Fractional dynamics based-enhancing control scheme of a delayed predator-prey model

Jun Yuan[a], Lingzhi Zhao[a] *, Chengdai Huang[b], Min Xiao[c]

[a] *School of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China*

[b] *School of Mathematics and Statistics, Xinyang Normal University, Xinyang 464000, China*

[c] *College of Automation, Nanjing University of Posts and Telecommunications, Nanjing 210003, China*

Abstract: To retard the onset of undesired bifurcation, the bifurcation control has developed into a theme of centralized research activities in delayed fractional-order system. In this paper, the problem of bifurcation control for a delayed fractional-order predator-prey model is investigated by employing an enhancing feedback control technique. The bifurcation point is firstly established for controlled model by using delay as a bifurcation parameter. Then, a series of numerical comparative studies on the effects of bifurcation control are implemented covering the partial or total removal of the branch for feedback gains. It reveals that the stability performance of the proposed model can be overwhelmingly elevated via the devised approaches in comparison with the dislocated feedback ones. A numerical example with simulations is ultimately designed to confirm the merits of the proposed theoretical results.

Keywords: Fractional order, Time delay, Enhancing feedback control, Predator-prey model, Hopf bifurcation

1 Introduction

In ecological systems, the interactions between two or more species and their dynamics are affected by each other, which can be depicted by prey-predator system. The dynamics of prey-predator systems is one of the basic topics in ecology owing to the worldwide importance and existence, which constructs the complex food chains and food networks. The famous predator-prey model was established by [1, 2]. Afterwards, the dynamical behaviors of prey-predator models, such as chaos, stability, bifurcations and oscillations usually depend on the system parameters. Time delays have been incorporated into biological systems to describe and take into account the time required for resource regeneration time, maturation period, reaction time, feeding time, gestation period [3, 4]. Nowadays, a great deal of outstanding results have been derived on the analysis of predator-prey models [5, 6, 7, 8, 9].

Fractional order dynamical systems have attracted numerous researchers in various branches, especially science and engineering. Comparing to the orthodox integer order dynamical systems, the fundamental distinguished influence of fractional order is that infinite memory and more degrees of freedom because it has nonlocal and weakly singular kernels [10, 11]. Most biological systems display fractional

*Corresponding to: Lingzhi Zhao E-mail: lzhzhao@163.com (L. Zhao).

dynamics owing to memory effects. The presence of memory in the model describes the history of the process involved and carries its impact to present and future developments of the process. Consequently, the differential equations with fractional order can depict more accurately the real phenomena than those with conventional integer order. Normally, modeling population dynamical systems with fractional order can enrich the dynamics, augment the complexity of the models and ameliorate the performance of intricate systems. Recently, some scholars have incorporated fractional calculus into predator-prey models and developed fractional predator-prey ones. A large number of results related to fractional dynamics of delayed predator-prey without delays [12, 13, 14, 15] or with delays [16, 17, 18, 19] have been captured.

Hopf bifurcation analysis is an efficient tool to acquire more information around the equilibrium point of the complex dynamical system. Bifurcations have been extensively researched for procuring the properties of nonlinear complicated dynamical systems [20, 21, 22, 23, 24]. It is well-known that first-class good results on bifurcations have been derived in traditional integer-order models with time delays. In recent years, the bifurcation of fractional-order delayed models has attracted increasing attention [25, 26, 27]. In [27], a delayed generalized fractional-order prey-predator model with interspecific competition was studied, and the global asymptotic stability conditions and local bifurcation criterions of the equilibrium were derived by choosing time delay as a bifurcation parameter.

Especially, bifurcation control is an extremely essential and efficient method. By means of it, one can design a controller to suppress or reduce some existing bifurcation dynamics for a given nonlinear system, thereby extending the stability domain and achieving some desirable dynamical behaviors. It should be noted that the stability performance of fractional-order dynamical systems also can be overly improved because of using the active bifurcation control strategies. The issue of bifurcation control of delayed fractional models has attracted increasing attention [28, 29, 30, 31, 32, 33, 34]. In [28], the onset of bifurcation of a delayed fractional-order small-world networks was effectively controlled by using a fractional-order PD feedback controller. In [31], an neoteric extended delayed feedback strategy was developed of a delayed fractional predator-prey model to deal with the bifurcation dynamics by adjusting extended feedback delay or fractional order. In [32], a parametric delay feedback control approach was further proposed to cope with bifurcation control for a delayed fractional dual congestion model, and it was found that the stability performance can be extremely heightened by adopting the parametric delay feedback controller. Generally, there exist many bifurcation control approaches including dislocated feedback control, speed feedback control and enhancing feedback control [35, 36, 37], et al. In [37], it revealed that the enhancing feedback control approach is the best choice of among the addressed four feedback control methods in controlling hyperchaotic Lorenz system involving relatively simple external inputs and relatively small necessary feedback coefficient. In [36], the author detected that the feedback coefficients were smaller than the ones of ordinary feedback control during controlling hyperchaotic Lorenz system, and the control cost were reduced. It should be pointed out that it is hard for a complex system to be controlled by merely one feedback variable, and in such cases the feedback gain is always very large. Therefore, it is essential and urgent to employ the enhancing feedback control to control the onset of bifurcation for capturing high-quality performance of the addressed fractional-order dynamical systems. Up to present, the bifurcation control of fractional predator-prey systems with delays based on enhancing feedback control tool has been not properly investigated before.

Motivated by the aforemention discussions, we shall address a theoretical analysis on bifurcation control for a delayed fractional-order predator-prey model by taking advantage of enhancing feedback control technique in this paper. The key features of this paper are listed as follows:

1) Enhancing feedback control strategy is developed to deal with the bifurcation control in a fractional delayed predator-prey model.

2) The bifurcation point of the controlled model can be completely concluded by theoretical derivation.

3) The effects of fractional order on the bifurcation points are fully investigated by using enhancing feedback control strategy and dislocated feedback. It is found that the performance of control gradually becomes perfect with the decrement of fractional order.

4) We discover that enhancing feedback control strategy overmatches dislocated feedback ones in delaying the onset of bifurcation control for the considered controlled system for given fractional order.

The rest of the current paper is arranged as follows. Some mathematical preliminaries are presented in Section 2. In Section 3, the investigated model are addressed. Key bifurcation control results by using enhancing feedback control method are wholly determined in Section 4. The efficiency of the proposed control scheme is verified with the help of a simulation example in Section 5. Finally, the paper ends with a conclusion.

2 Preliminaries

Many fractional derivative definitions are applied to deal with some practical issues including the Riemann-Liouville definition and the Caputo definition, et al. It should be noted that the Caputo derivative has many advantages consisting of the consistence of given initial conditions with integer-order derivative, the description of well-understood features of physical situation. This paper employs the Caputo derivative to handle the dynamical fractional-order systems.

Definition 2.1 *[10] The Caputo fractional-order derivative is defined by*

$$D_t^{\phi} f(t) = \frac{1}{\Gamma(l-\phi)} \int_0^t (t-s)^{l-\psi-1} f^{(l)}(s) ds,$$

where $l-1 \leq \phi < l \in Z^+$, $\Gamma(\cdot)$ *is the Gamma function,* $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$.

The Laplace transform of the Caputo fractional-order derivatives is

$$L\{D_t^{\phi} f(t); s\} = s^{\phi} F(s) - \sum_{k=0}^{l-1} s^{\phi-k-1} f^{(k)}(0), \quad l-1 \leq \phi < l \in Z^+.$$

If $f^k(0) = 0$, $k = 1, 2, \ldots, n$, then $L\{D_t^{\phi} f(t); s\} = s^{\phi} F(s)$.

Lemma 2.1 *[38] Consider the following* n*-dimensional linear fractional-order system*

$$\begin{cases} D^{\phi_1}\gamma_1(t) = \mathrm{k}_{11}\gamma_1(t) + \mathrm{k}_{12}\gamma_2(t) + \cdots + \mathrm{k}_{1n}\gamma_n(t), \\ D^{\phi_2}\gamma_2(t) = \mathrm{k}_{21}\gamma_1(t) + \mathrm{k}_{22}\gamma_2(t) + \cdots + \mathrm{k}_{2n}\gamma_n(t), \\ \quad \vdots \\ D^{\phi_n}\gamma_n(t) = \mathrm{k}_{n1}\gamma_1(t) + \mathrm{k}_{n2}\gamma_2(t) + \cdots + \mathrm{k}_{nn}\gamma_n(t), \end{cases} \tag{2.1}$$

where $0 < \phi_i < 1 (i = 1, 2, \dots, n)$. *It is assumed that* ϕ *is the lowest common multiple of the denominators* ψ_i *of* ϕ_i, *where* $\phi_i = \frac{\varphi_i}{\psi_i}$, $(\varphi_i, \psi_i) = 1$, $\varphi_i, \psi_i \in Z^+$, *for* $i = 1, 2, \dots, n$. *Define*

$$\triangle(\lambda) = \begin{bmatrix} \lambda^{\phi_1} - \mathrm{k}_{11} & -\mathrm{k}_{12} & \cdots & -\mathrm{k}_{1n} \\ -\mathrm{k}_{21} & \lambda^{\phi_2} - \mathrm{k}_{22} & \cdots & -\mathrm{k}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathrm{k}_{n1} & -\mathrm{k}_{n2} & \cdots & \lambda^{\phi_n} - \mathrm{k}_{nn} \end{bmatrix}.$$

Then the zero solution of system (2.1) is globally asymptotically stable in the Lyapunov sense if all roots λ *of the equation* $\det(\triangle(\lambda)) = 0$ *satisfy* $|\arg(\lambda)| > \phi_i \pi/2$.

Lemma 2.2 *[38] Consider the following* n*-dimensional linear fractional-order delayed system*

$$\begin{cases} D^{\phi_1}\gamma_1(t) = \mathrm{k}_{11}\gamma_1(t-\tau_{11}) + \mathrm{k}_{12}\gamma_2(t-\tau_{12}) + \cdots + \mathrm{k}_{1n}\gamma_n(t-\tau_{1n}), \\ D^{\phi_2}\gamma_2(t) = \mathrm{k}_{21}\gamma_1(t-\tau_{21}) + \mathrm{k}_{22}\gamma_2(t-\tau_{22}) + \cdots + \mathrm{k}_{2n}\gamma_n(t-\tau_{2n}), \\ \quad\vdots \\ D^{\phi_n}\gamma_n(t) = \mathrm{k}_{n1}\gamma_1(t-\tau_{n1}) + \mathrm{k}_{n2}\gamma_2(t-\tau_{n2}) + \cdots + \mathrm{k}_{nn}\gamma_n(t-\tau_{nn}), \end{cases} \tag{2.2}$$

where $\phi_i \in (0,1) (i = 1, 2, \dots, n)$, *the initial values* $\mathrm{V}_i(t) = \Psi_i(t)$ *are given for* $-\max_{i,j}, \tau_{i,j} = -\max_{i,j} \leq t \leq 0$ *and* $i = 1, 2, \dots, n$. *For system (2.2), time-delay matrix* $\tau = (\tau_{i,j}) \in (R^+)_{n\times n}$, *coefficient matrix* $H = (\mathrm{k}_{i,j})_{n\times n}$, *state variables* $\gamma_i(t)$, $\gamma_i(t-\tau_{i,j}) \in R$, *and initial values* $\Psi_i(t) \in C^0[-\tau_{\max}, 0]$. *Its fractional order is defined as* $\phi = (\phi_1, \phi_2, \dots, \phi_n)$. *It is defined as*

$$\triangle(s) = \begin{bmatrix} s^{\phi_1} - \mathrm{k}_{11}e^{-s\tau_{11}} & -\mathrm{k}_{12}e^{-s\tau_{12}} & \cdots & -\mathrm{k}_{1n}e^{-s\tau_{1n}} \\ -\mathrm{k}_{21}e^{-s\tau_{21}} & s^{\phi_2} - \mathrm{k}_{22}e^{-s\tau_{22}} & \cdots & -\mathrm{k}_{2n}e^{-s\tau_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathrm{k}_{n1}e^{-s\tau_{n1}} & -\mathrm{k}_{n2}e^{-s\tau_{n2}} & \cdots & s^{\phi_n} - \mathrm{k}_{nn}e^{-s\tau_{nn}} \end{bmatrix}.$$

Then the zero solution of system (2.2) is Lyapunov globally asymptotically stable if all the roots of the characteristic equation $\det(\triangle(s)) = 0$ *have negative real parts.*

3 Model formulation

In [39], the bifurcation of a ratio-dependent delayed predator-prey system with two delays was considered. The mathematical model was formulated by

$$\begin{cases} \dfrac{dN(t)}{dt} = r_1 N(t) - \varepsilon P(t) N(t), \\ \dfrac{dP(t)}{dt} = P(t)\Big[r_2 - \theta \dfrac{P(t-\tau_2)}{N(t-\tau_1)}\Big], \end{cases} \tag{3.1}$$

where the variables and parameters of system (3.1) are explained in Table.1.

For the sake of succinctness, we assume that $\tau_1 = \tau_2 = \tau$ in system (3.1), then the following system can be derived

$$\begin{cases} \dfrac{dN(t)}{dt} = r_1 N(t) - \varepsilon P(t) N(t), \\ \dfrac{dP(t)}{dt} = P(t)\Big[r_2 - \theta \dfrac{P(t-\tau)}{N(t-\tau)}\Big]. \end{cases} \tag{3.2}$$

Table 1: **The instructions relevant variables and parameters of system (3.1)**

Variables(Parameters)	Description
$N(t)$	Population densities of prey at time t
$P(t)$	Population densities of predator at time t
$N(t-\tau_1)$	Juveniles of prey who was born at time $t-\tau_1$ and survive at time t
$P(t-\tau_2)$	juveniles of prey and predator who were born at time $t-\tau_2$ and survive at time t
r_1	Predation rate of the mature predator
r_2	Conversion factor from the mature prey to the immature predator
ε	Death rates of the immature prey
θ	Death rates of the mature prey

In this paper, we add the enhancing feedback controllers $K_1[N(t)-N(t-\tau)], K_2[P(t)-P(t-\tau)$ to the model (3.2), that is the following fractional-order version predator-prey model with feedback controllers

$$\begin{cases} D^{\phi}N(t) = r_1N(t) - \varepsilon P(t)N(t) + K_1[N(t) - N(t-\tau)], \\ D^{\phi}P(t) = P(t)\Big[r_2 - \theta\dfrac{P(t-\tau)}{N(t-\tau)}\Big] + K_2[P(t) - P(t-\tau)], \end{cases} \tag{3.3}$$

where ϕ is fractional order, K_1, K_2 denote feedback gains. It is easy to see that the enhancing feedback controllers preserves the equilibrium point of the system (3.3).

Noting that system (3.3) degenerates into the uncontrolled integer-order version system (3.2) when $\phi = 1$, $K_1 = K_2 = 0$. It is not difficult to see that the positive equilibrium point $E^* = (N^*, P^*)$ of system (3.3) is consistent with system (3.1) and (3.2), which can be acquired by solving the following equations:

$$\begin{cases} r_1 - \varepsilon P^* = 0, \\ r_2N^* - \theta P^* = 0. \end{cases}$$

It implies that $N^* = \dfrac{\theta r_1}{\varepsilon r_2}$, $P^* = \dfrac{r_1}{\varepsilon}$. Obviously, system (3.3) has a unique positive equilibrium point E^*.

To obtain the better control effects, we address the following essential assumption:

(**H1**) $K_1 \leq 0$, $K_2 \leq 0$.

The core objective of this paper is to discuss the problem of bifurcation control for system (3.3) by taking time delay as a bifurcation parameter and the approach in [38]. Then, some comparative investigations on bifurcation control are executed. It is found that the stability performance of the controlled system can be extremely improved by enhancing feedback control than the dislocated feedback control.

4 Main results

In this section, time delay shall be selected as a bifurcation parameter to investigate the problem of bifurcation control for the predator-prey model (3.3). The existence bifurcation and bifurcation point for the proposed model shall be established.

Carrying out a transformation $\rho(t)=N(t)-N^*$, $\varrho(t)=P(t)-P^*$, then the system (3.3) becomes

$$\begin{cases} D^{\phi}\rho(t)=r_1(\rho(t)+N^*)-\varepsilon(\varrho(t)+P^*)(\rho(t)+N^*)+K_1[\rho(t)-\rho(t-\tau)], \\ D^{\phi}\varrho(t)=(\varrho(t)+P^*)\Big[r_2-\theta\dfrac{\varrho(t-\tau)+P^*}{\rho(t-\tau)+N^*}\Big]+K_2[\varrho(t)-\varrho(t-\tau)]. \end{cases} \tag{4.1}$$

The linearized system of network (4.1) can be gained that

$$\begin{cases} D^{\phi}\rho(t)=-\varepsilon N^*\varrho(t)+K_1\rho(t)-K_1\rho(t-\tau), \\ D^{\phi}\varrho(t)=\theta\Big(\dfrac{P^*}{N^*}\Big)^2\rho(t-\tau)-\theta\dfrac{P^*}{N^*}\varrho(t-\tau)+K_2\varrho(t)-K_2\varrho(t-\tau). \end{cases} \tag{4.2}$$

The associated characteristic equation of (4.2) is

$$\det\begin{bmatrix} s^{\phi}-K_1+K_1e^{-s\tau} & \varepsilon N^* \\ -\theta\Big(\dfrac{P^*}{N^*}\Big)^2e^{-s\tau} & s^{\phi}-K_2+\Big(K_2+\theta\dfrac{P^*}{N^*}\Big)e^{-s\tau} \end{bmatrix}=0. \tag{4.3}$$

Based on Eq.(4.3), we conclude that

$$P_1(s)+P_2(s)e^{-s\tau}+P_3(s)e^{-2s\tau}=0, \tag{4.4}$$

where

$$\begin{aligned} &P_1(s)=s^{2\phi}-(K_1+K_2)s^{\phi}+K_1K_2, \\ &P_2(s)=(K_1+K_2+\alpha)s^{\phi}+\beta-2K_1K_2-\alpha K_1, \\ &P_3(s)=K_1(K_2+\alpha), \\ &\alpha=\theta\frac{P^*}{N^*}, \\ &\beta=\alpha\varepsilon P^*. \end{aligned}$$

Multiply $e^{s\tau}$ on both sides of Eq.(4.4), then we get that

$$P_1(s)e^{s\tau}+P_2(s)+P_3(s)e^{-s\tau}=0. \tag{4.5}$$

Assume that $s=\varpi(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})(\varpi>0)$ is a purely imaginary root of Eq.(4.5), then we have

$$\begin{cases} (A_1+A_3)\cos\varpi\tau+(B_3-B_1)\sin\varpi\tau=-A_2, \\ (B_1+B_3)\cos\varpi\tau+(A_1-A_3)\sin\varpi\tau=-B_2, \end{cases} \tag{4.6}$$

where A_i, $B_i(i=1,2,3)$ are the real parts and imaginary parts of $P_i(s)$. A_i, B_i can be described as follows:

$$\begin{aligned} &A_1=\varpi^{2\phi}\cos\phi\pi-(K_1+K_2)\varpi^{\phi}\cos\frac{\phi\pi}{2}+K_1K_2, \\ &B_1=\varpi^{2\phi}\sin\phi\pi-(K_1+K_2)\varpi^{\phi}\sin\frac{\phi\pi}{2}, \\ &A_2=(K_1+K_2+\alpha)\varpi^{\phi}\cos\frac{\phi\pi}{2}+\beta-2K_1K_2-\alpha K_1, \\ &B_2=(K_1+K_2+\alpha)\varpi^{\phi}\sin\frac{\phi\pi}{2}, \\ &A_3=K_1(K_2+\alpha), \\ &B_3=0. \end{aligned}$$

We further label

$$
\begin{aligned}
&F_1(\varpi) = -A_2(A_1 - A_3) - B_1B_2, \\
&F_2(\varpi) = -B_2(A_1 + A_3) + B_1A_2, \\
&G(\varpi) = A_1^2 + B_1^2 - A_3^2.
\end{aligned}
$$

It follows from Eq.(4.6) that

$$
\begin{cases}
\cos\varpi\tau = \dfrac{F_1(\varpi)}{G(\varpi)}, \\
\sin\varpi\tau = \dfrac{F_2(\varpi)}{G(\varpi)}.
\end{cases}
\tag{4.7}
$$

In terms of Eq.(4.7), we procure that

$$
G^2(\varpi) = F_1^2(\varpi) + F_2^2(\varpi). \tag{4.8}
$$

It can be defined from Eq.(4.8) that

$$
H(\varpi) = G^2(\varpi) - F_1^2(\varpi) - F_2^2(\varpi) = 0. \tag{4.9}
$$

Under Eq.(4.9), we can obtain that

$$
H(\varpi) = \varpi^{8\phi} + l_1\varpi^{7\phi} + l_2\varpi^{6\phi} + l_3\varpi^{5\phi} + l_4\varpi^{4\phi} + l_5\varpi^{3\phi} + l_6\varpi^{2\phi} + l_7\varpi^{\phi} + l_8 = 0, \tag{4.10}
$$

where $l_i (i = 1, 2, \dots, 8)$ are computed in Appendix $A - B$.

In order to guarantee the occurrence of Hopf bifurcation for system (3.3), we further give the additional assumption:

(**H2**) Eq.(4.10) has at least positive real roots.

It should be noted that the assumption (**H2**) is only a necessary condition for the bifurcation of the system (3.3), not a sufficient condition.

According to $\cos\varpi\tau = \dfrac{F_1(\varpi)}{G(\varpi)}$, we can get

$$
\tau^{(k)} = \frac{1}{\varpi}\Big[\arccos\frac{F_1(\varpi)}{G(\varpi)} + 2k\pi\Big], \quad k = 0, 1, 2, \dots. \tag{4.11}
$$

Define the bifurcation point

$$
\tau_0 = \min\{\tau^{(k)}\}, \quad k = 0, 1, 2, \dots,
$$

where $\tau^{(k)}$ is defined by Eq.(4.11).

In what follows, we will consider the stability of system (3.3) when $\tau = 0$. If τ is removed, the characteristic Eq. (4.4) becomes

$$
\lambda^{2\phi} + \alpha\lambda^{\phi} + \beta = 0. \tag{4.12}
$$

It is obvious from $\alpha > 0$, $\beta > 0$ that the two roots of Eq.(4.12) have negative parts which satisfying Lemma 2.1. Hence, the positive equilibrium of the fractional system (3.3) is asymptotically stable.

In order to acquire the transversality condition of the occurrence for Hopf bifurcation, the following necessary assumption is needed for system (3.3):

(**H3**) $\dfrac{\chi_1 \upsilon_1 + \chi_2 \upsilon_2}{\upsilon_1^2 + \upsilon_2^2} \neq 0,$

where χ_1, χ_2, υ_1, υ_2 are defined by Eq.(4.15).

Lemma 4.1 *Let $s(\tau) = \xi(\tau) + i\varpi(\tau)$ be the root of Eq.(4.4) near $\tau = \tau_j$ satisfying $\xi(\tau_j) = 0$, $\varpi(\tau_j) = \varpi_0$, then the following transversality condition holds*

$$\mathrm{Re}\Big[\frac{ds}{d\tau}\Big]\Big|_{(\varpi=\varpi_0,\tau=\tau_0)} \neq 0,$$

where ϖ_0, τ_0 represent the critical frequency and bifurcation point of system (3.3).

Proof: By using implicit function theorem and differentiating (4.4) with respect to τ, we can get

$$P_1'(s)\frac{ds}{d\tau} + \Big[P_2'(s)\frac{ds}{d\tau}e^{-s\tau} + P_2(s)e^{-s\tau}\Big(-\tau\frac{ds}{d\tau} - s\Big)\Big] + \Big[P_3'(s)\frac{ds}{d\tau}e^{-2s\tau} + 2P_3(s)e^{-2s\tau}\Big(-\tau\frac{ds}{d\tau} - s\Big)\Big] = 0. \tag{4.13}$$

It is clear from Eq.(4.12) that $P_3'(s) = 0$. By mathematical derivation, it follows from Eq.(4.13) that

$$\frac{d\varsigma}{d\tau} = \frac{\chi(s)}{\upsilon(s)}, \tag{4.14}$$

where

$$\begin{aligned}
\chi(s) &= s[P_2(s)e^{-s\tau} + 2P_3(s)e^{-2s\tau}],\\
\upsilon(s) &= P_1'(2) + [P_2'(s) - \tau P_2(s)]e^{-s\tau} - 2\tau P_3(s)e^{-2s\tau}.
\end{aligned}$$

Let P_i^R, P_i^I stand for the real parts and the imaginary parts of $P_i(s)$. Let $P_i^{'R}$, $P_i^{'I}$ denote the real parts and the imaginary parts of $P_i'(s)$. Then by some computation, it can be deduced from (4.14) that

$$\mathrm{Re}\Big[\frac{ds}{d\tau}\Big]\Big|_{(\varpi=\varpi_0,\tau=\tau_0)} = \frac{\chi_1\upsilon_1 + \chi_2\upsilon_2}{\upsilon_1^2 + \upsilon_2^2}, \tag{4.15}$$

where

$$\begin{aligned}
\chi_1 &= \varpi_0(P_2^R \sin\varpi_0\tau_0 - P_2^I\cos\varpi_0\tau_0 + 2P_3^R\sin 2\varpi_0\tau_0 - 2P_3^I\cos 2\varpi_0\tau_0),\\
\chi_2 &= \varpi_0(P_2^R \cos\varpi_0\tau_0 + P_2^I\sin\varpi_0\tau_0 + 2P_3^R\cos 2\varpi_0\tau_0 + 2P_3^I\sin 2\varpi_0\tau_0),\\
\upsilon_1 &= P_1^{'R} + (P_2^{'R} - \tau_0 P_2^R)\cos\varpi_0\tau_0 + (P_2^{'I} - \tau_0 P_2^I)\sin\varpi_0\tau_0,\\
&\quad - 2\tau_0(P_3^R\cos 2\varpi_0\tau_0 + P_3^I\sin 2\varpi_0\tau_0),\\
\upsilon_2 &= P_1^{'I} + (P_2^{'I} - \tau_0 P_2^I)\cos\varpi_0\tau_0 - (P_2^{'R} - \tau_0 P_2^R)\sin\varpi_0\tau_0,\\
&\quad - 2\tau_0(P_3^I\cos 2\varpi_0\tau_0 - P_3^R\sin 2\varpi_0\tau_0).
\end{aligned}$$

(**H3**) indicates that transversality condition hold. We accomplish the proof of Lemma 4.1.

Based on the assumptions (**H1**)-(**H3**) and previous analysis, the following theorem can be derived.

Theorem 4.1 *Under (**H1**)-(**H3**), the following results hold:*

1) E^* of the fractional system (3.3) is asymptotically stable when $\tau \in [0, \tau_0)$;

2) System (3.3) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$, i.e., it has a branch of periodic solutions bifurcating from E^* near $\tau = \tau_0$.

Remark 4.1 *It is difficult to theoretically analyze all the positive real roots. Nevertheless, these positive real roots of Eq.(4.10) can be easily computed by using Maple numerical software. Hence, the critical frequency ϖ_0 and bifurcation point τ_0 can be accurately established.*

Remark 4.2 *Some analogous models were analyze in [40, 41, 42, 43]. It is worth mentioning that these results only concentrated on the dynamics of integer-order predator-prey models. It is more realistic to explore the dynamics of delayed predator-prey models by fully considering the effects of fractional calculus for ecosystems in this paper.*

Remark 4.3 *In this paper, the effects of fractional order on the bifurcation point are adequately discussed by calculation. It implies that the better effects in delaying the onset of bifurcation can be achieved as fractional order decreases if feedback gain are established.*

Remark 4.4 *In [28, 29, 30, 31, 32], various bifurcation strategies were adopted to control the onset of bifurcation for delayed fractional-order systems. Noting that these remarkable results only were obtained all based on the dislocated feedback approaches. Different from existing methods, the enhancing feedback control strategy is delay onset of the bifurcation for fractional delayed predator-prey system and satisfactory bifurcation control effects are realized compared with the dislocated feedback approaches in this paper. This hints that the proposed enhancing controllers possess a superior performance in controlling bifurcation in delayed fractional-order systems. The derived results can be extended to deal with others fractional-order systems with time delay.*

5 Numerical simulations

In this section, a simulation example is exploited to exhibit the correctness of the addressed theory. In our simulations, Adama-Bashforth-Moulton predictor-corrector scheme is adopted in [44]. For convenience of comparison, the uniform parameters are taken from [39]: $r_1 = 0.45$, $r_2 = 0.1$, $\varepsilon = 0.03$, $\theta = 0.05$. The positive equilibrium point E^* can be obtained as $(N^*, P^*) = (7.5, 15)$. Step-length is chosen as $h = 0.01$, and the initial values are taken as $(N(0), P(0)) = (8, 16)$. Consider the controlled system

$$\begin{cases} D^{\phi}N(t) = 0.45N(t) - 0.03P(t)N(t) + K_1[(N(t) - N(t-\tau)], \\ D^{\phi}P(t) = P(t)\Big[0.1 - 0.05\dfrac{P(t-\tau)}{N(t-\tau)}\Big] + K_2[(P(t) - P(t-\tau)]. \end{cases} \tag{5.1}$$

Selecting $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$ in system (5.1), it is derived that $\varpi_0 = 0.1496$, then $\tau_0 = 4.4222$. In terms of Theorem 4.1, E^* of controlled system (5.1) is asymptotically stable when $\tau = 3.8 < \tau_0$, which, depicted in Figs.1-2, while Figs.3-4 display that E^* of controlled system (5.1) is unstable, Hopf bifurcation occurs when $\tau = 4.8 > \tau_0$. Bifurcation diagrams of system (5.1) are simulated in Figs.5-6.

The same order is chosen as $\phi = 0.98$. We first select $K_1 = K_2 = 0$, which means that the controllers are removed, we derive $\tau_0 = 2.3807$. We further choose $\tau = 3.8 > \tau_0 = 2.3807$, it is clear that system (5.1) becomes unstable, which is depicted in Figs.7-8. Then we choose $K_2 = 0$, $K_1 = -0.08$. This indicates that dislocated feedback control emerges, then we have $\tau_0 = 3.0079$. We choose $\tau = 3.8 > \tau_0 = 3.0079$, it is obvious that system (5.1) becomes unstable, which is simulated in Figs.9-10. If $K_1 = 0$, $K_2 = -0.15$, it suggests that dislocated feedback control occurs, then we have $\tau_0 = 2.8650$. We also choose $\tau = 3.8 > \tau_0 = 2.8650$, it is obvious that system (5.1) becomes unstable, which is simulated in Figs.11-12.

In brief, system (5.1) will turn unstable once the controllers all are removed or dislocated feedback controller engenders.

In what follows, we shall fully consider the effects of the proposed enhancing control scheme.

Case 1. Selecting three sets of parameters $K_1 = -0.08$, $K_2 = -0.15$; $K_1 = -0.08$, $K_2 = -0.15$; $K_1 = -0.08$, $K_2 = 0$; $K_1 = 0$, $K_2 = -0.15$, respectively. By varying ϕ, we derive the values of τ_0, the comparative results are addressed in Figs.13-15. Fig.13 describes that the bifurcation point is more larger with $K_1 = -0.08$, $K_2 = -0.15$ than that one with $K_1 = K_2 = 0$ for given ϕ. This verifies that the effectiveness of the devised controllers. Figs.14-15 reveal that the performance of designed enhancing controllers overmatch the single controller.

Case 2. Fixing $\phi = 0.98$ and select two sets of parameters $K_2 = -0.15$ and $K_2 = 0$, then we derive the values of τ_0 with the change of K_1, which is demonstrated in Fig.16. Fig.16 indicates that the control effects are more better with the present of feedback gain K_2 than the absence of it.

Case 3. Taking $\phi = 0.98$ and select two sets of parameters $K_1 = -0.08$ and $K_1 = 0$, then we derive the values of τ_0 with the change of K_2, which is demonstrated in Fig.17. Fig.17 discloses that the control effects are more better with the present of feedback gain K_1 than the absence of it.

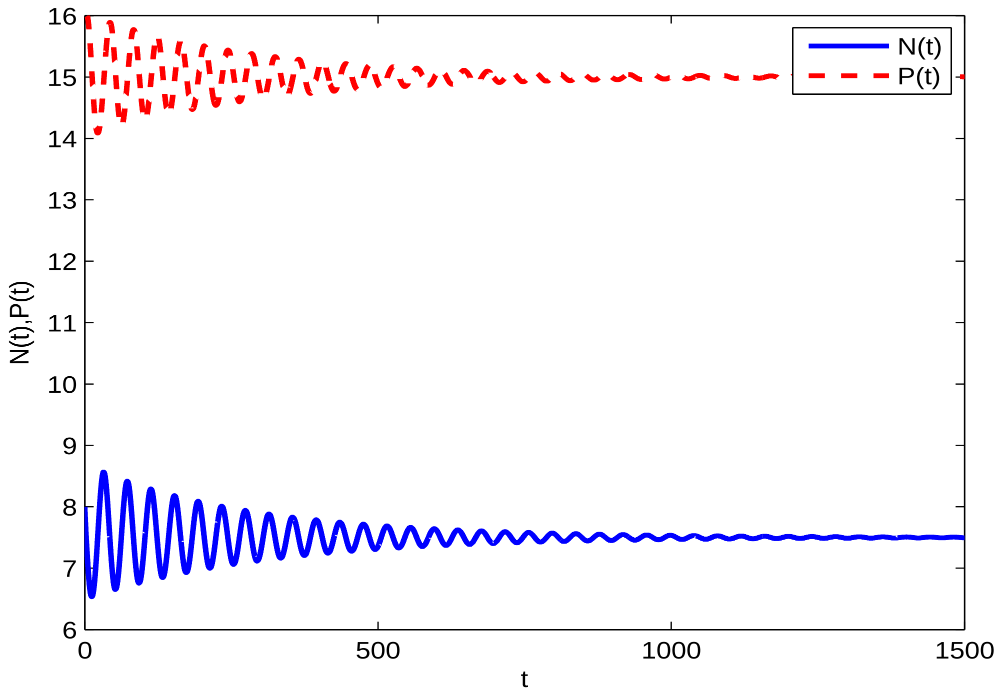

Figure 1: Time series of system (5.1) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 3.8 < \tau_0 = 4.4222$.

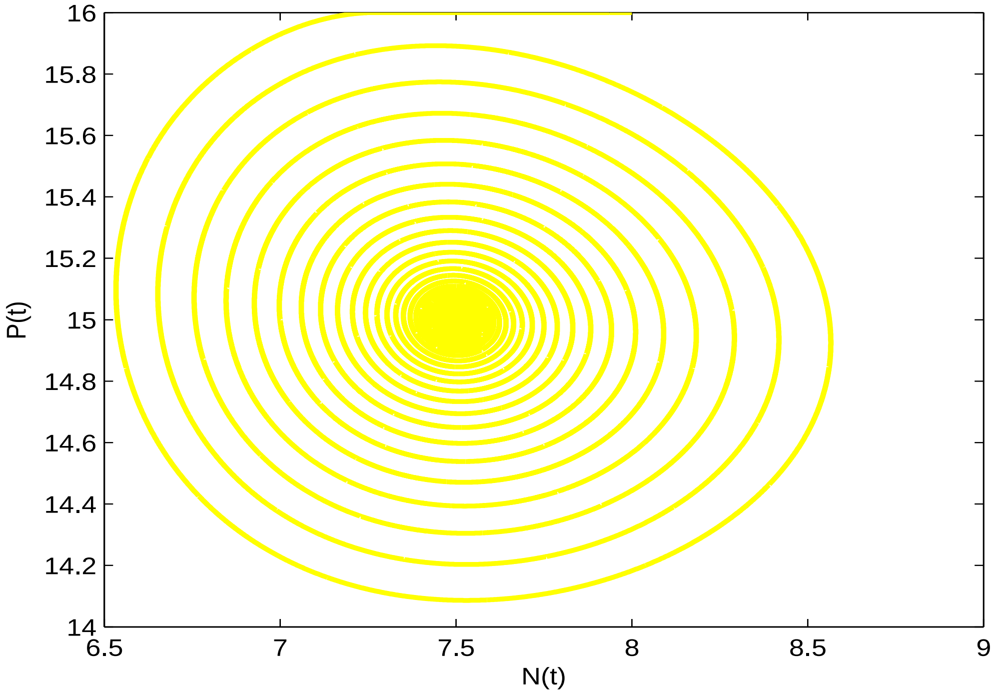

Figure 2: Portrait diagram of system (5.1) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 3.8 < \tau_0 = 4.4222$.

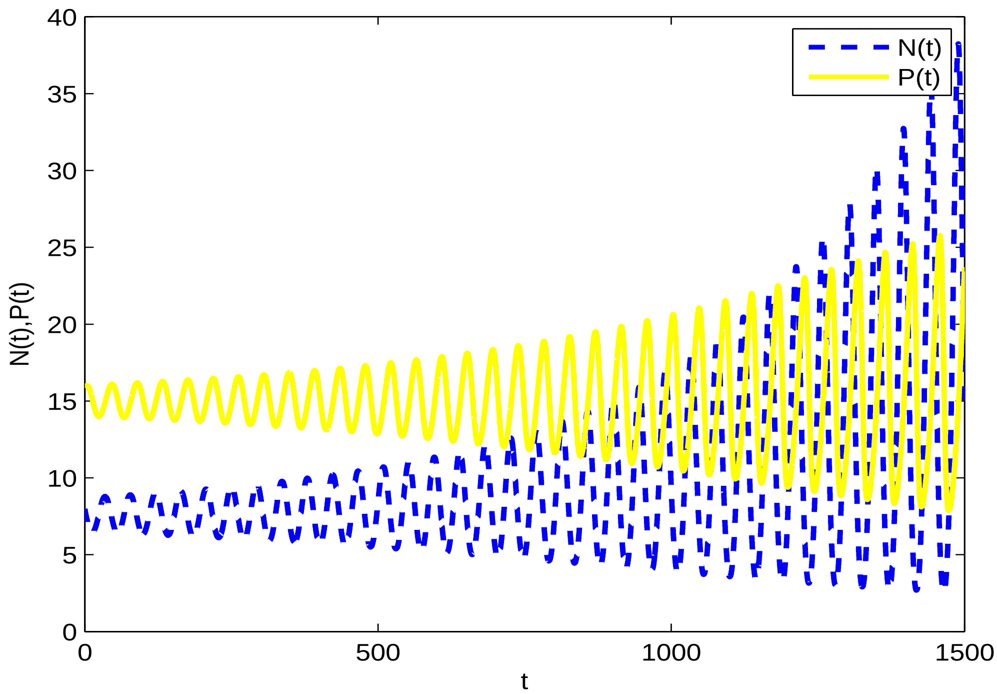

Figure 3: Time series of system (5.1) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 4.8 > \tau_0 = 4.4222$.

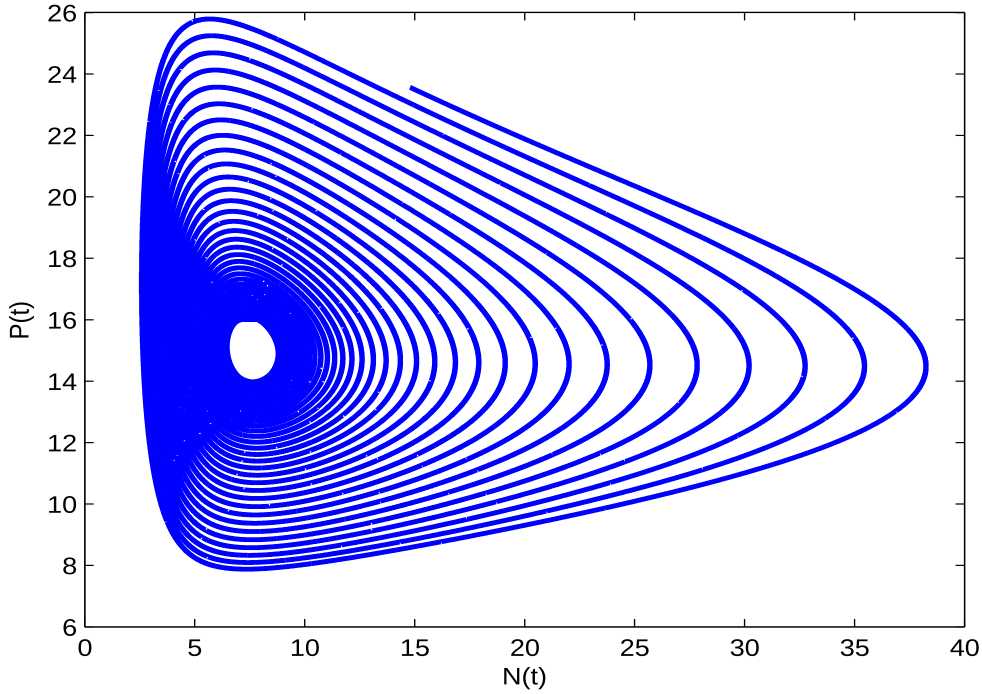

Figure 4: Portrait diagram of system (5.1) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = -0.15$, $\tau = 4.8 > \tau_0 = 4.4222$.

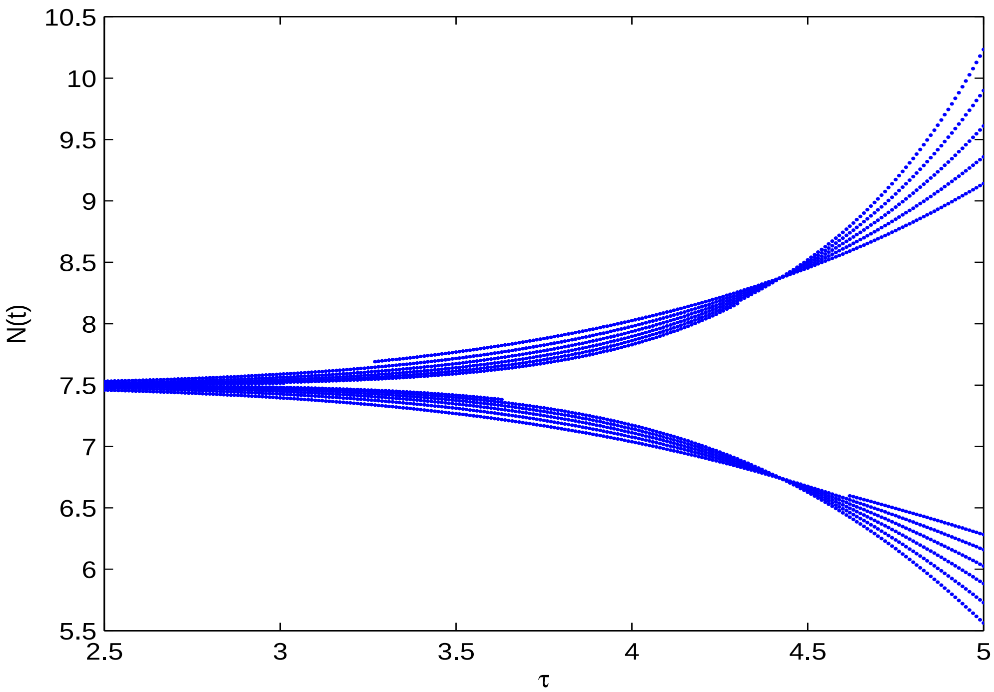

Figure 5: Bifurcation diagram of $N(t)$ for system (5.1).

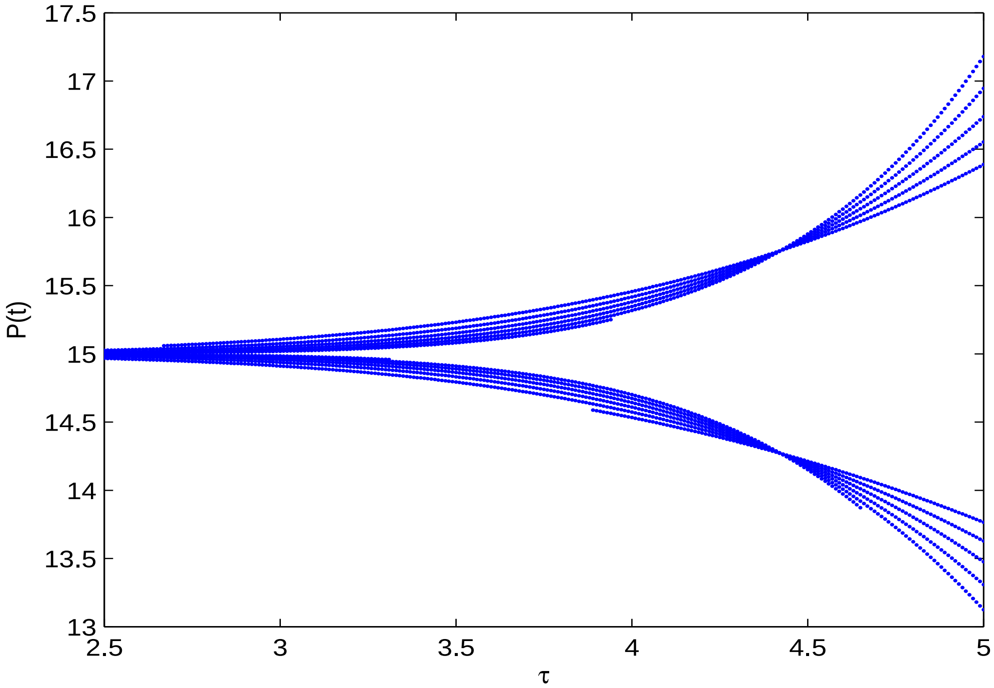

Figure 6: Bifurcation diagram of $P(t)$ for system (5.1).

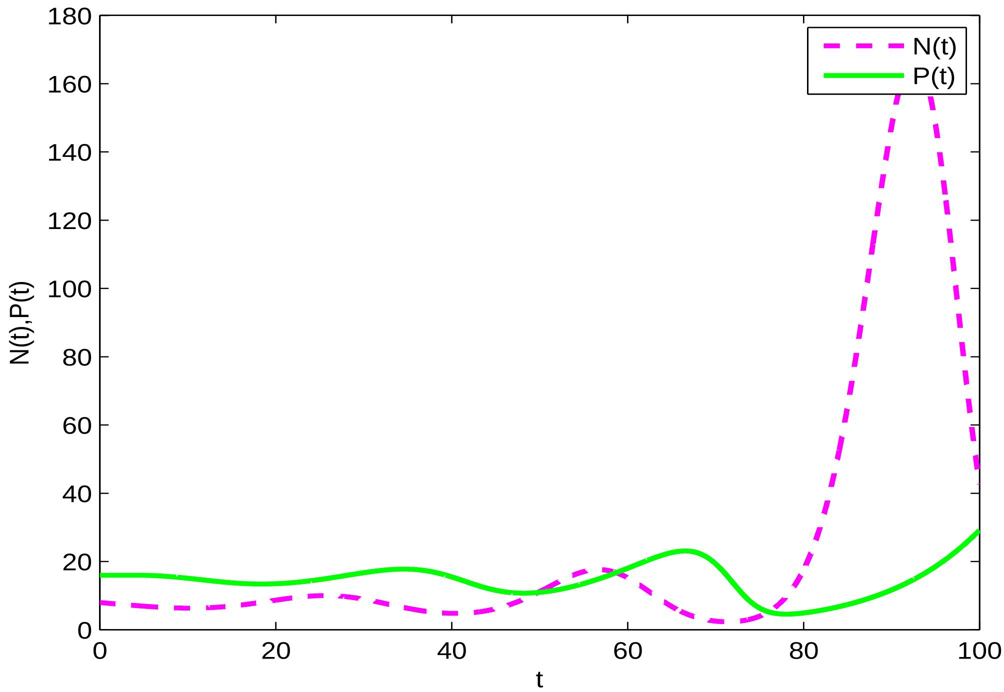

Figure 7: Time series of system (5.1) with $\phi = 0.98$, $K_1 = K_2 = 0$, $\tau = 3.8$.

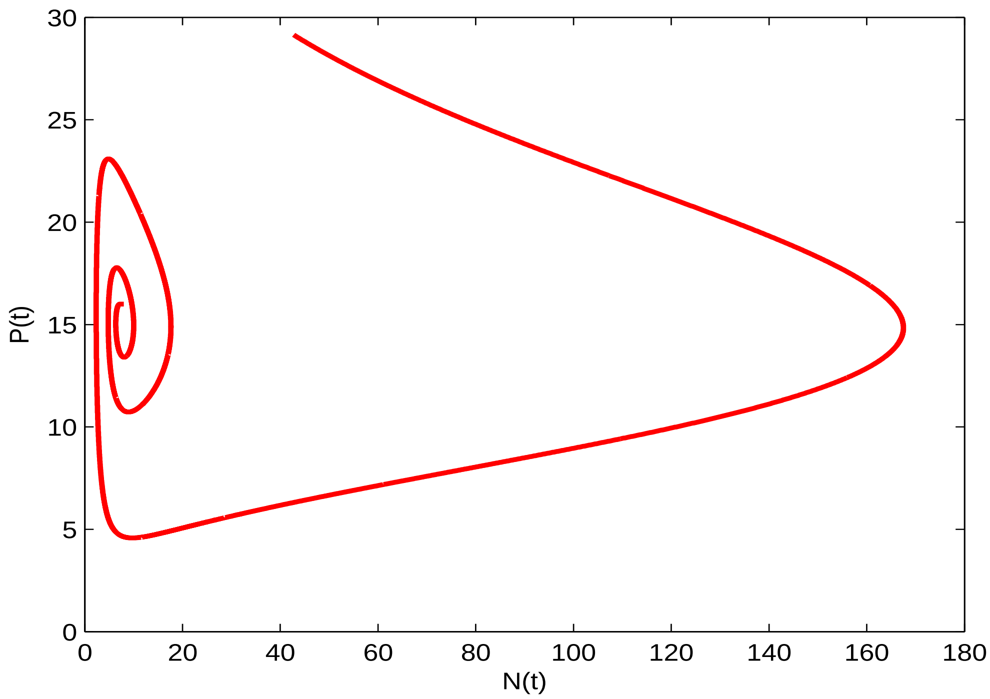

Figure 8: Portrait diagram of system (5.1) with $\phi = 0.98$, $K_1 = K_2 = 0$, $\tau = 3.8$.

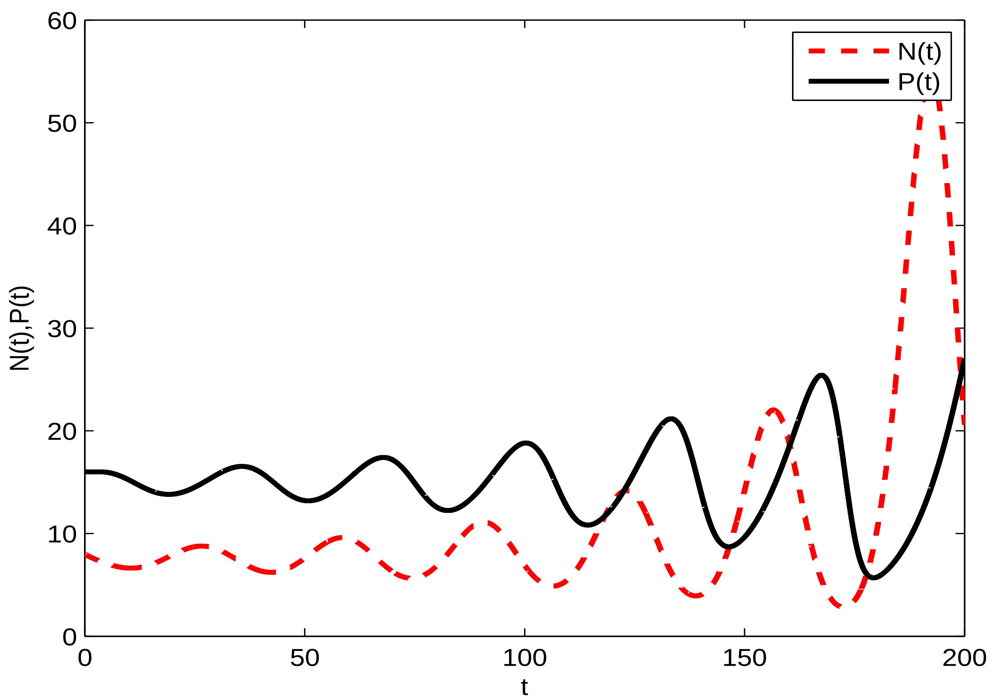

Figure 9: Time series of system (5.1) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = 0$, $\tau = 3.8$.

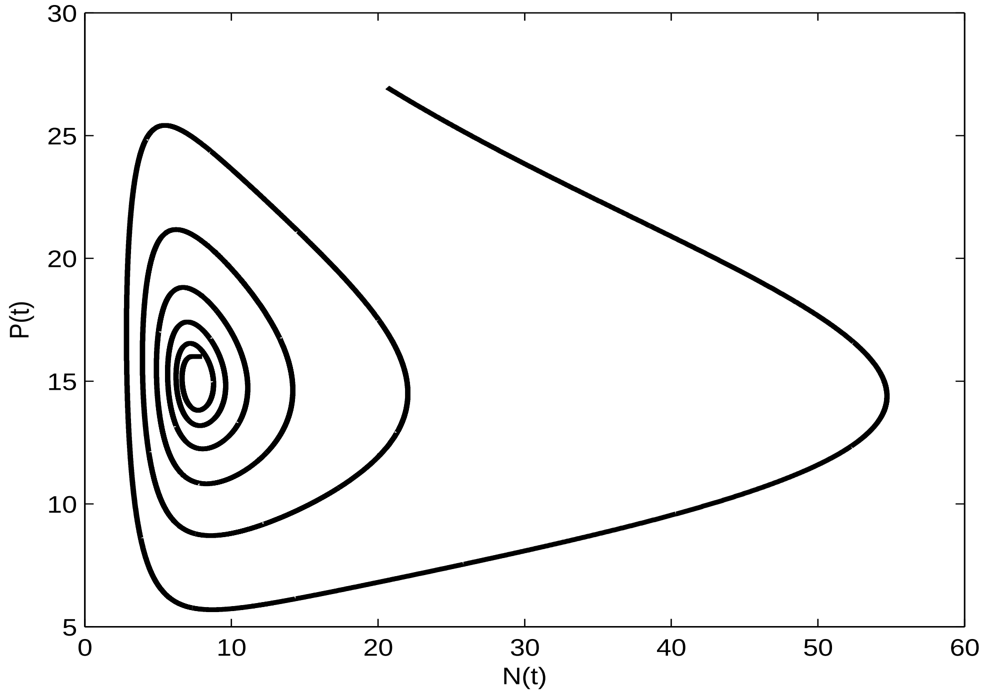

Figure 10: Portrait diagram of system (5.1) with $\phi = 0.98$, $K_1 = -0.08$, $K_2 = 0$, $\tau = 3.8$.

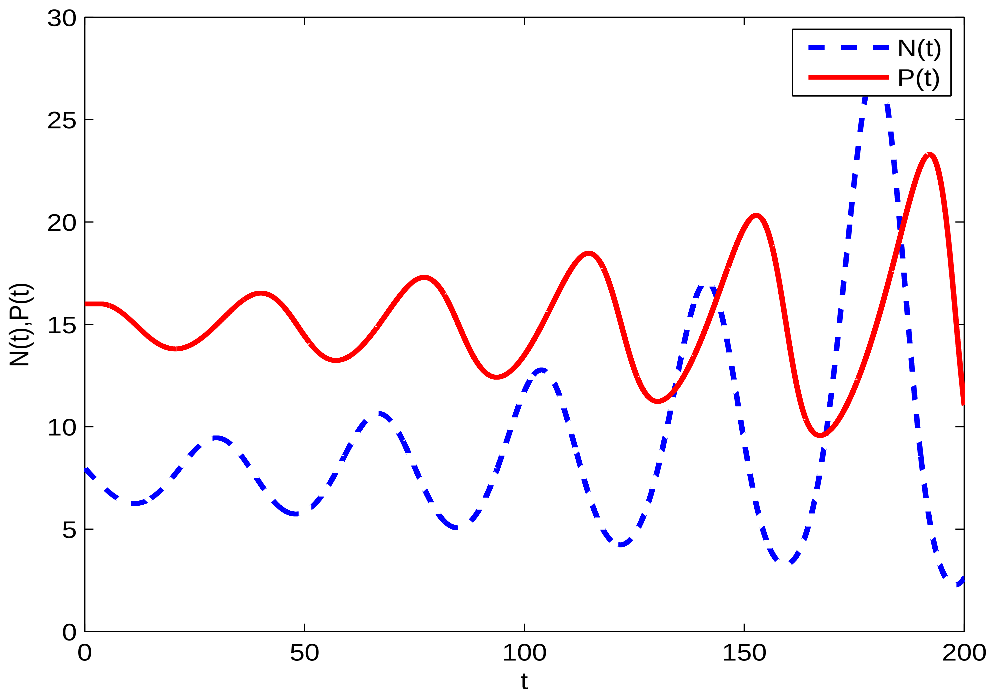

Figure 11: Time series of system (5.1) with $\phi = 0.98$, $K_1 = 0$, $K_2 = -0.15$, $\tau = 3.8$.

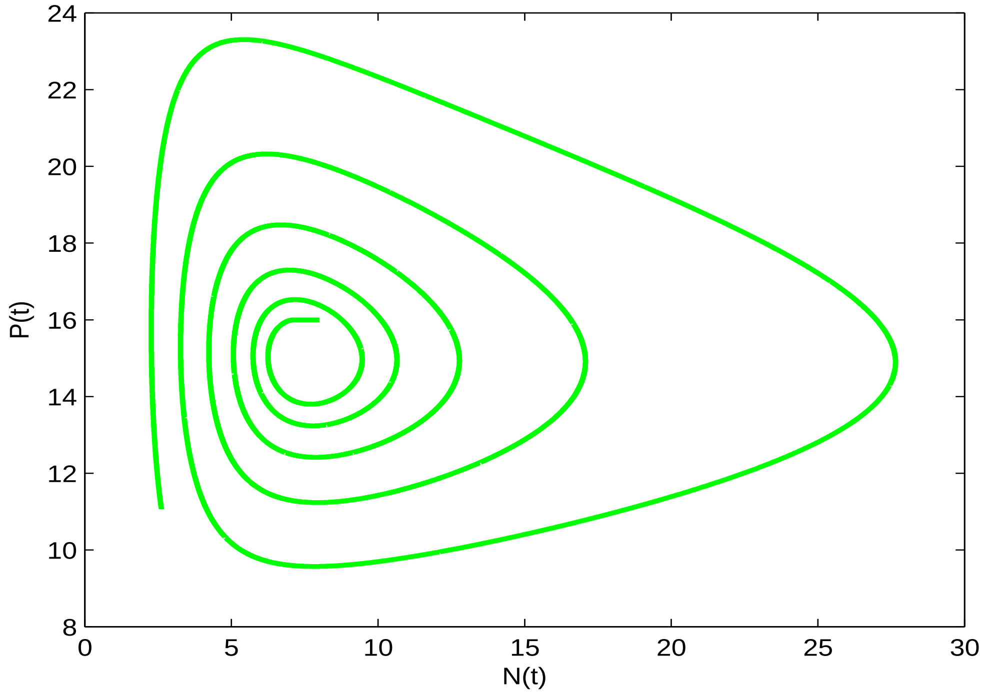

Figure 12: Portrait diagram of system (5.1) with $\phi = 0.98$, $K_1 = 0$, $K_2 = -0.15$, $\tau = 3.8$.

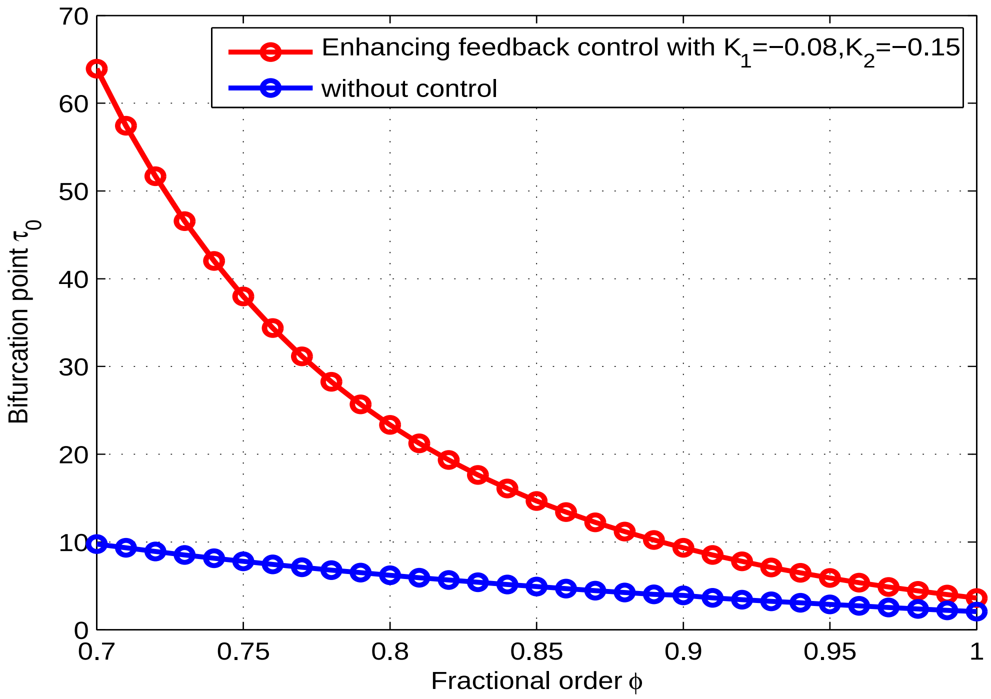

Figure 13: Comparison on the values of τ_0 versus ϕ for system (5.1).

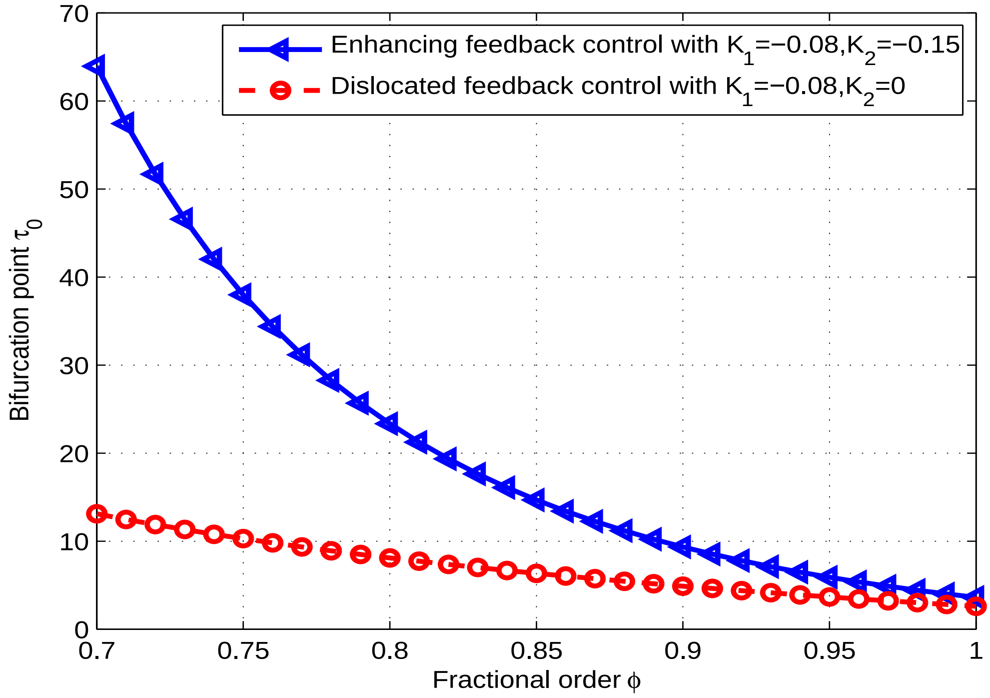

Figure 14: Comparison on the values of τ_0 versus ϕ for system (5.1).

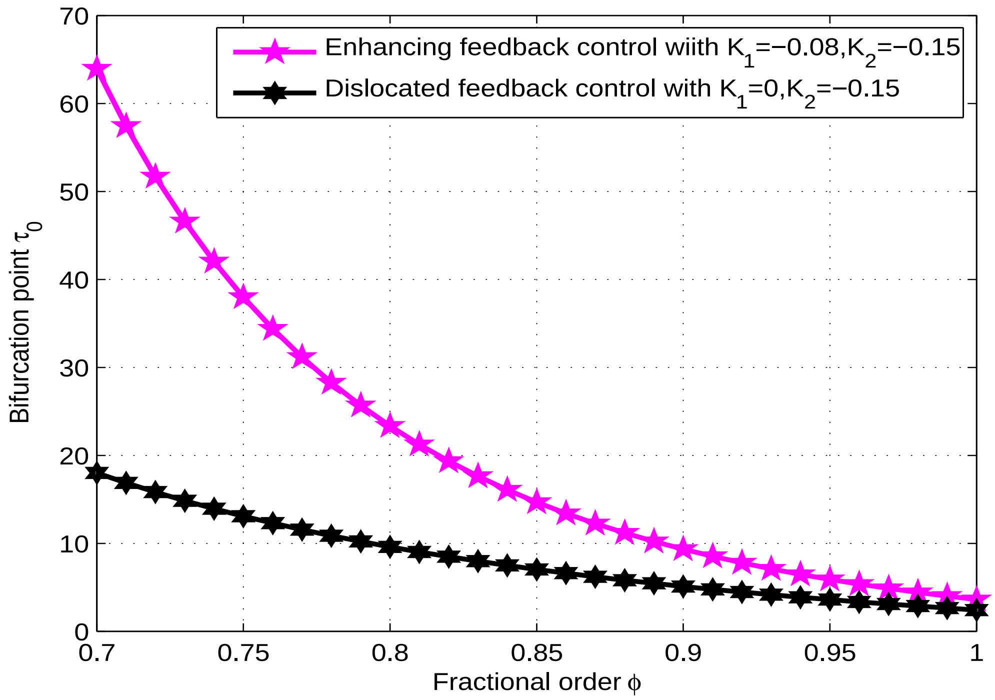

Figure 15: Comparison on the values of τ_0 versus ϕ for system (5.1).

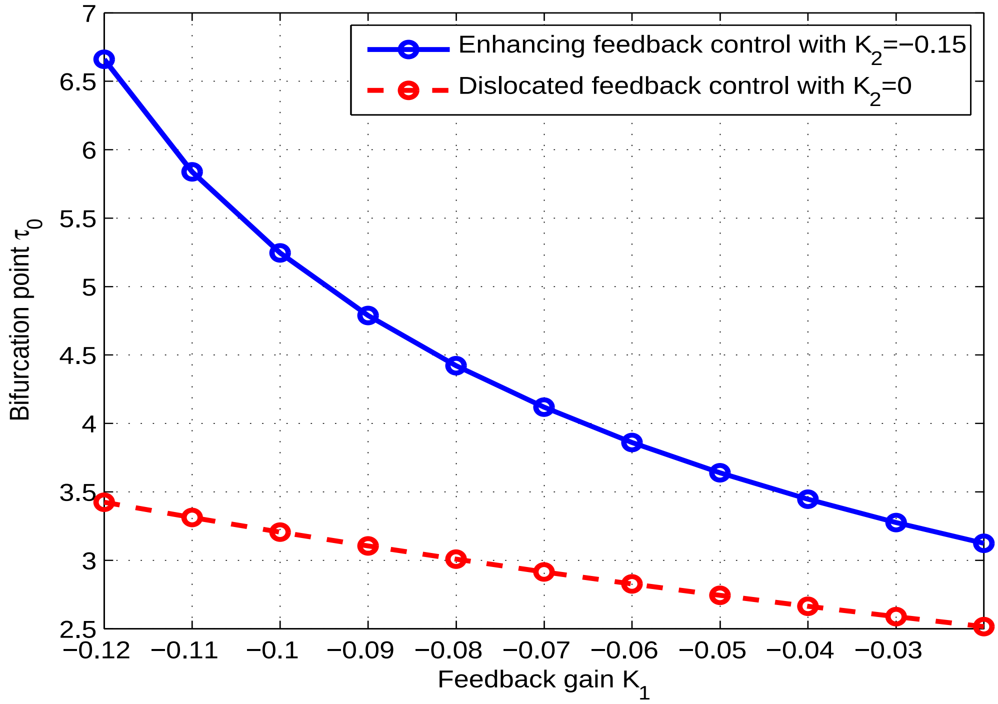

Figure 16: Comparison on the values of τ_0 versus K_1 for system (5.1) with $\phi = 0.98$.

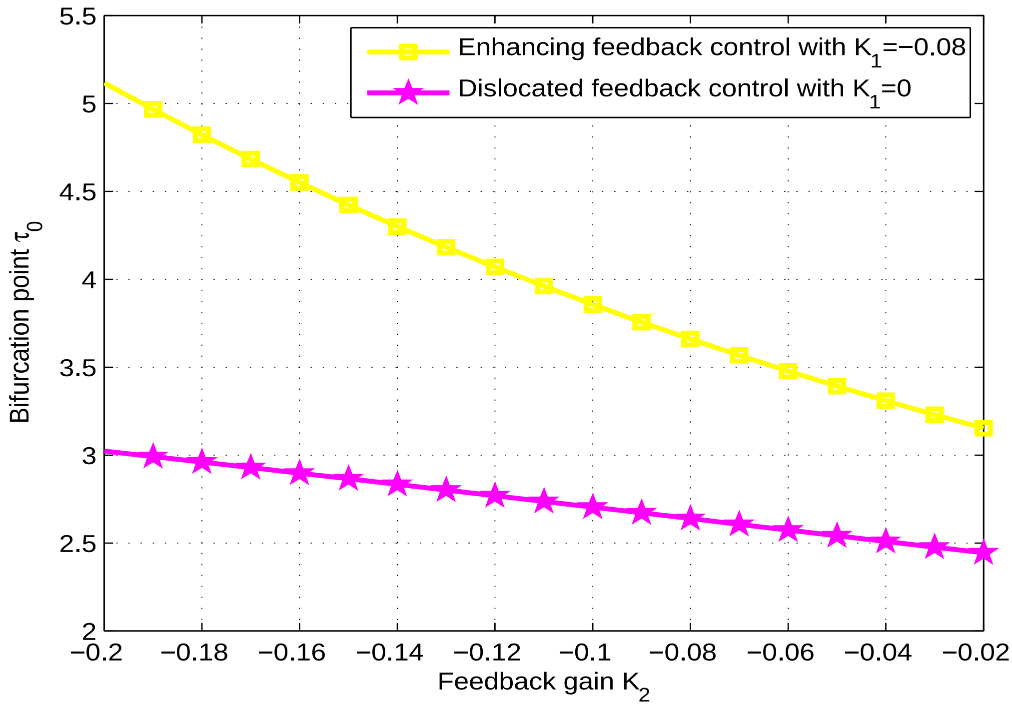

Figure 17: Comparison on the values of τ_0 versus K_2 for system (5.1) with $\phi = 0.98$.

6 Conclusion

In this paper, we address a theoretical analysis on bifurcation control for a delayed fractional-order predator-prey model by taking advantage of enhancing feedback control technique. An enhancing feedback control strategy is firstly developed to deal with the bifurcation control in a fractional delayed predator-prey model. This hints that the proposed enhancing controllers possess a superior performance in controlling bifurcation in delayed fractional-order systems. Then the bifurcation point of the controlled model can be completely concluded by theoretical derivation. The effects of fractional order on the bifurcation points are fully investigated by using enhancing feedback control strategy and dislocated feedback. It is found that the performance of control gradually becomes perfect with the decrement of fractional order. It implies that the better effects in delaying the onset of bifurcation can be achieved as fractional order decreases if feedback gain are established. We discover that enhancing feedback control strategy overmatches dislocated feedback ones in delaying the onset of bifurcation control for the considered controlled system for given fractional order. Finally, numerical simulations validate the efficiency of the derived theoretical results.

Acknowledgements

The work was supported by the National Natural Science Foundation of China (Grant No. 61573194), the Natural Science Foundation of Jiangsu Province of China (No. BK20181389) and the Natural Science Foundation of Jiangsu Higher Education Institutions of China (Grant No. 19KJB110017).

Appendix A

$$
\begin{aligned}
l_1 &= -4(K_1+K_2)\cos\frac{\phi\pi}{2},\\
l_2 &= (K_1^2+K_2^2-\alpha^2-2K_1K_2-2\alpha K_1-2\alpha K_2)(\sin^2\phi\pi\cos^2\frac{\phi\pi}{2}+\cos^2\phi\pi\sin^2\frac{\phi\pi}{2})\\
&\quad+(5K_1^2+5K_2^2-\alpha^2+10K_1K_2-2\alpha K_1-2\alpha K_2)(\sin^2\phi\pi\sin^2\frac{\phi\pi}{2}+\cos^2\phi\pi\cos^2\frac{\phi\pi}{2})\\
&\quad+2(K_1+K_2)^2\sin\phi\pi\sin 2\phi\pi+4K_1K_2(\cos\phi\pi\sin^2\phi\pi+\cos^3\phi\pi\Big),\\
l_3 &= (-6K_1^2K_2-6K_1K_2^2+2\alpha^2K_1+2\alpha^2K_2+4\alpha K_1^2+4\alpha K_2^2+8\alpha K_1K_2-2K_1^3-2K_2^3)\\
&\quad\cdot(\sin\phi\pi\sin^3\frac{\phi\pi}{2}+\cos\phi\pi\cos^3\frac{\phi\pi}{2})+(2\alpha^2K_1-2\beta K_1-2\beta K_2-2\alpha\beta-8K_2^2K_1-8K_1^2K_2\\
&\quad+2\alpha K_1^2+6\alpha K_1K_2)\cos^2\phi\pi\cos\frac{\phi\pi}{2}+(-2\beta K_1-2\beta K_2-2\alpha\beta+2\alpha^2K_1+2\alpha K_1^2+6\alpha K_1\\
&\quad\cdot K_2)\sin^2\phi\pi\cos\frac{\phi\pi}{2}+(-2K_1^3-2K_2^3+2\alpha^2K_1+2\alpha^2K_2+4\alpha K_1^2-6K_1^2K_2-6K_1K_2^2\\
&\quad+4\alpha K_2^2+8\alpha K_1K_2)(\cos\phi\pi\cos\frac{\phi\pi}{2}\sin^2\frac{\phi\pi}{2}+\sin\phi\pi\sin\frac{\phi\pi}{2}\cos^2\frac{\phi\pi}{2})-8K_1K_2(K_1+K_2)\\
&\quad\cdot\sin\phi\pi\cos\phi\pi\sin\frac{\phi\pi}{2},\\
l_4 &= -\alpha(2\alpha K_1K_2+6K_1^2K_2+6K_2^2K_1+2K_1^3+2K_2^3+\alpha K_2^2+\alpha K_1^2)(\sin^4\frac{\phi\pi}{2}+\cos^4\frac{\phi\pi}{2})\\
&\quad+(4\beta K_1K_2+2\alpha\beta K_1-8\alpha K_1^2K_2-4K_1^2K_2^2-\beta^2-3\alpha^2K_1^2)\sin^2\phi\pi+(2\alpha\beta K_1-8\alpha K_1^2K_2\\
&\quad+4\beta K_1K_2-\beta^2-3\alpha^2K_1^2)\cos^2\phi\pi+(4K_2^2K_1+4\beta K_2^2+4\beta K_1^2+2\alpha^3K_1-10\alpha K_2^2K_1\\
&\quad+8\beta K_1K_2+4\alpha\beta K_2+4\alpha\beta K_2+4K_1^3K_2+8K_1^2K_2^2-2\alpha K_1^3-12\alpha K_1^3K_2)\cos\phi\pi\cos^2\frac{\phi\pi}{2}\\
&\quad-\alpha(2\alpha K_2^2+2\alpha K_1^2+4K_2^3+4K_1^3+4\alpha K_1K_2+12K_2^2K_1+12K_1^2K_2)\cos^2\frac{\phi\pi}{2}\sin^2\frac{\phi\pi}{2}\\
&\quad-(2\alpha^3K_1+2\alpha K_1^3+4K_1^2+8\alpha^2K_1K_2+10K_1K_2^2+12\alpha K_1^2K_2)\cos\phi\pi\sin^2\frac{\phi\pi}{2}+4(\beta K_1^2\\
&\quad+\beta K_2^2+K_2^2K_1+\alpha^2K_1+\alpha\beta K_1+\alpha\beta K_2+\alpha^2K_1^2+2\beta K_1K_2+2\alpha^2K_1K_2+2K_1^2K_2^2)\\
&\quad\cdot\sin\phi\pi\sin\frac{\phi\pi}{2}\cos\frac{\phi\pi}{2}+4K_1^3K_2\sin\phi\pi\sin\frac{\phi\pi}{2}.
\end{aligned}
$$

Appendix B

$$
\begin{aligned}
l_5 = {} & (-4\alpha^2 K_1^2 K_2 - 4\alpha\beta K_1 K_2 + 4\alpha K_1^3 K_2 + 8\alpha K_1^2 K_2^2 + 4\alpha K_2^3 K_1 - 2\alpha^2 K_1^3 - 6\beta K_1^2 K_2 - 2\alpha^3 K_1^2 \\
& - 2\alpha\beta K_1^2 - 2\alpha^3 K_1 K_2 - 2\alpha\beta K_2^2 - 2\alpha^2 K_2^2 K_1 - 6\beta K_2^2 K_1 - 2\beta K_2^3 - 2\beta K_1^3)\cos^3\frac{\phi\pi}{2} + (4\alpha K_2^3 K_1 \\
& - 2\alpha^2 K_1^3 - 6\beta K_1^2 K_2 - 2\alpha^3 K_1 - 2\alpha\beta K_1^2 - 2\alpha\beta K_2^2 - 2\alpha^2 K_2^2 K_1 - 6\beta K_2^2 K_1 - 2\beta K_2^3 - 2\beta K_1^3 \\
& - 4\alpha\beta K_1 K_2 + 8\alpha K_1^2 K_2^2 + 4\alpha K_1^3 K_2 - 4\alpha^2 K_1^2 K_2 - 2\alpha^3 K_1 K_2)\cos\frac{\phi\pi}{2}\sin^2\frac{\phi\pi}{2} + (2\alpha^2 K_1^3 \\
& - 4\alpha K_1^2 K_2^2 - 10\alpha^2 K_2 K_1^2 - 4\beta K_1 K_2^2 + 4\alpha^2\beta K_1 + 2\beta^2 K_1 + 2\beta^2 K_2 - 4\alpha^3 K_1^2 + 4\alpha\beta K_1 K_2 \\
& - 4\beta K_1^2 K_2 + 4\alpha K_1^3 K_2)\sin\phi\pi\sin\frac{\phi\pi}{2} + (-8\beta K_2^2 K_1 + 4\alpha^2\beta K_1 - 6\alpha^2 K_2 K_1^2 + 8\alpha K_1^2 K_2^2 \\
& + 2\alpha^2 K_1^3 + 2\beta^2 K_1 + 2\beta^2 K_2 - 4\alpha^3 K_1^2 + 8\alpha K_1^3 K_2 - 8\beta K_1^2 K_2)\cos\phi\pi\cos\frac{\phi\pi}{2}, \\
l_6 = {} & (4\alpha^2 K_1^3 K_2 - 2\alpha\beta K_1^3 - 6\alpha\beta K_1 K_2^2 - 4\alpha^2\beta K_1^2 - 4\alpha^2\beta K_1 K_2 + 2\alpha^3 K_1^3 - 2\alpha^3 K_2 K_1^2 - 2\beta^2 K_1 K_2 \\
& - \beta^2 K_1^2 - \alpha^4 K_1^2 - \beta^2 K_2^2 - 8\alpha\beta K_1^2 K_2)\sin^2\frac{\phi\pi}{2} + (-2\alpha\beta K_2^2 K_1 - 4\alpha^2\beta K_1 K_2 - 2\alpha\beta K_1^3 - 4\alpha^2\beta K_1^2 \\
& + 8\alpha^2 K_1^3 K_2 + 4\beta K_1^3 K_2 + 8K_1^2 K_2^2 + 8\alpha^2 K_1^2 K_2^2 + 4\beta K_2^3 K_1 + 2\alpha^3 K_1^3 + 2\alpha^3 K_2 K_1^2 - 2\beta^2 K_1 K_2 \\
& - \beta^2 K_1^2 - \alpha^4 K_1^2 - \beta^2 K_2^2 - 4\alpha\beta K_1^2 K_2)\cos^2\frac{\phi\pi}{2} + \beta K_1(-8K_1 K_2 + 2\beta^2 - 4\alpha\beta K_1 + 4\alpha K_1^2 K_2 \\
& + 2\alpha^2 K_1^2)\cos\phi\pi, \\
l_7 = {} & \alpha K_1(2\alpha\beta K_1^2 - 2\alpha^2\beta K_1 + 8\beta K_1 K_2^2 + 4\alpha^2 K_1^2 K_2 - 2\beta^2 K_2 + 2\alpha^3 K_1^2 + 2\alpha\beta K_1 K_2 + 8\beta K_1^2 K_2 \\
& - 2\beta^2 K_1)\cos\frac{\phi\pi}{2}, \\
l_8 = {} & \alpha^2\beta K_1^2(4K_1 K_2 - \beta + 2\alpha K_1).
\end{aligned}
$$

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