GENERAL DECAY FOR WEAK VISCOELASTIC EQUATION OF KIRCHHOFF TYPE CONTAINING BALAKRISHNAN-TAYLOR DAMPING WITH NONLINEAR DELAY AND ACOUSTIC BOUNDARY CONDITIONS

MI JIN LEE[1], JONG YEOUL PARK[2], JUM-RAN KANG[3,*]

Abstract. In this paper, we consider the general energy decay for weak viscoelastic equation of Kirchhoff type containing Balakrishnan-Taylor damping with nonlinear delay and acoustic boundary conditions. By introducing suitable energy and Lyapunov functionals, we establish the general decay estimates for the energy, which depends on the behavior of both σ and g.

1. Introduction

The main purpose of this paper is to consider the general decay for weak viscoelastic equation of Kirchhoff type containing Balakrishnan-Taylor damping with nonlinear delay and acoustic boundary conditions

$$|u_t|^\rho u_{tt} - \big(a + b\|\nabla u\|^2 + \delta(\nabla u, \nabla u_t)\big)\Delta u - \Delta u_{tt} + \sigma(t)\int_0^t g(t-s)\Delta u(s)ds$$

$$= |u|^{p-2}u \quad \text{in } \Omega\times\mathbb{R}^+, \tag{1.1}$$

$$u = 0 \quad \text{on } \Gamma_0\times\mathbb{R}^+, \tag{1.2}$$

$$\big(a + b\|\nabla u\|^2 + \delta(\nabla u, \nabla u_t)\big)\frac{\partial u}{\partial\nu} + \frac{\partial u_{tt}}{\partial\nu} - \sigma(t)\int_0^t g(t-s)\frac{\partial u(s)}{\partial\nu}ds + \mu_1|u_t(x,t)|^{q-1}u_t(x,t)$$

$$+\mu_2|u_t(x,t-\tau)|^{q-1}u_t(x,t-\tau) = m(x)y_t \quad \text{on } \Gamma_1\times\mathbb{R}^+, \tag{1.3}$$

$$u_t + f(x)y_t + h(x)y = 0 \quad \text{on } \Gamma_1\times\mathbb{R}^+, \tag{1.4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.5}$$

$$y(x,0) = y_0(x) \quad \text{on } \Gamma_1, \tag{1.6}$$

$$u_t(x,t-\tau) = f_0(x,t-\tau) \quad \text{on } \Gamma_1, \quad 0<t<\tau, \tag{1.7}$$

where Ω be a bounded domain of $\mathbb{R}^n, n\geq 1$, with a smooth boundary $\Gamma = \Gamma_0\cup\Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint and $\nu = (\nu_1,\nu_2,...,\nu_n)$ represents the outward normal to Γ. The constants $\rho, a, b, \delta, p, q, \mu_1, \mu_2 > 0$, the functions $m, f, h : \Gamma_1 \to \mathbb{R}$ are essential bounded, g represents the kernel of the memory term. Moreover, $\tau > 0$ represents the time delay and u_0, u_1, y_0, f_0 are given functions belonging to suitable space that will be precisely later.

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Department of Mathematics, Pusan National University, Busan 609-735, South Korea[1,2], Department of Applied Mathematics, Pukyong National University, Busan 608-737, South Korea[3].
E-mail addresses: jin0624@pusan.ac.kr[1], jyepark@pusan.ac.kr[2], jrkang@pknu.ac.kr[3] .
* corresponding author.

The physical applications of the system (1.1)-(1.7) is related to the problem of noise control and suppression in practical applications [4, 7, 8, 9, 10]. Balakrishnan-Taylor damping which arises from a wind tunnel experiment at supersonic speeds was introduced by Balakrishnan and Taylor [1]. Later, several authors have investigated the existence and asymptotic behavior of solution for equations with Balakrishnan-Taylor damping (see [11, 16] and references and therein). Also time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences [3, 12, 14, 15].

Feng and Li [5] studied the nonlinear viscoelastic Kirchhoff plate equation with a time delay term in the internal feedback. The authors established the general energy decay for a viscoelastic Kirchhoff plate equation with a delay term. Recently, Lee et al. [6] proved the general energy decay of the system (1.1)-(1.7) with $\sigma(t) = 1$ and $q = 1$.

Motivated by previous work, in this paper, we study the general energy decay of the system (1.1)-(1.7) for relaxation function g and potential σ satisfying the suitable conditions.

To the best of our knowledge, the general decay of solution for weak viscoelastic equation of Kirchhoff type containing Balakrishnan-Taylor damping with nonlinear delay and acoustic boundary conditions. The outline of this paper is as follows. In Section 2, we give some preparations and hypotheses for our main result. In Section 3, we establish the general decay result of the energy by using energy perturbation method.

2. Preliminary

In this section, we present some material that we shall use in order to preset our result. We denote by

$$V = \{u \in H^1(\Omega) : u = 0 \ \text{ on } \ \Gamma_0\}$$

the closed subspace of $H^1(\Omega)$ equipped with the norm equivalence to the usual norm in $H^1(\Omega)$. The Poincare inequality holds in V, i.e., there exist a constant C_* such that

$$\|u\|_r \le C_* \|\nabla u\|, \quad 2 \le r \le \frac{2N}{N-2}, \quad \forall u \in V, \tag{2.1}$$

and there exists a constant $\tilde{C}_*$ such that

$$\|u\|_{r,\Gamma_1} \le \tilde{C}_* \|\nabla u\|. \tag{2.2}$$

For our study of problem (1.1)-(1.7), we will need the following assumptions

(H1) ρ and q satisfy

$$0 < \rho, q \le \frac{2}{N-2} \ \text{ if } \ N \ge 3, \qquad \rho, q > 0 \ \text{ if } \ N = 1, 2, \tag{2.3}$$

and p satisfies

$$0 < p \le \frac{4}{N-2} \ \text{ if } \ N \ge 3, \qquad p > 2 \ \text{ if } \ N = 1, 2. \tag{2.4}$$

(H2) $g,\ \sigma : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing differentiable functions such that g is a C^2 function and σ is C^1 function satisfying

$$g(0) > 0,\quad \int_0^\infty g(s)ds = l_0 < \infty,\quad \sigma(t) > 0,\quad a - \sigma(t)\int_0^t g(s)ds > l > 0,\quad \forall\ t \geq 0, \tag{2.5}$$

and there exists a nonincreasing differentiable function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$\zeta(t) > 0,\quad g'(t) \leq -\zeta(t)g(t),\quad \forall\ t \geq 0,\quad \lim_{t\to\infty} \frac{-\sigma'(t)}{\sigma(t)} = 0, \tag{2.6}$$

$$\big(\sigma(t)\int_0^t g(s)ds\big)' \geq 0. \tag{2.7}$$

(H3) There exist three positive constants m_1, f_1 and h_1 such that

$$m_1 \leq m(x),\quad f_1 \leq f(x),\quad h_1 \leq h(x),\quad x \in \Gamma_1. \tag{2.8}$$

As in [13], let us introduce the function

$$z(x,k,t) = u_t(x, t - \tau k),\quad x \in \Omega,\quad k \in (0,1),\quad \forall t > 0. \tag{2.9}$$

Then problem (1.1)-(1.7) is equivalent to

$$\begin{cases} |u_t|^\rho u_{tt} - \big(a + b\|\nabla u\|^2 + \delta(\nabla u, \nabla u_t)\big)\Delta u - \Delta u_{tt} + \sigma(t)\displaystyle\int_0^t g(t-s)\Delta u(s)ds \\ \quad = |u|^{p-2}u \text{ in } \Omega \times \mathbb{R}^+, \\ u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \\ \big(a + b\|\nabla u\|^2 + \delta(\nabla u, \nabla u_t)\big)\dfrac{\partial u}{\partial \nu} + \dfrac{\partial u_{tt}}{\partial \nu} - \sigma(t)\displaystyle\int_0^t g(t-s)\frac{\partial u(s)}{\partial \nu}ds + \mu_1|u_t(x,t)|^{q-1}u_t(x,t) \\ \quad +\mu_2|z(x,1,t)|^{q-1}z(x,1,t) = m(x)y_t \quad \text{on} \quad \Gamma_1 \times (0,1) \times \mathbb{R}^+, \\ \tau z_t(x,k,t) + z_k(x,k,t) = 0 \quad \text{on} \quad \Gamma_1 \times (0,1) \times \mathbb{R}^+, \\ u_t + f(x)y_t + h(x)y = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega, \\ y(x,0) = y_0(x) \quad \text{on} \quad \Gamma_1, \\ z(x,k,0) = f_0(x, -\tau k) \quad \text{on} \quad \Gamma_1 \times (0,1). \end{cases} \tag{2.10}$$

By combining with the argument of [2], we now state the local existence result of problem (2.10), which can be established.

Theorem 2.1. *Suppose that (H1)-(H3) hold and that* $(u_0, u_1) \in (H^2(\Omega) \cap V) \times V,\ y_0 \in L^2(\Gamma_1)$ *and* $f_0 \in L^2(\Gamma_1 \times (0,1))$. *Then for any* $T > 0$, *there exists a unique solution* (u, y, z) *of problem* (2.10) *on* $[0, T]$ *such that*

$$u \in L^\infty(0,T; H^2(\Omega) \cap V),\ u_t \in L^\infty(0,T;V) \cap L^{q+1}(\Gamma_1 \times (0,T)),$$
$$m^{1/2}y \in L^\infty(0,T;L^2(\Gamma_1)),\ m^{1/2}y_t \in L^2(0,T;L^2(\Gamma_1)).$$

3. Main Result

In this section, we shall state and show our main result. For this purpose, we define

$$J(t) = \frac{1}{2}\Big(a - \sigma(t)\int_0^t g(s)ds\Big)\|\nabla u(t)\|^2 + \frac{b}{4}\|\nabla u(t)\|^4 + \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\sigma(t)(g \circ \nabla u)(t)$$

$$+\frac{\xi}{2}\int_{\Gamma_1}\int_0^1 z^{q+1}(x,k,t)dkd\Gamma+\frac{1}{2}\int_{\Gamma_1}h(x)m(x)y^2(t)d\Gamma-\frac{1}{p}\|u(t)\|_p^p, \tag{3.1}$$

and

$$I(t)=\Big(a-\sigma(t)\int_0^t g(s)ds\Big)\|\nabla u(t)\|^2+\frac{b}{2}\|\nabla u(t)\|^4+\|\nabla u_t(t)\|^2+\sigma(t)(g\circ\nabla u)(t)$$
$$+\xi\int_{\Gamma_1}\int_0^1 z^{q+1}(x,k,t)dkd\Gamma+\int_{\Gamma_1}h(x)m(x)y^2(t)d\Gamma-\|u(t)\|_p^p, \tag{3.2}$$

where $(g\circ u)(t)=\int_0^t g(t-s)\|u(t)-u(s)\|^2ds$. From direct calculation, we find that

$$\sigma(t)(g*u,u_t)=-\frac{\sigma(t)}{2}g(t)\|u(t)\|^2-\frac{d}{dt}\left[\frac{\sigma(t)}{2}(g\circ u)(t)-\frac{\sigma(t)}{2}\Big(\int_0^t g(s)ds\Big)\|u(t)\|^2\right]$$
$$+\frac{\sigma(t)}{2}(g'\circ u)(t)+\frac{\sigma'(t)}{2}(g\circ u)(t)-\frac{\sigma'(t)}{2}\int_0^t g(s)ds\|u(t)\|^2, \tag{3.3}$$

and

$$(g*u,u)\le 2\Big(\int_0^t g(s)ds\Big)\|u(t)\|^2+\frac{1}{4}(g\circ u)(t), \tag{3.4}$$

where $(g*u)(t)=\int_0^t g(t-s)u(s)ds$. Now we denote the modified energy functional $E(t)$ associate with problem (2.10) by

$$E(t)=\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+\frac{1}{2}\Big(a-\sigma(t)\int_0^t g(s)ds\Big)\|\nabla u(t)\|^2+\frac{b}{4}\|\nabla u(t)\|^4+\frac{1}{2}\|\nabla u_t(t)\|^2$$
$$+\frac{1}{2}\sigma(t)(g\circ\nabla u)(t)+\frac{\xi}{2}\int_{\Gamma_1}\int_0^1 z^{q+1}(x,k,t)dkd\Gamma+\frac{1}{2}\int_{\Gamma_1}h(x)m(x)y^2(t)d\Gamma-\frac{1}{p}\|u(t)\|_p^p$$
$$=\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+J(t), \tag{3.5}$$

where ξ is positive constant such that

$$\frac{2\tau\mu_2 q}{q+1}<\xi<\frac{2\tau\mu_1(q+1)-2\tau\mu_2}{q+1}. \tag{3.6}$$

Lemma 3.1. *Let $\mu_2<\mu_1$ and assume that (H2) hold. Then for the solution of problem* (2.10)*, the energy functional defined by* (3.5) *satisfies*

$$E'(t)\le -C_1\|u_t(t)\|_{q+1,\Gamma_1}^{q+1}-C_2\int_{\Gamma_1}|z(x,1,t)|^{q+1}d\Gamma-\delta\Big(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\Big)^2$$
$$-\frac{\sigma(t)g(t)}{2}\|\nabla u(t)\|^2-\frac{\sigma'(t)}{2}\Big(\int_0^t g(s)ds\Big)\|\nabla u(t)\|^2+\frac{\sigma'(t)}{2}(g\circ\nabla u)(t)$$
$$+\frac{\sigma(t)}{2}(g'\circ\nabla u)(t)-\int_{\Gamma_1}m(x)f(x)y_t^2(t)d\Gamma\le 0, \tag{3.7}$$

where C_1 and C_2 are some positive constants.

Proof. Multiplying in the first equation of (2.10) by u_t integrating over Ω, using (3.3), we have

$$\frac{d}{dt}\left[\frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2}+\frac{1}{2}\Big(a-\sigma(t)\int_0^t g(s)ds\Big)\|\nabla u(t)\|^2+\frac{b}{4}\|\nabla u(t)\|^4+\frac{1}{2}\|\nabla u_t(t)\|^2\right.$$
$$\left.+\frac{1}{2}\sigma(t)(g\circ\nabla u)(t)-\frac{1}{p}\|u(t)\|_p^p+\frac{1}{2}\int_{\Gamma_1}h(x)m(x)y^2(t)d\Gamma\right]$$

$$= -\mu_1\|u_t(t)\|_{q+1,\Gamma_1}^{q+1} - \mu_2\int_{\Gamma_1}|z(x,1,t)|^{q-1}z(x,1,t)u_t(t)d\Gamma - \delta\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 - \frac{\sigma(t)}{2}g(t)\|\nabla u(t)\|^2$$
$$-\frac{\sigma'(t)}{2}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{\sigma'(t)}{2}(g\circ\nabla u)(t) + \frac{\sigma(t)}{2}(g'\circ\nabla u)(t) - \int_{\Gamma_1}m(x)f(x)y_t^2(t)d\Gamma. \quad (3.8)$$

Multiplying the equation in the fourth equation of (2.10) by $\xi|z|^{q-1}z$ and integrating the result over $\Gamma_1\times(0,1)$, we obtain

$$\frac{\xi}{2}\frac{d}{dt}\int_{\Gamma_1}\int_0^1|z(x,k,t)|^{q+1}dkd\Gamma = -\frac{\xi}{2\tau}\int_{\Gamma_1}\int_0^1\frac{\partial}{\partial k}|z(x,k,1)|^{q+1}dkd\Gamma$$
$$= -\frac{\xi}{2\tau}\int_{\Gamma_1}|z(x,1,t)|^{q+1}d\Gamma + \frac{\xi}{2\tau}\int_{\Gamma_1}|u_t(t)|^{q+1}d\Gamma. \quad (3.9)$$

By using Young's inequality, we get

$$\left|\mu_2\int_{\Gamma_1}|z(x,1,t)|^{q-1}z(x,1,t)u_t(t)\Gamma\right|$$
$$\leq \frac{\mu_2 q}{q+1}\int_{\Gamma_1}|z(x,1,t)|^{q+1}d\Gamma + \frac{\mu_2}{q+1}\int_{\Gamma_1}|u_t(t)|^{q+1}d\Gamma. \quad (3.10)$$

Thus from (3.8)-(3.10), (2.7) and definition of $E(t)$, we obtain

$$E'(t) \leq -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{q+1}\right)\|u_t(t)\|_{q+1,\Gamma_1}^{q+1} - \left(\frac{\xi}{2\tau} - \frac{\mu_2 q}{q+1}\right)\int_{\Gamma_1}|z(x,1,t)|^{q+1}d\Gamma$$
$$-\delta\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 - \frac{\sigma(t)}{2}g(t)\|\nabla u(t)\|^2 - \frac{\sigma'(t)}{2}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2$$
$$+\frac{\sigma'(t)}{2}(g\circ\nabla u)(t) + \frac{\sigma(t)}{2}(g'\circ\nabla u)(t) - \int_{\Gamma_1}m(x)f(x)y_t^2(t)d\Gamma.$$

Using (2.7) and (3.6), we take $C_1 = \mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{q+1} > 0$ and $C_2 = \frac{\xi}{2\tau} - \frac{\mu_2 q}{q+1} > 0$, which implies the desired inequality (3.7). □

Lemma 3.2. *Suppose that (H1)-(H2) holds. Let* (u, y, z) *be the solution of problem* (2.10). *Assume that* $I(0) > 0$ *and*

$$\alpha = \frac{C_*^p}{l}\left(\frac{2pE(0)}{l(p-2)}\right)^{\frac{p-2}{2}} < 1. \quad (3.11)$$

Then $I(t) > 0$ *for* $t\in[0,T]$, *where* $I(t)$ *is defined in* (3.2).

Proof. Since $I(0) > 0$ and continuity of $u(t)$, then there exists $T^* < T$ such that

$$I(t) \geq 0, \quad (3.12)$$

for all $t\in[0,T^*]$. Then (2.5), (3.1), (3.2) and (3.12) give

$$J(t) = \frac{p-2}{2p}\left[\left(a - \sigma(t)\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{b}{2}\|\nabla u(t)\|^4 + \|\nabla u_t(t)\|^2 + \sigma(t)(g\circ\nabla u)(t)\right.$$
$$\left.+\xi\int_{\Gamma_1}\int_0^1 z^{q+1}(x,k,t)dkd\Gamma + \int_{\Gamma_1}h(x)m(x)y^2(t)d\Gamma\right] + \frac{1}{p}I(t)$$
$$\geq \frac{p-2}{2p}l\|\nabla u(t)\|^2, \quad \forall t\in[0,T^*]. \quad (3.13)$$

Thus from (3.5), (3.7) and (3.13), we arrive at

$$l\|\nabla u(t)\|^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0), \quad \forall t \in [0,T^*]. \tag{3.14}$$

Applying (2.1), (2.5), (3.11) and (3.14), we have

$$\|u(t)\|_p^p \leq C_*^p\|\nabla u(t)\|^p \leq \alpha l\|\nabla u(t)\|^2 \leq \left(a - \sigma(t)\int_0^t g(s)ds\right)\|\nabla u(t)\|^2, \quad \forall t \in [0,T^*].$$

Consequently, we get

$$\begin{aligned} I(t) &= \left(a - \sigma(t)\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{b}{2}\|\nabla u(t)\|^4 + \|\nabla u_t(t)\|^2 + \sigma(t)(g \circ \nabla u)(t) \\ &+\xi\int_{\Gamma_1}\int_0^1 z^{q+1}(x,k,t)dkd\Gamma + \int_{\Gamma_1} h(x)m(x)y^2(t)d\Gamma - \|u(t)\|_p^p > 0, \quad \forall t \in [0,T^*]. \end{aligned}$$

By repeating this procedure, and using the fact that

$$\lim_{t\to T^*}\frac{C_*^p}{l}\left(\frac{2pE(t)}{l(p-2)}\right)^{\frac{p-2}{2}} \leq \alpha < 1,$$

T^* is extended to T. Thus the proof is complete. □

We state the global existence result, which can be obtained by the arguments of [3, 6, 16, 17].

Theorem 3.1. *Let $\mu_2 < \mu_1$ and suppose that (H1)-(H3) hold. Let $(u_0,u_1) \in (H^2(\Omega)\cap V)\times V$, $y_0 \in L^2(\Gamma_1)$, $f_0 \in L^2(\Gamma_1\times(0,1))$. If $I(0) > 0$ and satisfy* (3.11)*, then the solution (u,y,z) of* (2.10) *is bounded and global in time.*

Now we will establish the general decay property of the solution for problem (2.10) in the case $\mu_2 < \mu_1$. For this purpose, we define the functional

$$\mathcal{L}(t) = ME(t) + \varepsilon\sigma(t)\Phi_1(t) + \sigma(t)\Phi_2(t), \tag{3.15}$$

where M and ε are positive constants which will be specified later and

$$\begin{aligned} \Phi_1(t) &= \frac{1}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)u(t)dx + \frac{\delta}{4}\|\nabla u(t)\|^4 + \int_\Omega \nabla u_t(t)\nabla u(t)dx \\ &+\int_{\Gamma_1} m(x)u(t)y(t)d\Gamma + \frac{1}{2}\int_{\Gamma_1} m(x)f(x)y^2(t)d\Gamma, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \Phi_2(t) &= -\frac{1}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &-\int_\Omega \nabla u_t(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx. \end{aligned} \tag{3.17}$$

Before we show our main result, we need the following lemmas.

Lemma 3.3. *Let $u \in L^\infty([0,T];H_0^1(\Omega))$ and suppose that (H1) holds, then we get*

$$\begin{aligned} &\int_\Omega \left(\sigma(t)\int_0^t g(t-s)(u(t)-u(s))ds\right)^{\rho+2}dx \\ &\leq (a-l)^{\rho+1}\alpha_1\sigma(t)(g\circ\nabla u)(t), \end{aligned} \tag{3.18}$$

where $\alpha_1 = C_*^{\rho+2}\left(\frac{2pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}$.

Proof. From (2.1), (2.5), (3.14) and Hölder's inequality, we can derive

$$\begin{aligned}
&\int_\Omega \left(\sigma(t)\int_0^t g(t-s)(u(t)-u(s))ds\right)^{\rho+2} dx \\
&\leq \int_\Omega \left(\sigma(t)\int_0^t g(t-s)ds\right)^{\rho+1}\left(\sigma(t)\int_0^t g(t-s)|u(t)-u(s)|^{\rho+2}ds\right)dx \\
&\leq (a-l)^{\rho+1} C_*^{\rho+2}\left(\frac{2pE(0)}{l(p-2)}\right)^{\frac{\rho}{2}}\sigma(t)(g\circ\nabla u)(t).
\end{aligned}$$

□

Lemma 3.4. *Let* (u, y, z) *be the solution of* (2.10) *and suppose that (H1)-(H3) hold, then there exist two constants* β_1 *and* β_2 *such that*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{3.19}$$

Proof. Using (2.1), (2.2), (2.8), (3.14) and Young's inequality, we obtain

$$\left|\frac{1}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)u(t)dx\right| \leq \frac{1}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{\alpha_1}{(\rho+2)(\rho+1)}\|\nabla u(t)\|^2, \tag{3.20}$$

$$\left|\int_\Omega \nabla u_t(t)\nabla u(t)dx\right| \leq \frac{1}{2}\|\nabla u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2, \tag{3.21}$$

$$\left|\int_{\Gamma_1} m(x)u(t)y(t)d\Gamma\right| \leq \frac{\|m\|_\infty}{2h_1}\int_{\Gamma_1} h(x)m(x)y^2(t)d\Gamma + \frac{\tilde{C}_*^2}{2}\|\nabla u(t)\|^2. \tag{3.22}$$

Similarly, using (2.1), (2.5), (3.18) and Young's inequality, we can deduce

$$\begin{aligned}
&\left|\frac{1}{\rho+1}\int_\Omega \sigma(t)|u_t(t)|^\rho u_t(t)\int_0^t g(t-s)(u(t)-u(s))dsdx\right| \\
&\leq \frac{\sigma(t)}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+2)(\rho+1)}\int_\Omega\left(\sigma(t)\int_0^t g(t-s)(u(t)-u(s))ds\right)^{\rho+2}dx \\
&\leq \frac{\sigma(t)}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{(a-l)^{\rho+1}\alpha_1}{(\rho+2)(\rho+1)}\sigma(t)(g\circ\nabla u)(t),
\end{aligned} \tag{3.23}$$

and

$$\left|-\int_\Omega \sigma(t)\nabla u_t(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\right| \leq \frac{\sigma(t)}{2}\|\nabla u_t(t)\|^2 + \frac{l_0}{2}\sigma(t)(g\circ\nabla u)(t). \tag{3.24}$$

Combining (3.15)-(3.17), (3.20)-(3.24) and using (H2), we get

$$\begin{aligned}
&|\mathcal{L}(t) - ME(t)| \leq \varepsilon\sigma(t)|\Phi_1(t)| + \sigma(t)|\Phi_2(t)| \\
&\leq \frac{\sigma(t)(\varepsilon+1)}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} + \frac{\sigma(t)(\varepsilon+1)}{2}\|\nabla u_t(t)\|^2 + \varepsilon\sigma(t)\left(\frac{\alpha_1}{(\rho+2)(\rho+1)} + \frac{\tilde{C}_*^2}{2} + \frac{1}{2}\right)\|\nabla u(t)\|^2 \\
&+\varepsilon\sigma(t)\left(\frac{\delta}{4}\|\nabla u(t)\|^4 + \frac{\|m\|_\infty + \|f\|_\infty}{2h_1}\int_{\Gamma_1} h(x)m(x)y^2(t)d\Gamma\right) + \left(\frac{(a-l)^{\rho+1}\alpha_1}{(\rho+2)(\rho+1)} + \frac{l_0}{2}\right)\alpha(t)(g\circ\nabla u)(t) \\
&\leq CE(t),
\end{aligned}$$

where C is some positive constant. Choosing $M > 0$ sufficiently large and ε small, we obtain (3.19). □

The following theorem is our main result.

Theorem 3.2. *Let $\mu_2 < \mu_1$ and suppose that (H1)-(H3) and* (3.6) *hold. If $(u_0, u_1) \in (H^2(\Omega)\cap V)\times V$, $y_0 \in L^2(\Gamma_1)$, $f_0 \in L^2(\Gamma_1 \times (0,1))$ and satisfying* (3.11). *Then for any $t > t_0$, there exist positive constants K and κ such that the energy of the solution for problem* (2.10) *satisfies*

$$E(t) = Ke^{-\kappa \int_{t_0}^{t} \sigma(s)\zeta(s)ds}, \quad \forall t \geq t_0. \tag{3.25}$$

Proof. From Lemma 3.4, it suffices to prove that we obtain the estimate of $\mathcal{L}(t)$. For this purpose, first we estimate $\Phi_1'(t)$. It follows from (3.16) and (2.10) that

$$\begin{aligned}
\Phi_1'(t) &= \int_\Omega |u_t(t)|^\rho u_{tt}(t)u(t)dx + \frac{1}{\rho+1}\int_\Omega |u_t(t)|^{\rho+2}dx + \delta\int_\Omega \nabla u(t)\nabla u_t(t)dx\int_\Omega \nabla u(t)\nabla u(t)dx \\
&\quad + \int_\Omega \nabla u_{tt}(t)\nabla u(t)dx + \int_\Omega |\nabla u_t(t)|^2dx + \int_{\Gamma_1} m(x)u_t(t)y(t)d\Gamma + \int_{\Gamma_1} m(x)u(t)y_t(t)d\Gamma \\
&\quad + \int_{\Gamma_1} m(x)f(x)y(t)y_t(t)d\Gamma \\
&= -\Big(a + b\|\nabla u(t)\|^2\Big)\int_\Omega |\nabla u(t)|^2dx + \int_\Omega \nabla u(t)\sigma(t)\int_0^t g(t-s)\nabla u(s)dsdx + \int_\Omega |u(t)|^p dx \\
&\quad -\mu_1\int_{\Gamma_1} |u_t(t)|^{q-1}u_t(t)u(t)d\Gamma - \mu_2\int_{\Gamma_1} |z(x,1,t)|^{q-1}z(x,1,t)u(t)d\Gamma + \frac{1}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} \\
&\quad + \int_\Omega |\nabla u_t(t)|^2dx + 2\int_{\Gamma_1} m(x)u(t)y_t(t)d\Gamma - \int_{\Gamma_1} h(x)m(x)y^2(t)d\Gamma.
\end{aligned} \tag{3.26}$$

In what follows we will estimate the right hand side of (3.26). By using (2.1), (2.2), (2.5), (2.8), (3.5), (3.14) and Young's inequality, for any $\eta > 0$, we have

$$\begin{aligned}
&\left|\int_\Omega \nabla u(t)\sigma(t)\int_0^t g(t-s)\nabla u(s)dsdx\right| \\
&\leq \left|\int_\Omega \nabla u(t)\sigma(t)\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))dsdx\right| + \sigma(t)\int_0^t g(s)ds\int_\Omega |\nabla u(t)|^2dx \\
&\leq (1+\eta)(a-l)\|\nabla u(t)\|^2 + \frac{\sigma(t)}{4\eta}(g\circ\nabla u)(t),
\end{aligned} \tag{3.27}$$

$$\left|\mu_1\int_{\Gamma_1} |u_t(t)|^{q-1}u_t(t)u(t)d\Gamma\right| \leq \mu_1\eta\alpha_2\|\nabla u(t)\|^2 + \mu_1 C_\eta\|u_t(t)\|_{q+1,\Gamma_1}^{q+1} \tag{3.28}$$

$$\left|\mu_2\int_{\Gamma_1} |z(x,1,t)|^{q-1}z(x,1,t)u(t)d\Gamma\right| \leq \mu_2\eta\alpha_2\|\nabla u(t)\|^2 + \mu_2 C_\eta\|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1} \tag{3.29}$$

and

$$\left|2\int_{\Gamma_1} m(x)u(t)y_t(t)d\Gamma\right| \leq \eta\tilde{C}_*^2\|\nabla u(t)\|^2 + \frac{\|m\|_\infty}{\eta f_1}\int_{\Gamma_1} m(x)f(x)y_t^2(t)d\Gamma, \tag{3.30}$$

where $\alpha_2 = \tilde{C}_*^{q+1}\big(\frac{2pE(0)}{l(p-2)}\big)^{\frac{q-1}{2}}$. Choosing η small enough such that

$$\eta\big(a - l + \tilde{C}_*^2 + \mu_1\alpha_2 + \mu_2\alpha_2\big) \leq \frac{l}{2}$$

and substituting of (3.27)-(3.30) into (3.26), we obtain

$$\Phi_1'(t) \leq -\frac{l}{2}\|\nabla u(t)\|^2 - b\|\nabla u(t)\|^4 + \frac{1}{\rho+1}\|u_t(t)\|_{\rho+2}^{\rho+2} + \|\nabla u_t(t)\|^2 + \|u(t)\|_p^p$$

$$
\begin{aligned}
&+\frac{\sigma(t)}{4\eta}(g\circ\nabla u)(t)+\mu_1 C_\eta\|u_t(t)\|_{q+1,\Gamma_1}^{q+1}+\mu_2 C_\eta\|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1}\\
&+\frac{\|m\|_\infty}{\eta f_1}\int_{\Gamma_1} m(x)f(x)y_t^2(t)d\Gamma-\int_{\Gamma_1} h(x)m(x)y^2(t)d\Gamma. \qquad (3.31)
\end{aligned}
$$

Next, we would like to estimate $\Phi_2'(t)$. Taking the derivative of $\Phi_2(t)$ in (3.17) and using (2.10), we get

$$
\begin{aligned}
\Phi_2'(t) &= \big(a+b\|\nabla u(t)\|^2\big)\int_\Omega \nabla u(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\
&+\delta\int_\Omega \nabla u(t)\nabla u_t(t)dx\int_\Omega \nabla u(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\
&-\int_\Omega \sigma(t)\int_0^t g(t-s)\nabla u(s)ds\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\
&-\int_\Omega |u(t)|^{p-2}u(t)\int_0^t g(t-s)(u(t)-u(s))dsdx-\int_\Omega \nabla u_t(t)\int_0^t g'(t-s)(\nabla u(t)-\nabla u(s))dsdx\\
&-\frac{1}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx\\
&-\int_{\Gamma_1}\Big(m(x)y_t(t)-\mu_2|z(x,1,t)|^{q-1}z(x,1,t)-\mu_1|u_t(x,t)|^{q-1}u_t(x,t)\Big)\int_0^t g(t-s)(u(t)-u(s))dsd\Gamma\\
&-\int_\Omega \nabla u_t(t)\int_0^t g(t-s)\nabla u_t(t)dsdx-\frac{1}{\rho+1}\int_\Omega |u_t(t)|^\rho u_t(t)\int_0^t g(t-s)u_t(t)dsdx\\
&:= E_1+E_2+\cdots+E_9-\int_0^t g(s)ds||\nabla u_t(t)||^2-\frac{1}{\rho+1}\Big(\int_0^t g(s)ds\Big)||u_t(t)||_{\rho+2}^{\rho+2}. \qquad (3.32)
\end{aligned}
$$

From now we will estimate the right hand side of (3.32). By (2.1), (2.2), (2.5), (2.8), (3.7), (3.13), (3.14) and Young's inequality, for any $\gamma>0$, we derive the following inequalities

$$
\begin{aligned}
|E_1| &\le \left|\int_\Omega \Big(a+\frac{2bpE(0)}{l(p-2)}\Big)\nabla u(t)\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\right|\\
&\le \gamma\|\nabla u(t)\|^2+\frac{l_0}{4\gamma}\Big(a+\frac{2bpE(0)}{l(p-2)}\Big)^2(g\circ\nabla u)(t), \qquad (3.33)\\
|E_2| &\le \gamma\delta^2\Big(\int_\Omega \nabla u(t)\nabla u_t(t)dx\Big)^2||\nabla u(t)||^2+\frac{1}{4\gamma}\int_\Omega\Big(\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds\Big)^2dx\\
&\le -\frac{2\gamma\delta^2 pE(0)}{l(p-2)}E'(t)+\frac{l_0}{4\gamma}(g\circ\nabla u)(t), \qquad (3.34)\\
|E_3| &\le \gamma\int_\Omega \sigma(t)\Big(\int_0^t g(t-s)(|\nabla u(t)-\nabla u(s)|+|\nabla u(t)|)ds\Big)^2dx\\
&+\frac{1}{4\gamma}\int_\Omega \sigma(t)\Big(\int_0^t g(t-s)|\nabla u(t)-\nabla u(s)|ds\Big)^2dx\\
&\le \Big(2\gamma+\frac{1}{4\gamma}\Big)(a-l)(g\circ\nabla u)(t)+2\gamma(a-l)l_0\|\nabla u(t)\|^2, \qquad (3.35)\\
|E_4| &\le \gamma\int_\Omega |u(t)|^{2(p-1)}dx+\frac{C_*^2 l_0}{4\gamma}(g\circ\nabla u)(t)\le \gamma\alpha_3\|\nabla u(t)\|^2+\frac{C_*^2 l_0}{4\gamma}(g\circ\nabla u)(t), \qquad (3.36)\\
|E_5| &\le \gamma\|\nabla u_t(t)\|^2-\frac{g(0)}{4\gamma}(g'\circ\nabla u)(t), \qquad (3.37)\\
|E_6| &\le \frac{\gamma\alpha_4}{\rho+1}\|\nabla u_t(t)\|^2-\frac{g(0)C_*^2}{4\gamma(\rho+1)}(g'\circ\nabla u)(t), \qquad (3.38)
\end{aligned}
$$

$$|E_7| \le \frac{\gamma\|m\|_\infty}{f_1}\int_{\Gamma_1} m(x)f(x)y_t^2(t)d\Gamma + \frac{\tilde{C}_*^2 l_0}{4\gamma}(g\circ\nabla u)(t), \tag{3.39}$$

$$|E_8| \le \gamma\mu_2\|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1} + \mu_2 C_\gamma l_0^q \alpha_2 (g\circ\nabla u)(t), \tag{3.40}$$

and

$$|E_9| \le \gamma\mu_1\alpha_5\|\nabla u_t(t)\|^2 + \frac{\mu_1\tilde{C}_*^2 l_0}{4\gamma}(g\circ\nabla u)(t), \tag{3.41}$$

where $\alpha_3 = C_*^{2(p-1)}\big(\frac{2pE(0)}{l(p-2)}\big)^{p-2}$, $\alpha_4 = C_*^{2(\rho+1)}\big(\frac{2pE(0)}{p-2}\big)^{\rho}$ and $\alpha_5 = \tilde{C}_*^{2q}\big(\frac{2pE(0)}{p-2}\big)^{q-1}$. Thus from (3.32)-(3.41), we conclude that

$$\begin{aligned}\Phi_2'(t) \le & -\frac{1}{\rho+1}\big(\int_0^t g(s)ds\big)\|u_t(t)\|_{\rho+2}^{\rho+2} - \left(\int_0^t g(s)ds - \gamma\Big(1+\mu_1\alpha_5+\frac{\alpha_4}{\rho+1}\Big)\right)\|\nabla u_t(t)\|^2 \\ & + C_3(g\circ\nabla u)(t) - \frac{g(0)}{4\gamma}\left(1+\frac{C_*^2}{\rho+1}\right)(g'\circ\nabla u)(t) - \frac{2\gamma\delta^2 pE(0)}{l(p-2)}E'(t) + \gamma\mu_2\|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1} \\ & + \gamma\big(1+2(a-l)l_0+\alpha_3\big)\|\nabla u(t)\|^2 + \frac{\gamma\|m\|_\infty}{f_1}\int_{\Gamma_1} m(x)f(x)y_t^2(t)d\Gamma, \end{aligned} \tag{3.42}$$

where $C_3 = \frac{1}{4\gamma}\big\{l_0\big(a+\frac{2bpE(0)}{l(p-2)}\big)^2 + (8\gamma^2+1)(a-l) + l_0(1+C_*^2+\tilde{C}_*+\mu_1\tilde{C}_*^2) + 4\gamma\mu_2 C_\gamma l_0^q\alpha_2\big\}$. Similarly to Lemma 3.4, for any $\lambda>0$, we obtain

$$\begin{aligned}\sigma'(t)\Phi_1(t) \le & -\frac{\sigma'(t)}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} - C_4\sigma'(t)\|\nabla u(t)\|^2 - \frac{\sigma'(t)}{2}\|\nabla u_t(t)\|^2 \\ & -\frac{\sigma'(t)\delta}{4}\|\nabla u(t)\|^4 - \frac{\sigma'(t)(\|m\|_\infty+||f||_\infty)}{2h_1}\int_{\Gamma_1} h(x)m(x)y^2(t)d\Gamma \end{aligned} \tag{3.43}$$

and

$$\sigma'(t)\Phi_2(t) \le -\frac{\lambda\sigma'(t)}{\rho+2}\|u_t(t)\|_{\rho+2}^{\rho+2} - \lambda\sigma'(t)\|\nabla u_t(t)\|^2 - C_5\sigma'(t)(g\circ\nabla u)(t), \tag{3.44}$$

where $C_4 = \frac{1}{2} + \frac{\tilde{C}_*^2}{2} + \frac{\alpha_1}{(\rho+2)(\rho+1)}$ and $C_5 = \frac{C_\lambda {l_0}^{\rho+1}\alpha_1}{(\rho+2)(\rho+1)} + \frac{l_0}{4\lambda}$. Since g is positive, we have, for any $t_0>0$, $\int_0^t g(s)ds \ge \int_0^{t_0} g(s)ds := g_0 > 0$, for all $t\ge t_0$. Applying (3.7), (3.31) and (3.42)-(3.44), we see that for any $t\ge t_0$,

$$\begin{aligned}\mathcal{L}'(t) = & ME'(t) + \varepsilon\sigma'(t)\Phi_1(t) + \varepsilon\sigma(t)\Phi_1'(t) + \sigma'(t)\Phi_2(t) + \sigma(t)\Phi_2'(t) \\ \le & -\sigma(t)\left(\frac{g_0-\varepsilon}{\rho+1} + \frac{(\varepsilon+\lambda)\sigma'(t)}{(\rho+2)\sigma(t)}\right)\|u_t(t)\|_{\rho+2}^{\rho+2} - \sigma(t)\left(b\varepsilon + \frac{\varepsilon\delta\sigma'(t)}{4\sigma(t)}\right)\|\nabla u(t)\|^4 \\ & -\sigma(t)\left(\Big(\frac{g(t)}{2} + \frac{\sigma'(t)}{2\sigma(t)}\int_0^t g(s)ds\Big)M + \frac{\varepsilon C_4\sigma'(t)}{\sigma(t)} + \frac{\varepsilon l}{2} - \gamma\big(1+2(a-l)l_0+\alpha_3\big)\right)\|\nabla u(t)\|^2 \\ & -\sigma(t)\left(g_0 - \gamma\big(1+\mu_1\alpha_5+\frac{\alpha_4}{q+1}\big) - \varepsilon + \frac{(\varepsilon+2\lambda)\sigma'(t)}{2\sigma(t)}\right)\|\nabla u_t(t)\|^2 + \varepsilon\sigma(t)\|u(t)\|_p^p \\ & +\sigma(t)\left(\frac{M\sigma'(t)}{2\sigma(t)} + \frac{\varepsilon\sigma(t)}{4\eta} - \frac{C_5\sigma'(t)}{\sigma(t)} + C_3\right)(g\circ\nabla u)(t) - \sigma(t)\left(\frac{C_1 M}{\sigma(t)} - \varepsilon\mu_1 C_\eta\right)\|u_t(t)\|_{q+1,\Gamma_1}^{q+1} \\ & -\sigma(t)\left(\frac{C_2 M}{\sigma(t)} - \varepsilon\mu_2 C_\eta - \gamma\mu_2\right)\|z(x,1,t)\|_{q+1,\Gamma_1}^{q+1} + \sigma(t)\left(\frac{M}{2} - \frac{g(0)}{4\gamma}\Big(1+\frac{C_*^2}{\rho+1}\Big)\right)(g'\circ\nabla u)(t) \\ & -\sigma(t)\left(\varepsilon + \frac{\varepsilon\sigma'(t)(||m||_\infty+||f||_\infty)}{2h_1\sigma(t)}\right)\int_{\Gamma_1} h(x)m(x)y^2(t)d\Gamma - \sigma(t)\frac{2\gamma\delta^2 pE(0)}{l(p-2)}E'(t) \\ & -\sigma(t)\left(\frac{M}{\sigma(t)} - \frac{\varepsilon\|m\|_\infty}{\eta f_1} - \frac{\gamma\|m\|_\infty}{f_1}\right)\int_{\Gamma_1} m(x)f(x)y_t^2(t)d\Gamma. \end{aligned} \tag{3.45}$$

At this point, we choose ε and γ sufficiently small, M sufficiently large and then since $\lim_{t\to\infty} \frac{\sigma'(t)}{\sigma(t)} = 0$, we can choose $t_0 > 0$ sufficiently large such that

$$
\begin{aligned}
M_1 &= \frac{g_0 - \varepsilon}{\rho + 1} + \frac{(\varepsilon + \lambda)\sigma'(t)}{(\rho + 2)\sigma(t)} > 0, \\
M_2 &= \Big(\frac{g(t)}{2} + \frac{\sigma'(t)}{2\sigma(t)} \int_0^t g(s)ds\Big) M + \frac{\varepsilon C_4 \sigma'(t)}{\sigma(t)} + \frac{\varepsilon l}{2} - \gamma\big(1 + 2(a - l)l_0 + \alpha_3\big) > 0, \\
M_3 &= g_0 - \gamma\big(1 + \mu_1\alpha_5 + \frac{\alpha_4}{q + 1}\big) - \varepsilon + \frac{(\varepsilon + 2\lambda)\sigma'(t)}{2\sigma(t)} > 0, \\
M_4 &= \frac{M\sigma'(t)}{2\sigma(t)} + \frac{\varepsilon\sigma(t)}{4\eta} - \frac{C_5\sigma'(t)}{\sigma(t)} + C_3 > 0, \quad M_5 = \frac{C_1 M}{\sigma(t)} - \varepsilon\mu_1 C_\eta > 0, \\
M_6 &= \frac{C_2 M}{\sigma(t)} - \varepsilon\mu_2 C_\eta - \gamma\mu_2 > 0, \quad M_7 = \frac{M}{2} - \frac{g(0)}{4\gamma}\left(1 + \frac{C_*^2}{\rho + 1}\right) > 0,
\end{aligned}
$$

and

$$M_8 = \frac{M}{\sigma(t)} - \frac{\varepsilon\|m\|_\infty}{\eta f_1} - \frac{\gamma\|m\|_\infty}{f_1} > 0.$$

Then for any $t \geq t_0$, using (3.5) and (3.45), we deduce that

$$\mathcal{L}'(t) \leq -M_9\sigma(t)E(t) + M_{10}\sigma(t)(g \circ \nabla u)(t) - \sigma(t)\frac{2\gamma\delta^2 pE(0)}{l(p-2)}E'(t), \tag{3.46}$$

where M_9 and M_{10} are some positive constants. Multiplying (3.46) by $\zeta(t)$ and using (2.6), (3.7) and (3.14), we obtain for any $t \geq t_0$,

$$
\begin{aligned}
\zeta(t)\mathcal{L}'(t) &\leq -M_9\sigma(t)\zeta(t)E(t) - M_{10}\sigma(t)(g' \circ \nabla u)(t) - M_{11}\sigma(t)\zeta(t)E'(t) \\
&\leq -M_9\sigma(t)\zeta(t)E(t) - 2M_{10}E'(t) - M_{11}\sigma(t)\zeta(t)E'(t) \\
&\leq -M_9\sigma(t)\zeta(t)E(t) - (2M_{10} + M_{11}\sigma(t)\zeta(t))E'(t),
\end{aligned} \tag{3.47}
$$

where $M_{11} = \frac{2\gamma\delta^2 pE(0)}{l(p-2)}$. Now, we define

$$G(t) = \zeta(t)\mathcal{L}(t) + (2M_{10} + M_{11}\sigma(t)\zeta(t))E(t).$$

Using the fact that ζ and σ are nonincreasing positive functions and $\zeta'(t) \leq 0$ and $\sigma'(t) \leq 0$, (3.47) implies that

$$G'(t) \leq -M_9\sigma(t)\zeta(t)E(t) \leq -\kappa\sigma(t)\zeta(t)G(t)$$

where κ is a positive constant. Integrating the previous inequality between t_0 and t gives the following estimation for the function $G(t)$

$$G(t) \leq G(t_0)e^{-\kappa\int_{t_0}^t \sigma(s)\zeta(s)ds}, \quad \forall t \geq t_0.$$

Again, employing that $G(t)$ is equivalent to $E(t)$, we deduce

$$E(t) \leq Ke^{-\kappa\int_{t_0}^t \sigma(s)\zeta(s)ds}, \quad \forall t \geq t_0.$$

where K is a positive constant. Thus the proof of Theorem 3.2 is completed. □

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