| &\leq \int_V |h(u(x))||\varphi(x)||g_2|\, \mathrm{d}\mu \\ &\leq A \int_V |\varphi(x)||g_2(x)|\, \mathrm{d}\mu \\ &\leq A\|\varphi\|_2 \|g_2\|_2 \leq AC\|\varphi\|_{H_0^1(V)} \|g_2\|_2. \end{aligned} \tag{2.12}$$

Also, we have

$$|T(\varphi)| \leq \int_V |f(x)||\varphi(x)|\, \mathrm{d}\mu \leq \|f\|_2 \|\varphi\|_2 \leq C\, \|f\|_2 \|\varphi\|_{H_0^1(V)}, \tag{2.13}$$

where C is a constant arising out of the inequality (2.7). Now, $B_1(u,\cdot)$ and $B_2(u,\cdot)$ are linear and bounded. We define the operators

$$B, N : H_0^1(V) \to H^{-1}(V)$$

as

$$(Bu|\varphi) = B_1(u,\varphi), \quad (Nu|\varphi) = B_2(u,\varphi), \quad \text{for } u,\varphi \in H_0^1(V).$$

Then, (1.1) is equivalent to operator equation $Bu + Nu = T$, $u \in H_0^1(V)$.

3. Main results

In this section, we study the existence of a weak solution for (1.1).

Theorem 3.1. *Assume* (H_1) *and* (H_2). *Let* $\lambda > 0$ *not be an eigenvalue of*

$$\begin{aligned} -\Delta u - \lambda u(x) g_1(x) &= 0 \quad \text{in } V \setminus V_0, \\ u &= 0 \quad \text{on } V_0, \end{aligned} \tag{3.14}$$

and in addition, let

$$1 > \lambda\, C \big\| g_1 \big\|_\infty, \tag{3.15}$$

where, C is a constant arising out of the inequality (2.7). *Then, the BVP* (1.1) *has a weak solution* $u \in H_0^1(V)$. *Moreover, every (weak) solution u of* (1.1) *satisfies*

$$\|u\|_{H_0^1(V)} \leq \frac{C\big\{A\|g_2\|_2 + \|f\|_2\big\}}{(1 - C\lambda\|g_1\|_\infty)},$$

where A is a constant from hypotheses (H_1).

Proof. First we write the BVP (1.1) as operator equation

$$u \in H_0^1(V) : Bu + Nu = T \quad \text{in} \quad H^{-1}(V), \tag{3.16}$$

where $T \in H^{-1}(V)$, $B, N : H_0^1(V) \to H^{-1}(V)$ satisfies all the conditions given in Proposition 2.6. For convenience, we divide the proof into five steps.

Step-1 : From the previous section we know that the operator B is linear and continuous. By Lemma 2.3 the embedding of $H_0^1(V) \hookrightarrow L^2(V)$ is compact which shows that $B_1(\cdot,\cdot)$ is a strict regular Gårding form [29, p.364]. In fact, we have

$$\begin{aligned} B_1(u,u) =& \mathcal{W}(u,u) - \lambda \int_V u^2(x) g_1(x)\, \mathrm{d}\mu \\ &\geq \|u\|^2_{H_0^1(V)} - \lambda \|g_1\|_\infty \|u\|_2^2. \end{aligned} \tag{3.17}$$

Let $u_k \rightharpoonup u$ weakly in $H_0^1(V)$ and

$$\lim_{k\to\infty} (Bu_k - Bu | u_k - u) = 0, \tag{3.18}$$

Claim: B satisfies condition (S). Since B is linear, as in (3.17) we have,

$$\begin{aligned} (Bu_k - Bu|u_k - u) = (B(u_k - u)|u_k - u) &= B_1(u_k - u, u_k - u) \\ &\geq \|u_k - u\|^2_{H_0^1(V)} - \lambda \|g_1\|_\infty \|u_k - u\|_2^2 \\ &\geq \ (1 - C\lambda \|g_1\|_\infty) \|u_k - u\|^2_{H_0^1(V)}. \end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we note

$$0 \leq \ (1 - C\lambda\|g_1\|_\infty) \lim_{k\to\infty} \|u_k - u\|^2_{H_0^1(V)} \leq \lim_{k\to\infty} (Bu_k - Bu|u_k - u) = 0.$$

Since $(1-C\lambda\|g_1\|_\infty) > 0$, we have $\|u_k-u\|^2_{H^1_0(V)} \to 0$ as $k \to \infty$, which implies $\|u_k-u\|_{H^1_0(V)} \to 0$, as $k \to \infty$. Hence, B satisfies condition (S).

Step-2: Claim : $B+N$ is asymptotically linear. By (H_1), we have,

$$|(Nu|\varphi)| = |B_2(u,\varphi)| \leq \ AC\,\|g_2\|_2\|\varphi\|_{H^1_0(V)}, \quad \forall u \in H^1_0(V),$$

which implies

$$\|Nu\|_{H^{-1}(V)} \leq C',$$

where $C' = AC\|g_2\|_2$ is a constant depending on V. Consequently,

$$\frac{\|Nu\|_{H^{-1}(V)}}{\|u\|_{H^1_0(V)}} \to 0 \quad \text{as } \|u\|_{H^1_0(V)} \to \infty, \tag{3.20}$$

which shows that $B+N$ is asymptotically linear and the operator N is strongly continuous (we refer to [30, Corollary 26.14]).

Step-3: From Step-2, we note that the operator B satisfies condition (S). Since, N is strongly continuous, we note that $t(Nu-T)$ is strongly continuous, for $t \in [0,1]$. For each $t \in [0,1]$, the operator $A_t(u) = Bu + t(Nu-T)$ is a strongly continuous perturbation of the (S)-operator B. So, the operator $A_t(u)$ satisfies condition (S)(we refer to [30, Proposition 27.12]).

Step-4: Now, $Bu = 0$ implies

$$\mathcal{W}(u,u) - \lambda\int_V u^2(x)g_1(x)\ \mathrm{d}\mu = 0.$$

Consequently, we have

$$(1-C\lambda\|g_1\|_\infty)\|u\|^2_{H^1_0(V)} \leq 0$$

which shows that $u=0$, since $1-C\lambda\|g_1\|_\infty > 0$ and λ is not an eigenvalue of (3.14).

By Proposition 2.6, $Bu+Nu=T$ has a solution $u \in H^1_0(V)$ which equivalently shows, the BVP (1.1) has a solution $u \in H^1_0(V)$.

Step-5: As in (3.19) (with the help of embedding in Lemma 2.3), we obtain

$$B_1(u,u) \geq \big(1-C\lambda\|g_1\|_\infty\big)\|u\|^2_{H^1_0(V)}.$$

Since, $1 > C\lambda\|g_1\|_\infty$, we have

$$\|u\|^2_{H^1_0(V)} \leq \Big(\frac{1}{1-C\lambda\|g_1\|_\infty}\Big)B_1(u,u). \tag{3.21}$$

Also, we note that

$$|B_1(u,u)| \leq C\big\{A\|g_2\|_2 + \|f\|_2\big\}\|u\|_{H^1_0(V)}. \tag{3.22}$$

By (3.21) and (3.22), we have

$$\|u\|_{H^1_0(V)} \leq \frac{C\big\{A\|g_2\|_2 + \|f\|_2\big\}}{(1-C\lambda\|g_1\|_\infty)}.$$

$\square$

Next, we dispense with the condition (3.15) when g_1 does not change sign. The two results are related to the cases when $g_1 \geq 0$ with $\lambda \leq 0$ and $g_1 \leq 0$ with $\lambda > 0$. These results are similar to that of Theorem 3.1 but with suitable changes.

Theorem 3.2. *Suppose that* (H_1) *and* (H_2) *hold. Let* $g_1 \geq 0$ *and* $\lambda \leq 0$ *then, the BVP* (1.1) *has a solution* $u \in H^1_0(V)$ *and*

$$\|u\|_{H^1_0(V)} \leq C\big\{A\|g_2\|_2 + \|f\|_2\big\},$$

where C *is a constant arising out of the inequality* (2.7).

Proof. As in Theorem 3.1, the basic idea is to reduce the problem (1.1) into an operator equation $Bu+Nu=T$ and then, study the existence result with the help of Proposition 2.6. To do proceed, we define B,N and T, as in Theorem 3.1 and by a similar argument for estimates (2.11),(2.12) and (2.13), we have

$$
\begin{aligned}
|B_1(u,\varphi)| &\leq (1+C|\lambda|\|g_1\|_\infty)\|u\|_{H_0^1(V)}\|\varphi\|_{H_0^1(V)}\\
|B_2(u,\varphi)| &\leq CA\|g_2\|_2\|\varphi\|_{H_0^1(V)}\\
|T(\varphi)| &\leq C\|f\|_2\|\varphi\|_{H_0^1(V)}
\end{aligned}
$$

where C is the constant as in Lemma 2.3. The compact embedding of $H_0^1(V)\hookrightarrow L^2(V)$, shows that $B_1(\cdot,\cdot)$ is a strict regular Gårding form. Also, $\lambda\leq 0$ and $g_1\geq 0$ yields

$$
B_1(u,u)=\mathcal{W}(u,u)-\lambda\int_V u^2(x)g_1(x)\,\mathrm{d}\mu \geq \|u\|^2_{H_0^1(V)} \tag{3.23}
$$

Let $u_k\rightharpoonup u$ weakly in $H_0^1(V)$ and

$$
\lim_{k\to\infty}(Bu_k-Bu|u_k-u)=0. \tag{3.24}
$$

We claim that B satisfies condition (S). Since B is linear, as in (3.23) we have,

$$
\begin{aligned}
(Bu_k-Bu|u_k-u) &= (B(u_k-u)|u_k-u)\\
&= B_1(u_k-u,u_k-u)\geq\ \|u_k-u\|^2_{H_0^1(V)}.
\end{aligned} \tag{3.25}
$$

From (3.24) and (3.25), we note that

$$
0\leq \lim_{k\to\infty}\|u_k-u\|^2_{H_0^1(V)}\leq \lim_{k\to\infty}(Bu_k-Bu|u_k-u)=0
$$

which implies $\|u_k-u\|_{H_0^1(V)}\to 0$, as $k\to\infty$ and consequently, B satisfies condition (S). Next, we show that $B+N$ is asymptotically linear and N is strongly continuous. The proof is similar to that of Theorem 3.1 and we omit it for brevity. Since $\lambda\leq 0$, we get $Bu=0$ implies $u=0$ and hence, we note that $\lambda\leq 0$ is not an eigenvalue of (3.14). By Proposition 2.6, $Bu+Nu=T$ has a solution $u\in H_0^1(V)$ which equivalently shows that the BVP (1.1) has a solution $u\in H_0^1(V)$. Since $\lambda\leq 0$ and $g_1\geq 0$, we obtain (as in (3.23)) $B_1(u,u)\geq\|u\|^2_{H_0^1(V)}$. Then, by a similar argument as in Theorem 3.1, we have

$$
\|u\|_{H_0^1(V)}\leq C\big\{A\|g_2\|_2+\|f\|_2\big\},
$$

where C is a constant arising out of the inequality (2.7). □

With suitable modifications in the proof of Theorem 3.2, we have the following result.

Theorem 3.3. *Suppose that* (H_1) *and* (H_2) *hold. Let* $g_1\leq 0$ *and* $\lambda>0$ *then,* (1.1) *has a weak solution* $u\in H_0^1(V)$ *and there is a constant* k_0 *such that* $\|u\|_{H_0^1(V)}\leq k_0$ *for every (weak) solution* u.

4. Extensions

In Section 3, the nonlinearity h is assumed to be continuous and bounded. In this section, we extend these results for a class of functions h which are continuous only. Generalized Hölder's inequality comes handy for getting suitable estimates. We establish the existence of a weak solution for (1.1), where $h:\mathbb{R}\to\mathbb{R}$ is required to be continuous and to satisfy $|h(t)|\leq|t|^\epsilon, 0<\epsilon<1$, for all $t\in\mathbb{R}$. Again, we consider the cases $\lambda\leq 0$ and $\lambda>0$ separately. Although the proofs are similar to the ones in Section 3, we restrict ourselves to sketch the differences wherever needed. The result in [29] is not applicable here since h is not bounded. We collect the common hypotheses for convenience.

(H_1') Suppose that $h : \mathbb{R} \to \mathbb{R}$ defined by $|h(t)| \le |t|^\epsilon, t \in \mathbb{R}, 0 < \epsilon < 1$;

(H_2') $g_1 \in L^\infty(V)$, $g_2 \in L^{\frac{2}{1-\epsilon}}(V), 0 < \epsilon < 1$ and $f \in L^2(V)$.

Theorem 4.1. *Let the hypotheses (H_1'), (H_2') hold. Let $g_1 \ge 0$ and $\lambda \le 0$ then, (1.1) has a weak solution $u \in H_0^1(V)$ and there is a constant k_0 such that $\|u\|_{H_0^1(V)} \le k_0$ for every (weak) solution u.*

Proof. We give only a sketch of the proof since it is similar to the proof of Theorem 3.2. For $u \in H_0^1(V)$, from the hypotheses and by Lemma 2.3, we note that

$$\begin{aligned} |B_1(u,\varphi)| &\le \big(1 + C|\lambda|\|g_1\|_\infty\big)\|u\|_{H_0^1(V)}\|\varphi\|_{H_0^1(V)}, \\ |T(\varphi)| &\le C\|f\|_2\|\varphi\|_{H_0^1(V)}, \end{aligned} \tag{4.26}$$

where the constant C comes from Lemma 2.3. Again, by Lemma 2.3 and generalized Hölder's inequality [23, p.67], we have

$$|B_2(u,\varphi)| \le \int_V |h(u(x))||\varphi(x)||g_2| \,\mathrm{d}\mu \le \|u\|_2^\epsilon \|\varphi\|_2 \|g_2\|_{\frac{2}{1-\epsilon}}.$$

By a similar argument as in Theorem 3.2 (also refer to [30, Proposition 27.12]) we observe that the operator B_1 satisfies condition (S). We also observe that

$$|(Nu|\varphi)| = |B_2(u,\varphi)| \le C\|u\|^\epsilon_{H_0^1(V)}\|\varphi\|_{H_0^1(V)}\|g_2\|_{\frac{2}{1-\epsilon}}$$

which implies

$$\|Nu\|_{H^{-1}(V)} \le C\|u\|^\epsilon_{H_0^1(V)}\|g_2\|_{\frac{2}{1-\epsilon}} = c\|u\|^\epsilon_{H_0^1(V)},$$

where the constant $c = C\|g_2\|_{\frac{2}{1-\epsilon}}$. So

$$\frac{\|Nu\|_{H^{-1}(V)}}{\|u\|_{H_0^1(V)}} \le \frac{c\|u\|^\epsilon_{H_0^1(V)}}{\|u\|_{H_0^1(V)}} \to 0 \quad \text{as } \|u\|_{H_0^1(V)} \to \infty. \tag{4.27}$$

This shows that $B + N$ is asymptotically linear. Also, $u \in L^2(V)$ implies that $h(u) \in L^{\frac{2}{\epsilon}}(V)$ and define the Nemytskii operator

$$F : L^2(V) \to L^{\frac{2}{\epsilon}}(V) \tag{4.28}$$

by $(Fu)(x) = h(u(x))$; we have F is continuous (by [22, Theorem 2.1]). Now, the hypotheses (H_1'), (H_2') and generalized Hölder's inequality imply that

$$\begin{aligned} |(Nu_n|\varphi) - (Nu|\varphi)| &\le \int_V |h(u_n) - h(u)||g_2||\varphi| \,\mathrm{d}\mu \\ &\le C\|h(u_n) - h(u)\|_{\frac{2}{\epsilon}}\|g_2\|_{\frac{2}{1-\epsilon}}\|\varphi\|_{H_0^1(V)}. \end{aligned}$$

Let $u_n \rightharpoonup u$ weakly in $H_0^1(V)$. Then, by the continuity of F in $L^{\frac{2}{\epsilon}}(V)$ and by the compact embedding $H_0^1(V) \hookrightarrow L^2(V)$, we have

$$\|Nu_n - Nu\|_{H^{-1}(V)} \to 0 \quad \text{as } n \to \infty. \tag{4.29}$$

By a similar argument as in Theorem 3.1, we can show that the operator $A_t(u) = Bu + t(Nu - T)$ satisfies condition (S). If $\lambda \le 0$ then, $Bu = 0$ implies $u = 0$ and $\lambda \le 0$ is not an eigenvalue of the linear problem (3.14). By Proposition 2.6 the operator equation $Bu + Nu = T$ and consequently, (1.1) has a solution $u \in H_0^1(V)$, which completes the proof of existence result.

Now, as in (3.23), we have

$$B_1(u,u) \ge \|u\|^2_{H_0^1(V)}. \tag{4.30}$$

Also, we note that

$$|B_1(u,u)| \leq C\{\|u\|^{\epsilon}_{H^1_0(V)}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\}\|u\|_{H^1_0(V)}. \tag{4.31}$$

By (4.30) and (4.31), we have

$$\|u\|_{H^1_0(V)} \leq C\{\|u\|^{\epsilon}_{H^1_0(V)}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\} \tag{4.32}$$

If $\|u\|_{H^1_0(V)} \geq 1$, then from (4.32), we have

$$\|u\|_{H^1_0(V)} \leq C\big(\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)\|u\|^{\epsilon}_{H^1_0(V)}$$

which implies

$$\|u\|^{1-\epsilon}_{H^1_0(V)} \leq c$$

$$\text{or, } \|u\|_{H^1_0(V)} \leq c^{\frac{1}{1-\epsilon}},$$

where $c = C\big(\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)$ and $0 < \epsilon < 1$. If $\|u\|_{H^1_0(V)} \leq 1$, we have nothing to prove. Let $k_0 = \max\{1, c^{\frac{1}{1-\epsilon}}\}$. Hence, we have

$$\|u\|_{H^1_0(V)} \leq k_0.$$

□

Remark. Theorem 4.1 hold if $g_1 \leq 0$ and $\lambda > 0$ with the remaining intact. But when $\lambda > 0$ and g_1 changes sign, we need additional conditions on λ and g_1 (stated below) as in Theorem 3.1. We state these results below in Theorem 4.2 but we give a sketch of the proof. We note that in (4.27) the required asymptotic linearity of $B + N$ is a consequence of ϵ lying between 0 and 1.

Theorem 4.2. *Let the hypotheses (H_1'), (H_2') hold. Also, let $\lambda > 0$ not be an eigenvalue of (3.14) and in addition, let $1 > C\lambda\|g_1\|_\infty$. Then, the BVP (1.1) has a weak solution $u \in H^1_0(V)$ and there is a constant k_0 such that $\|u\|_{H^1_0(V)} \leq k_0$ for every (weak) solution u.*

Proof. The proof for existence of a weak solution $u \in H^1_0(V)$ for (1.1) is similar to the argument in Theorem 4.1 and Theorem 3.1 and hence, omitted. As in Theorem 3.1, we note that

$$(1 - C\lambda\|g_1\|_\infty)\|u\|^2_{H^1_0(V)} \leq C\{\|u\|^{\epsilon}_{H^1_0(V)}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\}\|u\|_{H^1_0(V)},$$

where C is a constant. Since $1 > C\lambda\|g_1\|_\infty$, we obtain

$$\|u\|_{H^1_0(V)} \leq \frac{C\big(\|u\|^{\epsilon}_{H^1_0(V)}\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)}{(1 - C\lambda\|g_1\|_\infty)} \tag{4.33}$$

If $\|u\|_{H^1_0(V)} \geq 1$, from (4.33), we have

$$\|u\|_{H^1_0(V)} \leq \frac{C\big(\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)\|u\|^{\epsilon}_{H^1_0(V)}}{(1 - C\lambda\|g_1\|_\infty)}$$

which implies that

$$\|u\|^{1-\epsilon}_{H^1_0(V)} \leq c$$

$$\text{or, } \|u\|_{H^1_0(V)} \leq c^{\frac{1}{1-\epsilon}},$$

where $c = \dfrac{C\big(\|g_2\|_{\frac{2}{1-\epsilon}} + \|f\|_2\big)}{(1 - C\lambda\|g_1\|_\infty)}$ and $0 < \epsilon < 1$. If $\|u\|_{H^1_0(V)} \leq 1$, we have nothing to prove. Let $k_0 = \max\{1, c^{\frac{1}{1-\epsilon}}\}$. Then, we have

$$\|u\|_{H^1_0(V)} \leq k_0.$$

□

References

[1] Ambrosetti, A. and Rabinowitz, P., Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349—381.

[2] Barlow, M. T. and Bass, R. F., Transition densities for Brownian motion on the Sierpiński carpet, Probab. Theory Related Fields 91 (1992), 307–330.

[3] Barlow, M. T. and Bass, R. F., Brownian motion and harmonic analysis on Sierpiński carpet, Canad. J. Math. 51 (1999), 673–744.

[4] Bisci, G. M., and Rădulescu, V., A characterization for elliptic problems on fractal sets, Proc. Amer. Math. Soc. 143 (2015), 2959-2968.

[5] Bisci, G. M., Repovš, D., and Servadei, R., Nonlinear problems on the Sierpiński gasket, J. Math. Anal. Appl. 452 (2017), 883–895.

[6] Breckner, B. E., Real-valued functions of finite energy on the Sierpiński gasket, Mathematica. 55(78), (2013), 142–158.

[7] Breckner, B. E., A short note on harmonic functions and zero divisors on the Sierpiński fractal, Arch. Math. (Basel), 106 (2016), 183–188.

[8] Breckner, B. E., Rădulescu, V., and Varga, C., Infinitely many solutions for the Dirichlet problem on the Sierpiński gasket, Anal. Appl. 9 (2011), 235–248.

[9] Breckner, B. E., Repovš, D. and Varga, C., On the existence of three solutions for the Dirichlet problem on the Sierpiński gasket, Nonlinear Anal. 73 (2010), 2980-2990.

[10] Breckner, B. E. and Varga, C., A note on gradient-type systems on fractals, Nonlinear Anal. Real World Appl. 21 (2015), 142–152.

[11] Breckner, B. E. and Varga, C., Multiple solutions of Dirichlet problems on the Sierpiński gasket, J. Optim. Theory Appl. 167 (2015), 842–861.

[12] Hess, P., On the Fredholm alternative for nonlinear functional equations in Banach spaces, Proc. Amer. Math. Soc. 33 (1972), 55–61.

[13] Falconer, K. J., Semilinear PDEs on self-similar fractals, Comm. Math. Phys. 206 (1999), $235-245$.

[14] Falconer, K. J., Fractal Geometry: Mathematical Foundations and Applications, 2nd edition, (John Wiley & Sons, 2003).

[15] Falconer, K. J. and Hu, J., Nonlinear elliptic equations on the Sierpiński gasket, J. Math. Anal. Appl. 240 (1999), 552–573.

[16] Faraci, F. and Kristály, A., *One-dimensional scalar field equations involving an oscillatory nonlinear term*, Discrete Contin. Dynam. Syst. **18**(1) (2007) 107–120.

[17] Fukushima, M. and Shima, T., On a spectral analysis for the Sierpiński gasket, Potential Anal. 1 (1992), 1–35.

[18] He, Z., Sublinear elliptic equation on fractal domains, J. Partial Differ. Equ. 24 (2011), 97–113.

[19] Hu, J., Multiple solutions for a class of nonlinear elliptic equations on the Sierpiński gasket, Sci. China Ser. A 47 (2004), 772–786.

[20] Kigami, J., Harmonic calculus on p.c.f. self-similar sets, Trans. Amev. Math. Soc. 335 (1993), 721–755.

[21] Kozlov, S. M., Harmonization and homogenization on fractals, Comm. Math. Phys. 153 (1993), 339–357.

[22] Krasnolsel'skii, M. A., Topological Methods in the Theory of Nonlinear Integral Equations, (GITTL, Moscow, 1956).

[23] Kufner, A., John, O. and Fučik, S., Functions Spaces, (Noordhoff, Leyden, 1977).

[24] Rabinowitz, P.H., Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65, American Mathematical Society, Providence, RI, 1986.

[25] Ricceri, B., *On a classical existence theorem for nonlinear elliptic equations*, in: M. Théra (Ed.), Esperimental, Constructive and Nonlinar Analysis, in: CMS Conf. Proc., vol.27, Canad. Math. Soc., 2000, pp.275–278.

[26] Strichartz, R. S., Some properties of Laplacian on fractals, J. Funct. Anal. 164 (1999), 181–208.

[27] Strichartz, R. S., Solvability for differential equations on fractals, J. Anal. Math. 96 (2005), 247–267.

[28] Struwe, M., Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3, (Springer Verlag, Berlin–Heidelberg, 1990).

[29] Zeidler, E., Nonlinear Functional anlysis and its Applications, Part II/A, (Springer- Verlag, New York, 1990).

[30] Zeidler, E., Nonlinear Functional anlysis and its Applications, Part II/B, (Springer-Verlag, New York, 1990).