Intra-layer Synchronization in a Duplex Networks with Noise

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Abstract

This paper concerns the impact of environmental noise on the intra-layer synchronization of the duplex networks. A duplex network contains two layers. Different from the previous works [1, 2], environmental noise is introduced into the dynamical system of the duplex network. We incorporate both the *inter-layer* delay and the *intra-layer* delay into the dynamical system. Both of the delays are time-varying. However, the paper [1] only considered the intra-layer delays and they are assumed as the constants. While the paper [2] did not consider the inter-layer delay or intra-layer delay. When the system does not achieve automatic intra-layer synchronization, we introduce two controllers: one is the state-feedback controller, the other is the adaptive state-feedback controller. Interestingly, we find that the intra-layer synchronization will achieve automatically if the inter-layer coupling strength is large enough and the intra-layer coupling strength is small enough under the situation that the time-varying inter-layer delays are absent. Finally, some interesting simulation results are obtained for the Chua-Chua chaotic system with application of our theoretic results, which show the feasibility and effectiveness of our control schemes.

Keywords: intra-layer synchronization, duplex network, adaptive control

1 Introduction

Nowadays, the complexity of the networks grows rapidly. Usually, the scale of the networks is unbelievable large. A single network is not very reasonable in the real world application. It does not involve a single network isolatedly, but depends on the interactions between subnetworks. In the past few years, scholars paid attention to synchronization problems in multi-layer networks and multi-agent systems. However, we focus on multiplex networks in this paper. Multiplex networks are a special kind of multi-layer networks. The interactions on the different layers in multiplex networks have the same set of nodes. It is possible to be distinct for the state of the node on each layer, and it is possible to be unique of the connectivity pattern on each layer.

Recently, Tang, Lu, L$\ddot{u}$ [1] proposed a duplex network setting of R$\ddot{o}$ssler oscillators as follows.

$$\begin{cases} \dot{x}_i(t) &= \big[f(x_i(t)) - c_2 \sum_{j=1}^{N} l_{ij} H(x_j(t-\zeta_1)) \\ & \quad -c_1[\Gamma(x_i(t-\zeta_2)) - \Gamma(y_i(t-\zeta_2))]\big], \\ \dot{y}_i(t) &= \big[f(y_i(t)) - c_2 \sum_{j=1}^{N} l_{ij} H(y_j(t-\zeta_1)) \\ & \quad -c_1[\Gamma(y_i(t-\zeta_2)) - \Gamma(x_i(t-\zeta_2))]\big], \end{cases} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T$ and $y_i(t) = (y_{i1}(t), y_{i2}(t), \cdots, y_{in}(t))^T$ are the x-state and the y-state of the i-th node, respectively. The states of all nodes in both layers exhibit the same isolated nodal dynamics, i.e., $\dot{x} = f(x)$ and $\dot{y} = f(y)$. c_1 is the coupling strength between the two layers, and c_2 is the coupling strength within each layer. $H(\cdot)$ is the intra-layer coupling function within each layer, and $\Gamma(\cdot)$ is the inter-layer coupling function between the two layers. As you see, it is possible that the intra-layer coupling function is different from that of the inter-layer. ζ_1 is the constant coupling delay within each layer, and ζ_2 is the constant coupling delays between the two layers. Moreover, both layers have the same topological structure and the common Laplacian matrix is represented by $L = (l_{ij})_{N\times N}$. The element l_{ij} represents the link between node i and node j within x-layer and y-layer. That is $l_{ij} = -1$ if node i and node j are linked through x-layer and y-layer. Finally, the Laplacian matrix satisfies the diffusion property, i.e., the zero row sum condition. That is $l_{ii} = -\sum_{j=1}^{N} l_{ij}$.

In the last few decades, to control the chaos in the complex networks, a lot of synchronization schemes have been developed. For examples, global synchronization (see e.g. Cao et al [3], Wang et al [4],Duan et al [5], Lu et al. [6, 7]),), finite-time synchronization (e.g. Liu et al [8], Liu et al [9], Li et al [10, 11], Yang et al [12], Yang and Huang [13]), synchronization based on impulsive control (e.g. Li et al[14], Li et al [15], Peng et al [16], Zhao et al [17], Lu et al[6], Lu et al [7],Yang et al [18, 12], adaptive synchronization (see e.g [6, 19, 20]), periodically (aperiodically) intermittent adaptive control (Guo et al [21], pinning synchronization (Liu et al. [22], Wang et al [23]), fractional synchronization (e.g. Xu et al[24, 25], Tan et al [26], Zhu [27, 28]), anti-synchronization (Al-sawalha and Noorani [29]) and so on. In particular, a lot of researchers are interested in the multiplex networks. However, most of them concentrated on the complete synchronization. Few papers studied the intra-layer synchronization. Tang, Lu, and L$\ddot{u}$ [1] showed that the inter-layer coupling functions have great influence on intra-layer synchronization regions, as well as on the intra-layer synchronizability. However, the unpredicted external perturbations such as the white noise (it can be seen as the derivative of a stochastic nature of Brownian motion) can cause great uncertainty. Thus, many scientists argued that the stochastic differential equations (SDE) is a powerful tool to model the Brownian motion on the complex systems (see e.g. Wang et al [23], Tan et al [26], Raj et al [32], Yang et al [12], Shi et al [33],Zhu [27],Zhu and Wang[28], Zhuang et al [34]Li et al [35]).

So far, there is no result considering the intra-layer synchronization problem for duplex networks with stochastic perturbations. Therefore, we investigate a duplex dynamic networks with stochastic perturbations based on additive couplings in this paper. The contributions of this paper are listed below.

1. Environmental noise is introduced into the dynamical system on the duplex network, which is described by the stochastic differential equations. While in [1, 2, 36], the authors did not consider this factor.

2. When the system does not achieve automatic intra-layer synchronization, we introduce two controllers: one is the state-feedback controller, the other is the adaptive state-feedback controller.

3. We incorporate both the **inter-layer** delay and the **intra-layer** delay into the dynamical system. Both of the delays are time-varying. The paper [1] only considered the intra-layer delays and the delays are assumed as the constants. While the paper [2] did not consider the inter-layer delay or intra-layer delay.

4. Interestingly, we find that the intra-layer synchronization will be achieved automatically if the inter-layer coupling strength is large enough and the intra-layer coupling strength is small enough under the situation that the time-varying inter-layer delays are absent.

2 Preliminary and model formulation

In this paper, we will study the synchronization criteria of the following duplex networked system with time-varying delays and stochastic perturbations [1, 2, 36]

$$\begin{cases} dx_i(t) = \big[\tilde{f}^x(x_i(t)) - c_2 \sum_{j=1}^{N} l_{ij}^x H(x_j(t-\zeta_1(t))) + u_i^x(t) \\ \qquad -c_1[\Gamma(x_i(t-\zeta_2(t))) - \Gamma(y_i(t-\zeta_2(t)))]\big] dt + \tilde{\sigma}_i^x(t, x_i(t)) dB(t), \\ dy_i(t) = \big[\tilde{f}^y(y_i(t)) - c_2 \sum_{j=1}^{N} l_{ij}^y H(y_j(t-\zeta_1(t))) + u_i^y(t) \\ \qquad -c_1[\Gamma(y_i(t-\zeta_2(t))) - \Gamma(x_i(t-\zeta_2(t)))]\big] dt + \tilde{\sigma}_i^y(t, y(t)) dB(t), \end{cases} \tag{2}$$

$i = 1, 2, \cdots, N$, where $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{in}(t))^T$ is the x-state of the i-th node, $y_i(t) = (y_{i1}(t), y_{i2}(t), \cdots, y_{in}(t))^T$ is the y-state of the i-th node, $u_i^x(t)$, $u_i^y(t)$ are the controllers. $\dot{x} = \tilde{f}^x(x)$ and $\dot{y} = \tilde{f}^y(y)$ represent the isolated nodal x-dynamics and y-dynamics, respectively. Moreover, $\tilde{\sigma}_i^x$ and $\tilde{\sigma}_i^y$ are noise intensity functions, c_1 is the coupling strength between the two layers, and c_2 is the coupling strength within each layer. $\zeta_1(t)$ is the coupling delay within each layer, and $\zeta_2(t)$ is the coupling delays between the two layers. The Laplacian matrices of the two layers are represented by $L^x = (l_{ij}^x)_{N\times N}$ and $L^y = (l_{ij}^y)_{N\times N}$, respectively. The elements l_{ij}^x and l_{ij}^y represent the link between node i and node j within x-layer and y-layer respectively. That is $l_{ij}^x = -1$ if node i and node j are linked within x-layer, otherwise $l_{ij}^x = 0$. l_{ij}^y's are similarly defined. Both the Laplacian matrices satisfy the diffusion property, i.e., the zero row sum condition. That is $l_{ii}^x = -\sum_{j=1}^N l_{ij}^x$, $l_{ii}^y = -\sum_{j=1}^N l_{ij}^y$. Similar to [1] or [2], we assume that both the intra-layer coupling function $H(\cdot)$ and the inter-layer coupling function $\Gamma(\cdot)$ are linear, i.e., $H(x) = Hx$ and $\Gamma(x) = \Gamma x$, where H and Γ are two matrices.

System (2) achieves intra-layer synchronization, when the states of all nodes in x-layer approach an identical state $s^x(t)$, and those of all nodes in y-layer approach an identical state $s^y(t)$ in the mean time. Then the dynamics of the intra-layer synchronous states can be described as follows.

$$\begin{cases} ds^x(t) = \big[\tilde{f}^x(s^x(t)) - c_1[\Gamma(s^x(t-\zeta_2(t))) - \Gamma(s^y(t-\zeta_2(t)))]\big] dt, \\ ds^y(t) = \big[\tilde{f}^y(s^y(t)) - c_1[\Gamma(s^y(t-\zeta_2(t))) - \Gamma(s^x(t-\zeta_2(t)))]\big] dt. \end{cases} \tag{3}$$

Let $e_i^x(t) = x_i(t) - s^x(t)$ and $e_i^y(t) = y_i(t) - s^y(t)$ be the x-layer and y-layer intra-layer synchronization errors respectively. Moreover, set $\sigma_i^x(t, e_i^x(t)) =$

$\tilde{\sigma}_i^x(t, s^x(t)+e_i^x(t))$ and $\sigma_i^y(t, e_i^y(t)) = \tilde{\sigma}_i^y(t, s^y(t)+e_i^y(t))$. Then system (2) can be rewritten as

$$\begin{cases} dx_i(t) = \big[\tilde{f}^x(x_i(t)) - c_2 \sum\limits_{j=1}^{N} l_{ij}^x H(x_j(t-\zeta_1(t))) + u_i^x(t) \\ \qquad -c_1[\Gamma(x_i(t-\zeta_2(t))) - \Gamma(y_i(t-\zeta_2(t)))]\big]dt + \sigma_i^x(t, e_i^x(t))dB(t), \\ dy_i(t) = \big[\tilde{f}^y(y_i(t)) - c_2 \sum\limits_{j=1}^{N} l_{ij}^y H(y_j(t-\zeta_1(t))) + u_i^y(t) \\ \qquad -c_1[\Gamma(y_i(t-\zeta_2(t))) - \Gamma(x_i(t-\zeta_2(t)))]\big]dt + \sigma_i^y(t, e_i^y(t))dB(t), \end{cases} \tag{4}$$

for $i = 1, 2, \cdots, N$. For convenience, we may also call system (3) as the (virtual) drive system and system (4) as the (virtual) response system, respectively.

Substracting (4) from (3), we obtain the dynamical system of the errors,

$$\begin{cases} de_i^x(t) = \big[f^x(e_i^x(t)) - c_2 \sum\limits_{j=1}^{N} l_{ij}^x H e_j^x(t-\zeta_1(t)) + u_i^x(t) \\ \qquad -c_1[\Gamma(e_i^x(t-\zeta_2(t))) - \Gamma(e_i^y(t-\zeta_2(t)))]\big]dt + \sigma_i^x(t, e_i^x(t))dB(t), \\ de_i^y(t) = \big[f^y(e_i^y(t)) - c_2 \sum\limits_{j=1}^{N} l_{ij}^y H(e_j^y(t-\zeta_1(t))) + u_i^y(t) \\ \qquad -c_1[\Gamma(e_i^y(t-\zeta_2(t))) - \Gamma(e_i^x(t-\zeta_2(t)))]\big]dt + \sigma_i^y(t, e_i^y(t))dB(t), \end{cases} \tag{5}$$

where $f^x(e_i^x(t) = \tilde{f}^x(x_i(t)) - \tilde{f}^x(s^x(t))$, $f^x(e_i^y(t) = \tilde{f}^y(y_i(t)) - \tilde{f}^y(s^y(t))$, $i = 1, 2, \cdots, N$. The system can also be written in the following compact forms.

$$\begin{cases} de^x(t) = \big[f^x(e^x(t)) - c_2(L^x \otimes H)e^x(t-\zeta_1(t)) + u^x(t) \\ \qquad -c_1[(I_N \otimes \Gamma)(e^x(t-\zeta_2(t)) - e^y(t-\zeta_2(t)))]\big]dt + \sigma^x(t, e^x(t))dB(t), \\ de^y(t) = \big[f^y(e^y(t)) - c_2(L^y \otimes H)e^y(t-\zeta_1(t)) + u^y(t) \\ \qquad -c_1[(I_N \otimes \Gamma)(e^y(t-\zeta_2(t)) - e^x(t-\zeta_2(t)))]\big]dt + \sigma^y(t, e^y(t))dB(t), \end{cases} \tag{6}$$

where $e^x(t) = (e_1^{xT}(t), \cdots, e_N^{xT}(t))^T, e^y(t) = (e_1^{yT}(t), \cdots, e_N^{yT}(t))^T$, $f^x(e^x(t)) = (f^{xT}(e_1^x(t)), \cdots, f^{xT}(e_N^x(t)))^T$, $f^y(e^y(t)) = (f^{yT}(e_1^y(t)), \cdots, f^{yT}(e_N^y(t)))^T$, $\sigma^x(t, e^x(t)) = (\sigma_1^{xT}(t, e_1^x(t)), \cdots, \sigma_N^{xT}(t, e_N^x(t)))^T$, $\sigma^y(t, e^y(t)) = (\sigma_1^{yT}(t, e_1^y(t)), \cdots, \sigma_N^{yT}(t, e_N^y(t)))^T$. Or

$$\begin{aligned} de(t) \;=\; & \Bigg[\begin{pmatrix} f^x(e(t)) \\ f^y(e(t)) \end{pmatrix} - c_2 \begin{pmatrix} (L^x \otimes H)e^x(t-\zeta_1(t)) \\ (L^y \otimes H)e^y(t-\zeta_1(t)) \end{pmatrix} \\ & -c_1 \begin{pmatrix} (I_N \otimes \Gamma)(e^x(t-\zeta_2(t)) - e^y(t-\zeta_2(t))) \\ (I_N \otimes \Gamma)(e^y(t-\zeta_2(t)) - e^x(t-\zeta_2(t))) \end{pmatrix} \end{aligned}$$

$$+\begin{pmatrix} u^x(t) \\ u^y(t) \end{pmatrix}\Bigg]dt + \begin{pmatrix} \sigma^x(t, e^x(t)) \\ \sigma^y(t, e^y(t)) \end{pmatrix} dB(t)$$

For simplicity, we also set $e(t) = (e^{xT}(t), e^{yT}(t))^T$, $f(e(t)) = (f^{xT}(e^x(t)), f^{yT}(e^y(t)))^T$, and $\sigma(t, e(t)) = (\sigma^{xT}(t, e^x(t)), \sigma^{yT}(t, e^y(t)))^T$.

We also make the following assumptions.

Assumption (A1) Each system of (3) and (4) admits a unique solution, respectively.

Assumption (A2) There exist diagonal matrices U_1 and U_2 such that

$$\begin{array}{c}(\tilde{f}_i^x(z_1) - \tilde{f}_i^x(z_2))^T(\tilde{f}_i^x(z_1) - \tilde{f}_i^x(z_2)) \le (z_1 - z_2)^T U_1 (z_1 - z_2), \\ (\tilde{f}_i^y(z_1) - \tilde{f}_i^y(z_2))^T(\tilde{f}_i^y(z_1) - \tilde{f}_i^y(z_2)) \le (z_1 - z_2)^T U_2 (z_1 - z_2),\end{array}$$

for all $z_1, z_2 \in \mathbb{R}^n$, $i = 1, 2, \cdots, N$.

Assumption (A3)[20, 37] There exist positive constants ζ, ρ_1 and ρ_2 satisfying $0 \le \zeta_1(t) \le \zeta$, $0 \le \zeta_2(t) \le \zeta$, $0 \le \dot{\zeta}_1(t) \le \rho_1 < 1$ and $0 \le \dot{\zeta}_2(t) \le \rho_2 < 1$.

Assumption (A4)[20, 32, 37] There are positive semi-definite matrices Σ^x, Σ^y such that

$$\mathrm{trace}[\sigma_i^{xT}(t, z)\sigma_i^x(t, z)] \le z^T \Sigma^x z \text{ and } \mathrm{trace}[\sigma_i^{yT}(t, z)\sigma_i^y(t, z)] \le z^T \Sigma^y z$$

for all $z \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, and $i = 1, 2, \cdots, N$.

Definition 2.1 *The duplex networked system* (2) *achieves stochastic intra-layer synchronization almost surely, if*

$$\lim_{t\to+\infty}(x_i(t) - x_j(t)) = 0, \ a.s. \text{ and } \lim_{t\to+\infty}(y_i(t) - y_j(t)) = 0, \ a.s.$$

for $i, j = 1, 2, \cdots, N$, *or quivalently,*

$$\lim_{t\to+\infty}(x_i(t) - s^x(t)) = 0, \ a.s. \text{ and } \lim_{t\to+\infty}(y_i(t) - s^y(t)) = 0, \ a.s.$$

for $i = 1, 2, \cdots, N$.

In what follows, the LaSalle-type invariance lemma [38] for stochastic differential delay equations is recalled to proved our results. Consider the following n-dimensional stochastic differential delay equation

$$dx(t) = f(t, x(t), x(t - \zeta))dt + \sigma(t, x(t), x(t - \zeta))dB(t). \quad (7)$$

Lemma 2.1 *[38] Assume that system* (7) *has a unique solution* $x(t,\xi)$ *on* $t>0$ *for any given initial data* $\{x(\theta): -\zeta \leq \theta\} = \xi \in C^b_{\mathcal{F}_0}([-\zeta,0];\mathbb{R}^n)$*. Moreover, both* $f(x,y,t)$ *and* $\sigma(t,x,y)$ *are locally bounded in* (x,y) *and uniformly bounded in* t*. If there exist a function* $V \in C^{2,1}(\mathbb{R}^+ \times \mathbb{R}^n;\mathbb{R}^+)$*,* $\beta \in L^1(\mathbb{R}^+,\mathbb{R}^+)$ *and* $\omega_1,\omega_2 \in C(\mathbb{R}^n;\mathbb{R}^+)$ *such that*

$$\mathcal{L}V(t,x,y) \leq \beta(t) - \omega_1(x) + \omega_2(y),\ (t,x,y) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \tag{8}$$

$$\omega_1(x) > \omega_2(x), \forall x \neq 0, \tag{9}$$

$$\lim_{\|x\|\to\infty} \inf_{0\leq t<\infty} V(t,x) = \infty. \tag{10}$$

Then $\lim_{t\to\infty} x(t,\xi) = 0$ *a.s. for every* $\xi \in C^b_{\mathcal{F}_0}([-\zeta,0];\mathbb{R}^n)$*.*

3 Main Results

3.1 Synchronization with state-feedback controller

Assume that the controller is defined as follows

$$\begin{cases} u_i^x(t) = -k_i^x e_i^x(t), \\ u_i^y(t) = -k_i^y e_i^y(t) \end{cases}$$

for $i = 1,2,\cdots,N$, i.e.,

$$\begin{pmatrix} u^x(t) \\ u^y(t) \end{pmatrix} = \begin{pmatrix} -(K^x \otimes I_n)e^x(t) \\ -(K^y \otimes I_n)e^y(t) \end{pmatrix}, \tag{11}$$

where k_i^x's and k_i^y's are constant control gains, $u^x(t) = (u_1^{xT}(t),\cdots,u_N^{xT}(t))^T$, $u^y(t) = (u_1^{yT}(t),\cdots,u_N^{yT}(t))^T$, $K^x = \text{diag}\{k_1^x,\cdots,k_N^x\}$ and $K^y = \text{diag}\{k_1^y,\cdots,k_N^y\}$

Theorem 3.1 *Assume that* **(A1)**-**(A4)** *hold. Under the controller* (11)*, the duplex network* (4) *and* (3) *are stochastically synchronized almost surely if there exist two positive numbers* λ_1,μ_1 *and positive definite matrices* P_1,P_2,P_3, Q_1,Q_2,Q_3*, such that*

$$P_1 \leq \lambda_1 I_n \text{ and } Q_1 \leq \mu_1 I_n \tag{12}$$

$$\Pi = \begin{pmatrix} \Pi_{11} & 0 & I_N\otimes P_1 & 0 & \Pi_{15} & 0 & \Pi_{17} & \Pi_{18} \\ 0 & \Pi_{22} & 0 & I_N\otimes Q_1 & 0 & \Pi_{26} & \Pi_{27} & \Pi_{28} \\ * & 0 & -I_N\otimes I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & -I_N\otimes I_n & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & \Pi_{55} & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \Pi_{66} & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & \Pi_{77} & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & \Pi_{88} \end{pmatrix} < 0, \tag{13}$$

where

$$
\begin{aligned}
\Pi_{11} &= -2K^x \otimes P_1 + I_N \otimes (\lambda_1 \Sigma^x + P_2 + P_3 + U_1),\\
\Pi_{15} &= -c_2 L^x \otimes P_1 H,\\
\Pi_{17} &= -c_1 I_N \otimes P_1 \Gamma,\\
\Pi_{18} &= c_1 I_N \otimes P_1 \Gamma,\\
\Pi_{22} &= -2K^y \otimes Q_1 + I_N \otimes (\mu_1 \Sigma^y + Q_2 + Q_3 + U_2),\\
\Pi_{26} &= -c_2 L^y \otimes Q_1 H,\\
\Pi_{27} &= c_1 I_N \otimes Q_1 \Gamma,\\
\Pi_{28} &= -c_1 I_N \otimes Q_1 \Gamma,\\
\Pi_{55} &= -(1-\rho_1)(I_N \otimes P_2),\\
\Pi_{66} &= -(1-\rho_1)(I_N \otimes Q_2),\\
\Pi_{77} &= -(1-\rho_2)(I_N \otimes P_3),\\
\Pi_{88} &= -(1-\rho_2)(I_N \otimes Q_3).
\end{aligned}
$$

Proof. We choose a Lyapunov functional

$$
\begin{aligned}
& V(t, e(t), e(t-\zeta_1(t)), e(t-\zeta_2(t))) = e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} e(t)\\
& + \int_{t-\zeta_1(t))}^{t} e^T(s) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(s) ds\\
& + \int_{t-\zeta_2(t))}^{t} e^T(s) \begin{pmatrix} I_N \otimes P_3 & 0 \\ 0 & I_N \otimes Q_3 \end{pmatrix} e(s) ds.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \mathcal{L}V(t, e(t), e(t-\zeta_1(t)), e(t-\zeta_2(t)))\\
= \; & 2e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \Bigg[\begin{pmatrix} f^x(e^x(t)) \\ f^y(e^y(t)) \end{pmatrix}\\
& - c_2 \begin{pmatrix} (L^x \otimes H) e^x(t-\zeta_1(t)) \\ (L^y \otimes H) e^y(t-\zeta_1(t)) \end{pmatrix} - c_1 \begin{pmatrix} (I_N \otimes \Gamma)(e^x(t-\zeta_2(t)) - e^y(t-\zeta_2(t))) \\ (I_N \otimes \Gamma)(e^y(t-\zeta_2(t)) - e^x(t-\zeta_2(t))) \end{pmatrix}\\
& - \begin{pmatrix} (K^x \otimes I_n) e^x(t) \\ (K^y \otimes I_n) e^y(t) \end{pmatrix} \Bigg] + \mathrm{trace}\Big[\sigma^T(t, e(t)) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \sigma(t, e(t)) \Big]\\
& + e^T(t) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(t)\\
& - (1 - \dot{\zeta}_1(t)) e^T(t-\zeta_1(t)) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(t-\zeta_1(t))
\end{aligned}
$$

$$
\begin{aligned}
&+e^T(t)\begin{pmatrix} I_N\otimes P_3 & 0\\ 0 & I_N\otimes Q_3\end{pmatrix}e(t)\\
&-(1-\dot{\zeta}_2(t))e^T(t-\zeta_2(t))\begin{pmatrix} I_N\otimes P_3 & 0\\ 0 & I_N\otimes Q_3\end{pmatrix}e(t-\zeta_2(t))\\
\leq\ & 2e^{xT}(t)(I_N\otimes P_1)f^x(e^x(t))+2e^{yT}(t)(I_N\otimes Q_1)f^y(e^y(t))\\
&-2c_2e^{xT}(t)(I_N\otimes P_1)(L^x\otimes H)e^x(t-\zeta_1(t))-2c_2e^{yT}(t)(I_N\otimes Q_1)(L^y\otimes H)e^y(t-\zeta_1(t))\\
&-2e^{xT}(t)(I_N\otimes P_1)(K^x\otimes I_n)e^x(t)-2e^{yT}(I_N\otimes Q_1)(K^y\otimes I_n)e^y(t)\\
&-2c_1e^{xT}(t)(I_N\otimes P_1)(I_N\otimes\Gamma)(e^x(t-\zeta_2(t))-e^y(t-\zeta_2(t)))\\
&-2c_1e^{yT}(t)(I_N\otimes Q_1)(I_N\otimes\Gamma)(e^y(t-\zeta_2(t))-e^x(t-\zeta_2(t)))\\
&+\mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N\otimes P_1)\sigma^x(t,e^x(t))\big]+\mathrm{trace}\big[\sigma^{yT}(t,e^y(t))(I_N\otimes Q_1)\sigma^y(t,e^y(t))\big]\\
&+e^{xT}(t)(I_N\otimes P_2)e^x(t)+e^{yT}(t)(I_N\otimes Q_2)e^y(t)\\
&-(1-\dot{\zeta}_1(t))(e^{xT}(t-\zeta_1(t))(I_N\otimes P_2)e^x(t-\zeta_1(t))+e^{yT}(t-\zeta_1(t))(I_N\otimes Q_2)e^y(t-\zeta_1(t)))\\
&+e^{xT}(t)(I_N\otimes P_3)e^x(t)+e^{yT}(t)(I_N\otimes Q_3)e^y(t)\\
&-(1-\dot{\zeta}_2(t))(e^{xT}(t-\zeta_2(t))(I_N\otimes P_3)e^x(t-\zeta_2(t))+e^{yT}(t-\zeta_2(t))(I_N\otimes Q_3)e^y(t-\zeta_2(t)))\\
&+e^{xT}(t)(I_N\otimes U_1)e^x(t)-f^{xT}(e^x(t))f^x(e^x(t))+e^{yT}(t)(I_N\otimes U_2)e^y(t)-f^{yT}(e^y(t))f^y(e^y(t)),
\end{aligned}
$$

using Assumption **(A2)**. Moreover, with the assumption **(A4)**, one obtains that

$$
\begin{aligned}
&\mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N\otimes P_1)\sigma^x(t,e^x(t))\big]\\
=\ &\mathrm{trace}\left(\sum_{i=1}^N\sigma_i^{xT}(t,e_i^x(t))P_1\sigma_i^x(t,e_i^x(t))\right)\\
\leq\ &\lambda_{\max}(P_1)\sum_{i=1}^N\mathrm{trace}\Big(\sigma_i^{xT}(t,e_i(t))\sigma_i^x(t,e_i(t))\Big)\\
\leq\ &\lambda_1\sum_{i=1}^N e_i^{xT}(t)\Sigma^x e_i(t)\leq\lambda_1 e^{xT}(t)(I_N\otimes\Sigma^x)e^x(t).
\end{aligned}
$$

Similarly, one also obtains that

$$
\mathrm{trace}\big[\sigma^{yT}(t,e^y(t))(I_N\otimes Q_1)\sigma^y(t,e^y(t))\big]\leq\mu_1 e^{yT}(t)(I_N\otimes\Sigma^y)e^y(t).
$$

Therefore,

$$
\begin{aligned}
&\mathcal{L}V(t,e(t),e(t-\zeta_1(t)),e(t-\zeta_2(t)))\\
\leq\ & 2e^{xT}(t)(I_N\otimes P_1)f^x(e^x(t))+2e^{yT}(t)(I_N\otimes Q_1)f^y(e^y(t))\\
&-2c_2e^{xT}(t)(L^x\otimes P_1H)e^x(t-\zeta_1(t))-2c_2e^{yT}(t)(L^y\otimes Q_1H)e^y(t-\zeta_1(t))\\
&-2e^{xT}(t)(K^x\otimes P_1)e^x(t)-2e^{yT}(K^y\otimes Q_1)e^y(t)
\end{aligned}
$$

$$
\begin{aligned}
& -2c_1 e^{xT}(t)(I_N \otimes P_1\Gamma)(e^x(t-\zeta_2(t)) - e^y(t-\zeta_2(t))) \\
& -2c_1 e^{yT}(t)(I_N \otimes Q_1\Gamma)(e^y(t-\zeta_2(t)) - e^x(t-\zeta_2(t))) \\
& +\lambda_1 e^{xT}(t)(I_N \otimes \Sigma^x)e^x(t) + \mu_1 e^{yT}(t)(I_N \otimes \Sigma^y)e^y(t) \\
& +e^{xT}(t)(I_N \otimes P_2)e^x(t) + e^{yT}(t)(I_N \otimes Q_2)e^y(t) \\
& -(1-\rho_1)(e^{xT}(t-\zeta_1(t))(I_N \otimes P_2)e^x(t-\zeta_1(t)) + e^{yT}(t-\zeta_1(t))(I_N \otimes Q_2)e^y(t-\zeta_1(t))) \\
& +e^{xT}(t)(I_N \otimes P_3)e^x(t) + e^{yT}(t)(I_N \otimes Q_3)e^y(t) \\
& -(1-\rho_2)(e^{xT}(t-\zeta_2(t))(I_N \otimes P_3)e^x(t-\zeta_2(t)) + e^{yT}(t-\zeta_2(t))(I_N \otimes Q_3)e^y(t-\zeta_2(t))) \\
& +e^{xT}(t)(I_N \otimes U_1)e^x(t) - f^{xT}(e^x(t))f^x(e^x(t)) + e^{yT}(t)(I_N \otimes U_2)e^y(t) - f^{yT}(e^y(t))f^y(e^y(t)) \\
= \ & \mathbf{z}^T(t)\Pi\mathbf{z}(t) \le \lambda_{\max}(\Pi)\mathbf{z}^T(t)\mathbf{z}(t) \triangleq -\omega(\mathbf{z})
\end{aligned}
$$

where $\mathbf{z}^T(t) = (e^{xT}(t), e^{yT}, f^{xT}(e^{xT}(t)), f^{yT}(e^{yT}(t)), e^{xT}(t-\zeta_1(t)), e^{yT}(t-\zeta_1(t)), e^{xT}(t-\zeta_2(t)), e^{yT}(t-\zeta_2(t)))$ and

$$
\Pi = \begin{pmatrix}
\Pi_{11} & 0 & I_N \otimes P_1 & 0 & \Pi_{15} & 0 & \Pi_{17} & \Pi_{18} \\
0 & \Pi_{22} & 0 & I_N \otimes Q_1 & 0 & \Pi_{26} & \Pi_{27} & \Pi_{28} \\
* & 0 & -I_N \otimes I_n & 0 & 0 & 0 & 0 & 0 \\
0 & * & 0 & -I_N \otimes I_n & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & \Pi_{55} & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 & \Pi_{66} & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & \Pi_{77} & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & \Pi_{88}
\end{pmatrix},
$$

where

$$
\begin{gathered}
\Pi_{11} = -2K^x \otimes P_1 + I_N \otimes (\lambda_1\Sigma^x + P_2 + P_3 + U_1), \\
\Pi_{15} = -c_2 L^x \otimes P_1 H, \\
\Pi_{17} = -c_1 I_N \otimes P_1\Gamma, \\
\Pi_{18} = c_1 I_N \otimes P_1\Gamma, \\
\Pi_{22} = -2K^y \otimes Q_1 + I_N \otimes (\mu_1\Sigma^y + Q_2 + Q_3 + U_2), \\
\Pi_{26} = -c_2 L^y \otimes Q_1 H, \\
\Pi_{27} = c_1 I_N \otimes Q_1\Gamma, \\
\Pi_{28} = -c_1 I_N \otimes Q_1\Gamma, \\
\Pi_{55} = -(1-\rho_1)(I_N \otimes P_2), \\
\Pi_{66} = -(1-\rho_1)(I_N \otimes Q_2), \\
\Pi_{77} = -(1-\rho_2)(I_N \otimes P_3), \\
\Pi_{88} = -(1-\rho_2)(I_N \otimes Q_3).
\end{gathered}
$$

By Lemma 2.1, the proof is complete.

Note that each node in one layer is the replica of the node in another layer. Therefore, $\zeta_2(t)$ should be significantly smaller than $\zeta_1(t)$. In the following result, we assume that $\zeta_2(t) \equiv 0$.

Theorem 3.2 *Assume that* **(A1)-(A4)** *hold. under the controller* (11)*, the duplex network* (4) *and* (3) *are stochastically synchronized almost surely, if there exist two positive numbers* λ_1, μ_1 *and positive definite matrices* P_1, P_2, Q_1, Q_2*, such that*

$$P_1 \leq \lambda_1 I_n \ \textit{and}\ Q_1 \leq \mu_1 I_n \tag{14}$$

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & I_N \otimes P_1 & 0 & \Sigma_{15} & 0 \\ * & \Sigma_{22} & 0 & I_N \otimes Q_1 & 0 & \Sigma_{26} \\ * & 0 & -I_N \otimes I_n & 0 & 0 & 0 \\ 0 & * & 0 & -I_N \otimes I_n & 0 & 0 \\ * & 0 & 0 & 0 & \Sigma_{55} & 0 \\ 0 & * & 0 & 0 & 0 & \Sigma_{66} \end{pmatrix} < 0, \tag{15}$$

where

$$\begin{aligned} \Sigma_{11} &= -2K^x \otimes P_1 + I_N \otimes (\lambda_1 \Sigma^x + P_2 + U_1 - 2c_1 P_1 \Gamma), \\ \Sigma_{12} &= c_1 I_N \otimes (P_1 + Q_1)\Gamma \\ \Sigma_{15} &= -c_2 L^x \otimes P_1 H, \\ \Sigma_{22} &= -2K^y \otimes Q_1 + I_N \otimes (\mu_1 \Sigma^y + Q_2 + U_2 - 2c_1 Q_1 \Gamma), \\ \Sigma_{26} &= -c_2 L^y \otimes Q_1 H, \\ \Sigma_{55} &= -(1-\rho_1)(I_N \otimes P_2), \\ \Sigma_{66} &= -(1-\rho_1)(I_N \otimes Q_2), \end{aligned}$$

Proof. Consider the Lyapunov functional

$$\begin{aligned} V(t, e(t), e(t-\zeta_1(t))) &= e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} e(t) \\ &+ \int_{t-\zeta_1(t))}^{t} e^T(s) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(s) ds. \end{aligned}$$

Thus,

$$\begin{aligned} & \mathcal{L}V(t, e(t), e(t-\zeta_1(t))) \\ = \ & 2e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \Bigg[\begin{pmatrix} f^x(e^x(t)) \\ f^y(e^y(t)) \end{pmatrix} \\ & -c_2 \begin{pmatrix} (L^x \otimes H) e^x(t-\zeta_1(t)) \\ (L^y \otimes H) e^y(t-\zeta_1(t)) \end{pmatrix} - c_1 \begin{pmatrix} (I_N \otimes \Gamma)(e^x(t) - e^y(t)) \\ (I_N \otimes \Gamma)(e^y(t) - e^x(t)) \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
& -\begin{pmatrix} (K^x \otimes I_n)e^x(t) \\ (K^y \otimes I_n)e^y(t) \end{pmatrix} \Bigg] + \mathrm{trace}\Big[\sigma^T(t,e(t)) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \sigma(t,e(t))\Big] \\
& + e^T(t) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(t) \\
& -(1-\dot{\zeta}_1(t))e^T(t-\zeta_1(t)) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(t-\zeta_1(t)) \\
\leq \; & 2e^{xT}(t)(I_N \otimes P_1)f^x(e^x(t)) + 2e^{yT}(t)(I_N \otimes Q_1)f^y(e^y(t)) \\
& -2c_2 e^{xT}(t)(I_N \otimes P_1)(L^x \otimes H)e^x(t-\zeta_1(t)) - 2c_2 e^{yT}(t)(I_N \otimes Q_1)(L^y \otimes H)e^y(t-\zeta_1(t)) \\
& -2e^{xT}(t)(I_N \otimes P_1)(K^x \otimes I_n)e^x(t) - 2e^{yT}(I_N \otimes Q_1)(K^y \otimes I_n)e^y(t) \\
& -2c_1 e^{xT}(t)(I_N \otimes P_1)(I_N \otimes \Gamma)(e^x(t) - e^y(t)) \\
& -2c_1 e^{yT}(t)(I_N \otimes Q_1)(I_N \otimes \Gamma)(e^y(t) - e^x(t)) \\
& +\mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N \otimes P_1)\sigma^x(t,e^x(t))\big] + \mathrm{trace}\big[\sigma^{yT}(t,e^y(t))(I_N \otimes Q_1)\sigma^y(t,e^y(t))\big] \\
& +e^{xT}(t)(I_N \otimes P_2)e^x(t) + e^{yT}(t)(I_N \otimes Q_2)e^y(t) \\
& -(1-\dot{\zeta}_1(t))(e^{xT}(t-\zeta_1(t))(I_N \otimes P_2)e^x(t-\zeta_1(t)) + e^{yT}(t-\zeta_1(t))(I_N \otimes Q_2)e^y(t-\zeta_1(t))) \\
& +e^{xT}(t)(I_N \otimes U_1)e^x(t) - f^{xT}(e^x(t))f^x(e^x(t)) + e^{yT}(t)(I_N \otimes U_2)e^y(t) - f^{yT}(e^y(t))f^y(e^y(t)),
\end{aligned}
$$

using Assumption **(A2)**. Moreover, with the assumption **(A4)**, one obtains that

$$
\begin{aligned}
& \mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N \otimes P_1)\sigma^x(t,e^x(t))\big] = \mathrm{trace}\left(\sum_{i=1}^{N} \sigma_i^{xT}(t,e_i^x(t))P_1\sigma_i^x(t,e_i^x(t))\right) \\
\leq \; & \lambda\mathrm{max}(P_1)\sum_{i=1}^{N}\mathrm{trace}\Big(\sigma_i^{xT}(t,e_i(t))\sigma_i^x(t,e_i(t))\Big) \\
\leq \; & \lambda_1 \sum_{i=1}^{N} e_i^{xT}(t)\Sigma^x e_i(t) \leq \lambda_1 e^{xT}(t)(I_N \otimes \Sigma^x)e^x(t).
\end{aligned}
$$

Similarly, one also obtains that

$$
\mathrm{trace}\big[\sigma^{yT}(t,e^y(t))(I_N \otimes Q_1)\sigma^y(t,e^y(t))\big] \leq \mu_1 e^{yT}(t)(I_N \otimes \Sigma^y)e^y(t).
$$

Therefore,

$$
\begin{aligned}
& \mathcal{L}V(t,e(t),e(t-\zeta_1(t))) \\
\leq \; & 2e^{xT}(t)(I_N \otimes P_1)f^x(e^x(t)) + 2e^{yT}(t)(I_N \otimes Q_1)f^y(e^y(t)) \\
& -2c_2 e^{xT}(t)(L^x \otimes P_1H)e^x(t-\zeta_1(t)) - 2c_2 e^{yT}(t)(L^y \otimes Q_1H)e^y(t-\zeta_1(t)) \\
& -2e^{xT}(t)(K^x \otimes P_1)e^x(t) - 2e^{yT}(K^y \otimes Q_1)e^y(t) \\
& -2c_1 e^{xT}(t)(I_N \otimes P_1\Gamma)(e^x(t) - e^y(t))
\end{aligned}
$$

$$
\begin{aligned}
& -2c_1 e^{yT}(t)(I_N \otimes Q_1\Gamma)(e^y(t) - e^x(t)) \\
& +\lambda_1 e^{xT}(t)(I_N \otimes \Sigma^x)e^x(t) + \mu_1 e^{yT}(t)(I_N \otimes \Sigma^y)e^y(t) \\
& +e^{xT}(t)(I_N \otimes P_2)e^x(t) + e^{yT}(t)(I_N \otimes Q_2)e^y(t) \\
& -(1-\rho_1)(e^{xT}(t-\zeta_1(t))(I_N \otimes P_2)e^x(t-\zeta_1(t)) \\
& +e^{yT}(t-\zeta_1(t))(I_N \otimes Q_2)e^y(t-\zeta_1(t))) \\
& +e^{xT}(t)(I_N \otimes U_1)e^x(t) - f^{xT}(e^x(t))f^x(e^x(t)) \\
& +e^{yT}(t)(I_N \otimes U_2)e^y(t) - f^{yT}(e^y(t))f^y(e^y(t)) \\
= \ & \mathbf{z}^T(t)\Sigma\mathbf{z}(t) \leq \lambda\max(\Sigma)\mathbf{z}^T(t)\mathbf{z}(t) \triangleq -\omega(\mathbf{z})
\end{aligned}
$$

where $\mathbf{z}^T(t) = (e^{xT}(t), e^{yT}, f^{xT}(e^x(t)), f^{yT}(e^y(t)), e^{xT}(t-\zeta_1(t)), e^{yT}(t-\zeta_1(t)))$ and

$$
\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} & I_N \otimes P_1 & 0 & \Sigma_{15} & 0 \\
* & \Sigma_{22} & 0 & I_N \otimes Q_1 & 0 & \Sigma_{26} \\
* & 0 & -I_N \otimes I_n & 0 & 0 & 0 \\
0 & * & 0 & -I_N \otimes I_n & 0 & 0 \\
* & 0 & 0 & 0 & \Sigma_{55} & 0 \\
0 & * & 0 & 0 & 0 & \Sigma_{66}
\end{pmatrix},
$$

where

$$
\begin{aligned}
\Sigma_{11} &= -2K^x \otimes P_1 + I_N \otimes (\lambda_1\Sigma^x + P_2 + U_1 - 2c_1P_1\Gamma), \\
\Sigma_{12} &= c_1 I_N \otimes (P_1 + Q_1)\Gamma \\
\Sigma_{15} &= -c_2 L^x \otimes P_1 H, \\
\Sigma_{22} &= -2K^y \otimes Q_1 + I_N \otimes (\mu_1\Sigma^y + Q_2 + U_2 - 2c_1Q_1\Gamma), \\
\Sigma_{26} &= -c_2 L^y \otimes Q_1 H, \\
\Sigma_{55} &= -(1-\rho_1)(I_N \otimes P_2), \\
\Sigma_{66} &= -(1-\rho_1)(I_N \otimes Q_2),
\end{aligned}
$$

By Lemma 2.1, the proof is complete.

Remark 3.1 *Comparing* (13) *and* (15)*, one can see that it is easier to obtain a feasible solution to the linear inequalities in the case* $\zeta_2(t) \equiv 0$*. Moreover, different from Theorem 3.1, one can easily find that* (15) *is diagonally dominant if the inter-layer coupling strength* c_1 *is large enough and the intra-layer coupling strength is small enough. In such situation, there always exist feasible solutions to* (14) *and* (15) *without control input which means that the intra-layer synchronization would be automatically achieved without any control input.*

3.2 Synchronization with adaptive state-feedback controller

Assume that the controller is defined as follows

$$\begin{cases} u_i^x(t) = -k_i^x(t)e_i^x(t), \\ u_i^y(t) = -k_i^y(t)e_i^y(t), \end{cases}$$

for $i = 1, 2, \cdots, N$, i.e.,

$$\begin{pmatrix} u^x(t) \\ u^y(t) \end{pmatrix} = \begin{pmatrix} -(K^x(t) \otimes I_n)e^x(t) \\ -(K^y(t) \otimes I_n)e^y(t) \end{pmatrix}, \tag{16}$$

where $u^x(t) = (u_1^{xT}(t), \cdots, u_N^{xT}(t))^T$, $u^y(t) = (u_1^{yT}(t), \cdots, u_N^{yT}(t))^T$, $K^x(t) = \mathrm{diag}\{k_1^x(t), \cdots, k_N^x(t)\}$ and $K^y(t) = \mathrm{diag}\{k_1^y(t), \cdots, k_N^y(t)\}$. Besides, $k_i^x(t)$'s and $k_i^y(t)$'s are adaptive control gains and updated by

$$\begin{cases} \dot{k}_i^x(t) = \delta_i^x e_i^{xT}(t)e_i^x(t), \\ \dot{k}_i^y(t) = \delta_i^y e_i^{yT}(t)e_i^y(t), \end{cases} \tag{17}$$

where δ_i^x, δ_i^y are some positive constants to be designed.

Theorem 3.3 *Assume that* **(A1)**-**(A4)** *hold. Under the controller* (16)*, the duplex network* (4) *and* (3) *are stochastically synchronized almost surely, if there exist two positive numbers λ_1, μ_1 and positive definite matrices $P_1 = p_1 I_n, P_2, P_3, Q_1 = q_1 I_n, Q_2, Q_3$, such that*

$$P_1 \le \lambda_1 I_n \text{ and } Q_1 \le \mu_1 I_n \tag{18}$$

$$\Xi = \begin{pmatrix} \Xi_{11} & 0 & I_N \otimes P_1 & 0 & \Xi_{15} & 0 & \Xi_{17} & \Xi_{18} \\ 0 & \Xi_{22} & 0 & I_N \otimes Q_1 & 0 & \Xi_{26} & \Xi_{27} & \Xi_{28} \\ * & 0 & -I_N \otimes I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & -I_N \otimes I_n & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & \Xi_{55} & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \Xi_{66} & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & \Xi_{77} & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & \Xi_{88} \end{pmatrix} < 0, \tag{19}$$

where

$$\Xi_{11} = -2\bar{K}^x \otimes P_1 + I_N \otimes (\lambda_1 \Sigma^x + P_2 + P_3 + U_1),$$
$$\Xi_{15} = -c_2 L^x \otimes P_1 H,$$
$$\Xi_{17} = -c_1 I_N \otimes P_1 \Gamma,$$

$$
\begin{aligned}
\Xi_{18} &= c_1 I_N \otimes P_1 \Gamma, \\
\Xi_{22} &= -2\bar{K}^y \otimes Q_1 + I_N \otimes (\mu_1 \Sigma^y + Q_2 + Q_3 + U_2), \\
\Xi_{26} &= -c_2 L^y \otimes Q_1 H, \\
\Xi_{27} &= c_1 I_N \otimes Q_1 \Gamma, \\
\Xi_{28} &= -c_1 I_N \otimes Q_1 \Gamma, \\
\Xi_{55} &= -(1-\rho_1)(I_N \otimes P_2), \\
\Xi_{66} &= -(1-\rho_1)(I_N \otimes Q_2), \\
\Xi_{77} &= -(1-\rho_2)(I_N \otimes P_3), \\
\Xi_{88} &= -(1-\rho_2)(I_N \otimes Q_3),
\end{aligned}
$$

and $\bar{K}^x = \mathrm{diag}\{\bar{k}_1^x, \cdots, \bar{k}_N^x\}$, $\bar{K}^y = \mathrm{diag}\{\bar{k}_1^y, \cdots, \bar{k}_N^y\}$.

Proof. Consider the Lyapunov functional

$$
\begin{aligned}
&V(t, e(t), e(t-\zeta_1(t)), e(t-\zeta_2(t))) = e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} e(t) \\
&+ \int_{t-\zeta_1(t))}^{t} e^T(s) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(s) ds + p_1 \sum_{i=1}^{N} \frac{1}{\delta_i^x} (k_i^x(t) - \bar{k}_i^x)^2 \\
&+ \int_{t-\zeta_2(t))}^{t} e^T(s) \begin{pmatrix} I_N \otimes P_3 & 0 \\ 0 & I_N \otimes Q_3 \end{pmatrix} e(s) ds + q_1 \sum_{i=1}^{N} \frac{1}{\delta_i^y} (k_i^y(t) - \bar{k}_i^y)^2
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& \mathcal{L}V(t, e(t), e(t-\zeta_1(t)), e(t-\zeta_2(t))) \\
= \; & 2e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \\
& \Bigg[\begin{pmatrix} f^x(e^x(t)) \\ f^y(e^y(t)) \end{pmatrix} - c_2 \begin{pmatrix} (L^x \otimes H) e^x(t-\zeta_1(t)) \\ (L^y \otimes H) e^y(t-\zeta_1(t)) \end{pmatrix} \\
& - c_1 \begin{pmatrix} (I_N \otimes \Gamma)(e^x(t-\zeta_2(t)) - e^y(t-\zeta_2(t))) \\ (I_N \otimes \Gamma)(e^y(t-\zeta_2(t)) - e^x(t-\zeta_2(t))) \end{pmatrix} \\
& - \begin{pmatrix} (K^x(t) \otimes I_n) e^x(t) \\ (K^y(t) \otimes I_n) e^y(t) \end{pmatrix} \Bigg] \\
& + \mathrm{trace}\Big[\sigma^T(t, e(t)) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \sigma(t, e(t)) \Big] \\
& + 2e^{xT}(t)(K^x(t) \otimes P_1) e^x(t) - 2e^{xT}(\bar{K}^x \otimes P_1) e^x(t) + 2e^{yT}(t)(K^y(t) \otimes Q_1) e^y(t) \\
& - 2e^{yT}(\bar{K}^y \otimes P_1) e^y(t)
\end{aligned}
$$

$$
\begin{aligned}
& +e^T(t)\begin{pmatrix} I_N\otimes P_2 & 0 \\ 0 & I_N\otimes Q_2 \end{pmatrix} e(t) \\
& -(1-\dot{\zeta}_1(t))e^T(t-\zeta_1(t))\begin{pmatrix} I_N\otimes P_2 & 0 \\ 0 & I_N\otimes Q_2 \end{pmatrix} e(t-\zeta_1(t)) \\
& +e^T(t)\begin{pmatrix} I_N\otimes P_3 & 0 \\ 0 & I_N\otimes Q_3 \end{pmatrix} e(t) \\
& -(1-\dot{\zeta}_2(t))e^T(t-\zeta_2(t))\begin{pmatrix} I_N\otimes P_3 & 0 \\ 0 & I_N\otimes Q_3 \end{pmatrix} e(t-\zeta_2(t)) \\
\leq\ & 2e^{xT}(t)(I_N\otimes P_1)f^x(e^x(t)) + 2e^{yT}(t)(I_N\otimes Q_1)f^y(e^y(t)) \\
& -2c_2e^{xT}(t)(I_N\otimes P_1)(L^x\otimes H)e^x(t-\zeta_1(t)) \\
& -2c_2e^{yT}(t)(I_N\otimes Q_1)(L^y\otimes H)e^y(t-\zeta_1(t)) \\
& -2e^{xT}(t)(I_N\otimes P_1)(\bar{K}^x\otimes I_n)e^x(t) \\
& -2e^{yT}(I_N\otimes Q_1)(\bar{K}^y\otimes I_n)e^y(t) \\
& -2c_1e^{xT}(t)(I_N\otimes P_1)(I_N\otimes\Gamma)(e^x(t-\zeta_2(t)) - e^y(t-\zeta_2(t))) \\
& -2c_1e^{yT}(t)(I_N\otimes Q_1)(I_N\otimes\Gamma)(e^y(t-\zeta_2(t)) - e^x(t-\zeta_2(t))) \\
& +\mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N\otimes P_1)\sigma^x(t,e^x(t))\big] \\
& +\mathrm{trace}\big[\sigma^{yT}(t,e^y(t))(I_N\otimes Q_1)\sigma^y(t,e^y(t))\big] \\
& +e^{xT}(t)(I_N\otimes P_2)e^x(t) + e^{yT}(t)(I_N\otimes Q_2)e^y(t) \\
& -(1-\dot{\zeta}_1(t))(e^{xT}(t-\zeta_1(t))(I_N\otimes P_2)e^x(t-\zeta_1(t)) \\
& +e^{yT}(t-\zeta_1(t))(I_N\otimes Q_2)e^y(t-\zeta_1(t))) \\
& +e^{xT}(t)(I_N\otimes P_3)e^x(t) + e^{yT}(t)(I_N\otimes Q_3)e^y(t) \\
& -(1-\dot{\zeta}_2(t))(e^{xT}(t-\zeta_2(t))(I_N\otimes P_3)e^x(t-\zeta_2(t)) \\
& +e^{yT}(t-\zeta_2(t))(I_N\otimes Q_3)e^y(t-\zeta_2(t))) \\
& +e^{xT}(t)(I_N\otimes U_1)e^x(t) - f^{xT}(e^x(t))f^x(e^x(t)) \\
& +e^{yT}(t)(I_N\otimes U_2)e^y(t) - f^{yT}(e^y(t))f^y(e^y(t)),
\end{aligned}
$$

using Assumption **(A2)**. Moreover, with the assumption **(A4)**, one obtains that

$$
\begin{aligned}
& \mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N\otimes P_1)\sigma^x(t,e^x(t))\big] \\
=\ & \mathrm{trace}\left(\sum_{i=1}^{N}\sigma_i^{xT}(t,e_i^x(t))P_1\sigma_i^x(t,e_i^x(t))\right) \\
\leq\ & \lambda\mathrm{max}(P_1)\sum_{i=1}^{N}\mathrm{trace}\Big(\sigma_i^{xT}(t,e_i(t))\sigma_i^x(t,e_i(t))\Big)
\end{aligned}
$$

$$\leq \quad \lambda_1 \sum_{i=1}^{N} e_i^{xT}(t)\Sigma^x e_i(t) \leq \lambda_1 e^{xT}(t)(I_N \otimes \Sigma^x)e^x(t).$$

Similarly, one also obtains that

$$\mathrm{trace}\big[\sigma^{yT}(t, e^y(t))(I_N \otimes Q_1)\sigma^y(t, e^y(t))\big] \leq \mu_1 e^{yT}(t)(I_N \otimes \Sigma^y)e^y(t).$$

Therefore,

$$\begin{aligned}
& \mathcal{L}V(t, e(t), e(t-\zeta_1(t)), e(t-\zeta_2(t))) \\
\leq \; & 2e^{xT}(t)(I_N \otimes P_1)f^x(e^x(t)) + 2e^{yT}(t)(I_N \otimes Q_1)f^y(e^y(t)) \\
& -2c_2 e^{xT}(t)(L^x \otimes P_1 H)e^x(t-\zeta_1(t)) - 2c_2 e^{yT}(t)(L^y \otimes Q_1 H)e^y(t-\zeta_1(t)) \\
& -2e^{xT}(t)(\bar{K}^x \otimes P_1)e^x(t) - 2e^{yT}(\bar{K}^y \otimes Q_1)e^y(t) \\
& -2c_1 e^{xT}(t)(I_N \otimes P_1\Gamma)(e^x(t-\zeta_2(t)) - e^y(t-\zeta_2(t))) \\
& -2c_1 e^{yT}(t)(I_N \otimes Q_1\Gamma)(e^y(t-\zeta_2(t)) - e^x(t-\zeta_2(t))) \\
& +\lambda_1 e^{xT}(t)(I_N \otimes \Sigma^x)e^x(t) + \mu_1 e^{yT}(t)(I_N \otimes \Sigma^y)e^y(t) \\
& +e^{xT}(t)(I_N \otimes P_2)e^x(t) + e^{yT}(t)(I_N \otimes Q_2)e^y(t) \\
& -(1-\rho_1)(e^{xT}(t-\zeta_1(t))(I_N \otimes P_2)e^x(t-\zeta_1(t)) \\
& +e^{yT}(t-\zeta_1(t))(I_N \otimes Q_2)e^y(t-\zeta_1(t))) \\
& +e^{xT}(t)(I_N \otimes P_3)e^x(t) + e^{yT}(t)(I_N \otimes Q_3)e^y(t) \\
& -(1-\rho_2)(e^{xT}(t-\zeta_2(t))(I_N \otimes P_3)e^x(t-\zeta_2(t)) \\
& +e^{yT}(t-\zeta_2(t))(I_N \otimes Q_3)e^y(t-\zeta_2(t))) \\
& +e^{xT}(t)(I_N \otimes U_1)e^x(t) - f^{xT}(e^x(t))f^x(e^x(t)) \\
& +e^{yT}(t)(I_N \otimes U_2)e^y(t) - f^{yT}(e^y(t))f^y(e^y(t)) \\
= \; & \mathbf{z}^T(t)\Xi\mathbf{z}(t) \leq \lambda_{\max}(\Xi)\mathbf{z}^T(t)\mathbf{z}(t) \triangleq -\omega(\mathbf{z})
\end{aligned}$$

where $\mathbf{z}^T(t) = (e^{xT}(t), e^{yT}, f^{xT}(e^{xT}(t)), f^{yT}(e^{yT}(t)), e^{xT}(t-\zeta_1(t)), e^{yT}(t-\zeta_1(t)), e^{xT}(t-\zeta_2(t)), e^{yT}(t-\zeta_2(t)))$ and

$$\Xi = \begin{pmatrix}
\Xi_{11} & 0 & I_N \otimes P_1 & 0 & \Xi_{15} & 0 & \Xi_{17} & \Xi_{18} \\
0 & \Xi_{22} & 0 & I_N \otimes Q_1 & 0 & \Xi_{26} & \Xi_{27} & \Xi_{28} \\
* & 0 & -I_N \otimes I_n & 0 & 0 & 0 & 0 & 0 \\
0 & * & 0 & -I_N \otimes I_n & 0 & 0 & 0 & 0 \\
* & 0 & 0 & 0 & \Xi_{55} & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 & \Xi_{66} & 0 & 0 \\
* & * & 0 & 0 & 0 & 0 & \Xi_{77} & 0 \\
* & * & 0 & 0 & 0 & 0 & 0 & \Xi_{88}
\end{pmatrix},$$

where

$$\Xi_{11} = -2\bar{K}^x \otimes P_1 + I_N \otimes (\lambda_1\Sigma^x + P_2 + P_3 + U_1),$$

$$\begin{aligned}
\Xi_{15} &= -c_2 L^x \otimes P_1 H, \\
\Xi_{17} &= -c_1 I_N \otimes P_1 \Gamma, \\
\Xi_{18} &= c_1 I_N \otimes P_1 \Gamma, \\
\Xi_{22} &= -2\bar{K}^y \otimes Q_1 + I_N \otimes (\mu_1 \Sigma^y + Q_2 + Q_3 + U_2), \\
\Xi_{26} &= -c_2 L^y \otimes Q_1 H, \\
\Xi_{27} &= c_1 I_N \otimes Q_1 \Gamma, \\
\Xi_{28} &= -c_1 I_N \otimes Q_1 \Gamma, \\
\Xi_{55} &= -(1-\rho_1)(I_N \otimes P_2), \\
\Xi_{66} &= -(1-\rho_1)(I_N \otimes Q_2), \\
\Xi_{77} &= -(1-\rho_2)(I_N \otimes P_3), \\
\Xi_{88} &= -(1-\rho_2)(I_N \otimes Q_3).
\end{aligned}$$

By Lemma 2.1, the proof is complete.

For the special case where $\zeta_2(t) \equiv 0$, we have the following result.

Theorem 3.4 *Assume that* **(A1)**-**(A4)** *hold. Under the controller* (16)*, the duplex network* (4) *and* (3) *are stochastically synchronized almost surely, if there exist two positive numbers* λ_1, μ_1 *and positive definite matrices* $P_1 = p_1 I_n, P_2, Q_1 = q_1 I_n, Q_2,$*, such that*

$$P_1 \le \lambda_1 I_n \text{ and } Q_1 \le \mu_1 I_n \tag{20}$$

$$\Lambda = \begin{pmatrix}
\Lambda_{11} & \Lambda_{12} & I_N \otimes P_1 & 0 & \Lambda_{15} & 0 \\
* & \Lambda_{22} & 0 & I_N \otimes Q_1 & 0 & \Lambda_{26} \\
* & 0 & -I_N \otimes I_n & 0 & 0 & 0 \\
0 & * & 0 & -I_N \otimes I_n & 0 & 0 \\
* & 0 & 0 & 0 & \Lambda_{55} & 0 \\
0 & * & 0 & 0 & 0 & \Lambda_{66}
\end{pmatrix} < 0, \tag{21}$$

where

$$\begin{aligned}
\Lambda_{11} &= -2\bar{K}^x \otimes P_1 + I_N \otimes (\lambda_1 \Sigma^x + P_2 + U_1 - 2c_1 P_1 \Gamma), \\
\Lambda_{12} &= c_1 I_N \otimes (P_1 + Q_1)\Gamma \\
\Lambda_{15} &= -c_2 L^x \otimes P_1 H, \\
\Lambda_{22} &= -2\bar{K}^y \otimes Q_1 + I_N \otimes (\mu_1 \Sigma^y + Q_2 + U_2 - 2c_1 Q_1 \Gamma), \\
\Lambda_{26} &= -c_2 L^y \otimes Q_1 H, \\
\Lambda_{55} &= -(1-\rho_1)(I_N \otimes P_2), \\
\Lambda_{66} &= -(1-\rho_1)(I_N \otimes Q_2),
\end{aligned}$$

and $\bar{K}^x = \mathrm{diag}\{\bar{k}_1^x, \cdots, \bar{k}_N^x\}$, $\bar{K}^y = \mathrm{diag}\{\bar{k}_1^y, \cdots, \bar{k}_N^y\}$.

Proof. Consider the Lyapunov functional

$$\begin{aligned}
& V(t, e(t), e(t-\zeta_1(t))) \\
= \; & e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} e(t) \\
& + \int_{t-\zeta_1(t))}^{t} e^T(s) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(s) ds \\
& + p_1 \sum_{i=1}^{N} \frac{1}{\delta_i^x} (k_i^x(t) - \bar{k}_i^x)^2 + q_1 \sum_{i=1}^{N} \frac{1}{\delta_i^y} (k_i^y(t) - \bar{k}_i^y)^2
\end{aligned}$$

Thus,

$$\begin{aligned}
& \mathcal{L}V(t, e(t), e(t-\zeta_1(t))) \\
= \; & 2e^T(t) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \\
& \Bigg[\begin{pmatrix} f^x(e^x(t)) \\ f^y(e^y(t)) \end{pmatrix} - c_2 \begin{pmatrix} (L^x \otimes H) e^x(t-\zeta_1(t)) \\ (L^y \otimes H) e^y(t-\zeta_1(t)) \end{pmatrix} \\
& - c_1 \begin{pmatrix} (I_N \otimes \Gamma)(e^x(t) - e^y(t)) \\ (I_N \otimes \Gamma)(e^y(t) - e^x(t)) \end{pmatrix} \\
& - \begin{pmatrix} (K^x(t) \otimes I_n) e^x(t) \\ (K^y(t) \otimes I_n) e^y(t) \end{pmatrix} \Bigg] \\
& + \mathrm{trace}\Big[\sigma^T(t, e(t)) \begin{pmatrix} I_N \otimes P_1 & 0 \\ 0 & I_N \otimes Q_1 \end{pmatrix} \sigma(t, e(t)) \Big] \\
& + 2e^{xT}(t)(K^x(t) \otimes P_1) e^x(t) \\
& - 2e^{xT}(\bar{K}^x \otimes P_1) e^x(t) + 2e^{yT}(t)(K^y(t) \otimes Q_1) e^y(t) \\
& - 2e^{yT}(\bar{K}^y \otimes P_1) e^y(t) + e^T(t) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(t) \\
& - (1 - \dot{\zeta}_1(t)) e^T(t-\zeta_1(t)) \begin{pmatrix} I_N \otimes P_2 & 0 \\ 0 & I_N \otimes Q_2 \end{pmatrix} e(t-\zeta_1(t)) \\
\leq \; & 2e^{xT}(t)(I_N \otimes P_1) f^x(e^x(t)) + 2e^{yT}(t)(I_N \otimes Q_1) f^y(e^y(t)) \\
& - 2c_2 e^{xT}(t)(I_N \otimes P_1)(L^x \otimes H) e^x(t-\zeta_1(t)) \\
& - 2c_2 e^{yT}(t)(I_N \otimes Q_1)(L^y \otimes H) e^y(t-\zeta_1(t)) \\
& - 2e^{xT}(t)(I_N \otimes P_1)(\bar{K}^x \otimes I_n) e^x(t) \\
& - 2e^{yT}(I_N \otimes Q_1)(\bar{K}^y \otimes I_n) e^y(t) \\
& - 2c_1 e^{xT}(t)(I_N \otimes P_1)(I_N \otimes \Gamma)(e^x(t) - e^y(t))
\end{aligned}$$

$$
\begin{aligned}
&-2c_1e^{yT}(t)(I_N\otimes Q_1)(I_N\otimes\Gamma)(e^y(t)-e^x(t))\\
&+\mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N\otimes P_1)\sigma^x(t,e^x(t))\big]\\
&+\mathrm{trace}\big[\sigma^{yT}(t,e^y(t))(I_N\otimes Q_1)\sigma^y(t,e^y(t))\big]\\
&+e^{xT}(t)(I_N\otimes P_2)e^x(t)+e^{yT}(t)(I_N\otimes Q_2)e^y(t)\\
&-(1-\dot{\zeta}_1(t))(e^{xT}(t-\zeta_1(t))(I_N\otimes P_2)e^x(t-\zeta_1(t))\\
&+e^{yT}(t-\zeta_1(t))(I_N\otimes Q_2)e^y(t-\zeta_1(t)))\\
&+e^{xT}(t)(I_N\otimes U_1)e^x(t)-f^{xT}(e^x(t))f^x(e^x(t))\\
&+e^{yT}(t)(I_N\otimes U_2)e^y(t)-f^{yT}(e^y(t))f^y(e^y(t)),
\end{aligned}
$$

using Assumption **(A2)**. Moreover, with the assumption **(A4)**, one obtains that

$$
\begin{aligned}
&\mathrm{trace}\big[\sigma^{xT}(t,e^x(t))(I_N\otimes P_1)\sigma^x(t,e^x(t))\big]\\
= \;&\mathrm{trace}\left(\sum_{i=1}^{N}\sigma_i^{xT}(t,e_i^x(t))P_1\sigma_i^x(t,e_i^x(t))\right)\\
\le \;&\lambda_{\max}(P_1)\sum_{i=1}^{N}\mathrm{trace}\Big(\sigma_i^{xT}(t,e_i(t))\sigma_i^x(t,e_i(t))\Big)\\
\le \;&\lambda_1\sum_{i=1}^{N}e_i^{xT}(t)\Sigma^x e_i(t)\le\lambda_1e^{xT}(t)(I_N\otimes\Sigma^x)e^x(t).
\end{aligned}
$$

Similarly, one also obtains that

$$
\mathrm{trace}\big[\sigma^{yT}(t,e^y(t))(I_N\otimes Q_1)\sigma^y(t,e^y(t))\big]\le\mu_1e^{yT}(t)(I_N\otimes\Sigma^y)e^y(t).
$$

Therefore,

$$
\begin{aligned}
&\mathcal{L}V(t,e(t),e(t-\zeta_1(t)),e(t))\\
\le \;&2e^{xT}(t)(I_N\otimes P_1)f^x(e^x(t))+2e^{yT}(t)(I_N\otimes Q_1)f^y(e^y(t))\\
&-2c_2e^{xT}(t)(L^x\otimes P_1H)e^x(t-\zeta_1(t))\\
&-2c_2e^{yT}(t)(L^y\otimes Q_1H)e^y(t-\zeta_1(t))\\
&-2e^{xT}(t)(\bar{K}^x\otimes P_1)e^x(t)\\
&-2e^{yT}(\bar{K}^y\otimes Q_1)e^y(t)\\
&-2c_1e^{xT}(t)(I_N\otimes P_1\Gamma)(e^x(t)-e^y(t))\\
&-2c_1e^{yT}(t)(I_N\otimes Q_1\Gamma)(e^y(t)-e^x(t))\\
&+\lambda_1e^{xT}(t)(I_N\otimes\Sigma^x)e^x(t)+\mu_1e^{yT}(t)(I_N\otimes\Sigma^y)e^y(t)\\
&+e^{xT}(t)(I_N\otimes P_2)e^x(t)+e^{yT}(t)(I_N\otimes Q_2)e^y(t)
\end{aligned}
$$

$$
\begin{aligned}
& -(1-\rho_1)(e^{xT}(t-\zeta_1(t))(I_N \otimes P_2)e^x(t-\zeta_1(t)) \\
& +e^{yT}(t-\zeta_1(t))(I_N \otimes Q_2)e^y(t-\zeta_1(t))) \\
& +e^{xT}(t)(I_N \otimes U_1)e^x(t) \\
& -f^{xT}(e^x(t))f^x(e^x(t)) + e^{yT}(t)(I_N \otimes U_2)e^y(t) - f^{yT}(e^y(t))f^y(e^y(t)) \\
= \; & \mathbf{z}^T(t)\Lambda\mathbf{z}(t) \le \lambda_{\max}(\Lambda)\mathbf{z}^T(t)\mathbf{z}(t) \triangleq -\omega(\mathbf{z})
\end{aligned}
$$

where $\mathbf{z}^T(t) = (e^{xT}(t), e^{yT}, f^{xT}(e^x(t)), f^{yT}(e^y(t)), e^{xT}(t-\zeta_1(t)), e^{yT}(t-\zeta_1(t)))$ and

$$
\Lambda = \begin{pmatrix}
\Lambda_{11} & \Lambda_{12} & I_N \otimes P_1 & 0 & \Lambda_{15} & 0 \\
* & \Lambda_{22} & 0 & I_N \otimes Q_1 & 0 & \Lambda_{26} \\
* & 0 & -I_N \otimes I_n & 0 & 0 & 0 \\
0 & * & 0 & -I_N \otimes I_n & 0 & 0 \\
* & 0 & 0 & 0 & \Lambda_{55} & 0 \\
0 & * & 0 & 0 & 0 & \Lambda_{66}
\end{pmatrix},
$$

where

$$
\begin{aligned}
\Lambda_{11} &= -2\bar{K}^x \otimes P_1 + I_N \otimes (\lambda_1\Sigma^x + P_2 + U_1 - 2c_1P_1\Gamma), \\
\Lambda_{12} &= c_1 I_N \otimes (P_1 + Q_1)\Gamma \\
\Lambda_{15} &= -c_2 L^x \otimes P_1 H, \\
\Lambda_{22} &= -2\bar{K}^y \otimes Q_1 + I_N \otimes (\mu_1\Sigma^y + Q_2 + U_2 - 2c_1Q_1\Gamma), \\
\Lambda_{26} &= -c_2 L^y \otimes Q_1 H, \\
\Lambda_{55} &= -(1-\rho_1)(I_N \otimes P_2), \\
\Lambda_{66} &= -(1-\rho_1)(I_N \otimes Q_2),
\end{aligned}
$$

By Lemma 2.1, the proof is complete.

Remark 3.2 *Comparing* (19) *and* (21)*, one finds that it is easier to obtain a feasible solution to the linear inequalities in the case* $\zeta_2(t) \equiv 0$*. Moreover, similar to Remark 3.1, the intra-layer synchronization would be automatically achieved without any control input when the inter-layer coupling strength* c_1 *is large enough and the intra-layer coupling strength* c_2 *is small enough.*

4 Numerical simulations

To verify our main results, we construct a Chua-Chua chaotic system in this section.

A Chua-Chua chaotic system. In the synchronous states (3), we choose the following settings of the Chua chaotic systems

$$
\tilde{f}^x(z) = A^x z - \alpha^x h^x(z) \text{ and } \tilde{f}^y(z) = A^y z - \alpha^y h^y(z),
$$

where $z = (z_1, z_2, z_3)^T$,

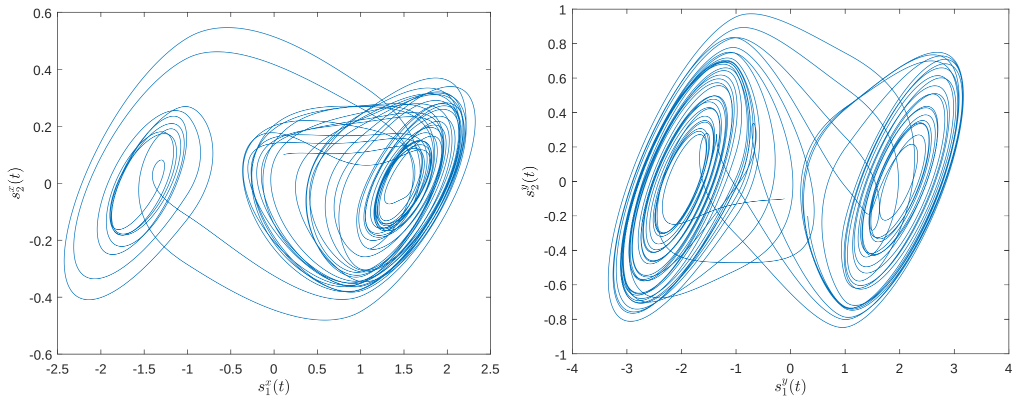
$$A^x = \begin{pmatrix} -\alpha^x & \alpha^x & 0 \\ 1 & -1 & 1 \\ 0 & \beta^x & 0 \end{pmatrix} \text{ and } A^y = \begin{pmatrix} -\alpha^y & \alpha^y & 0 \\ 1 & -1 & 1 \\ 0 & \beta^y & 0 \end{pmatrix},$$

and $h^x(z) = (m_1^x z_1 + \frac{1}{2}(m_0^x - m_1^x)(|z_1+1| - |z_1-1|), 0, 0)^T$, $h^y(z) = (m_1^y z_1 + \frac{1}{2}(m_0^y - m_1^y)(|z_1+1| - |z_1-1|), 0, 0)^T$. Moreover, we choose the following parameters $\alpha^x = 9$, $\beta^x = 100/7$, $m_0^x = -8/7$, $m_1^x = -5/7$, $\alpha^y = 8$, $\beta^x = 110/7$, $m_0^x = -10/7$, $m_1^x = -4/7$, $c_1 = 0.05$, $\zeta_2(t) = 0.4\sin(t) + 0.6$,

$$\Gamma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

With some efforts, we obtain that $U_1 = 2\text{diag}\{(\alpha^x)^2 + 3, (\alpha^x)^2 + (\beta^x)^2 + 1, 3\} + 2(\alpha^x)^2\text{diag}\{(m_1^x)^2 + (m_0^x - m_1^x)^2, 0, 0\} \leq \text{diag}\{281, 573, 3\}$ and $U_2 \leq \text{diag}\{270, 624, 3\}$. Figure 1 gives the dynamical behavior of the drive system (3).

Figure 1: The dynamic behavior of the drive system (3).



(a) Dynamic behavior x-state of the drive system.

(b) Dynamic behavior y-state of the drive system.

The duplex network. Our duplex network consists x-layer and y-layer, where each layer has 100 Chua chaotic oscillators. We use Watts-Strogatz small-world graph [39] to construct the x-layer by taking initial degree $d = 4$ and rewiring probability $p = 0.2$. The y-layer is a scale-free graph [40, 41] where the node degree follows a power law distribution with exponent 2.2. The intra-layer inner coupling matrix is chosen as the identity matrix $H = I_3$.

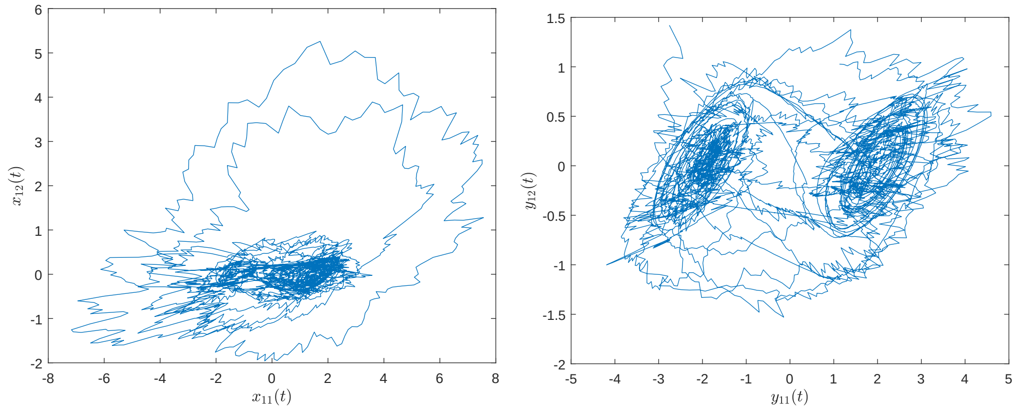
We also set $c_2 = 0.05$ and $\zeta_1(t) = 0.5\cos(t) + 0.5$. The noise intensity functions are chosen as

$$\begin{aligned} \sigma_i^x(t, e_i^x(t)) &= 0.5(e_i^x(t), \quad e_i^x(t)), \\ \sigma_i^y(t, e_i^y(t)) &= 0.3(e_i^y(t), \quad e_i^y(t)), \end{aligned}$$

for $i = 1, 2, \cdots, N$. Therefore, $\Sigma^x = 2 \cdot 0.5^2 I_3, \Sigma^y = 2 \cdot 0.3^2 I_3$. Moreover, we use the Euler-Maruyama method [42] to compute the trajectories of the stochastic differential equations.

The dynamical behaviors of the (4) are shown in Figure 2.

Figure 2: The dynamic behavior of the response system (4) with no control.



(a) Dynamic behavior x-state of node 1 of the response system.

(b) Dynamic behavior y-state of node 1 of the response system.

To evaluate the performance of the synchronization, the total x-layer and y-layer errors of the intra-layer synchronization are defined as

$$\|e^x(t)\| = \left(\sum_{i=1}^{N} (x_i(t) - s^x(t))^T (x_i(t) - s^x(t)) \right)^{1/2},$$

and

$$\|e^y(t)\| = \left(\sum_{i=1}^{N} (y_i(t) - s^y(t))^T (y_i(t) - s^y(t)) \right)^{1/2},$$

respectively.

The situation under controller (11)**.** For simplicity, we set $k_1^x = k_2^x = \cdots = k_N^x = k^x$, $k_1^y = k_2^y = \cdots = k_N^y = k^y$ and thus $K^x = k^x I_N, K^y = k^y I_N$. Note that (13) is not a linear matrix inequality. Even though, we can search the solution to (12) and (13) as follows.

- *Step 1*, we assign large numbers to k^x and k^y respectively, and then (13) becomes a linear matrix inequality of P_1, P_2, P_3, Q_1, Q_2, Q_3 which can be solved by the LMI toolbox provided by Matlab.

- *Step 2*, if (12) and (13) are feasible by the LMI toolbox, then we decrease the values of k^x and k^y.

Repeat *Step 2* before the inequalities (12) and (13) become infeasible, we will obtain appropriate control gains k^x and k^y.

Under the preceding settings, we finally choose $k^x = 3.5$ and $k^y = 5.5$ and the solution to (12) and (13) is listed as follows, $\mu_1 = 43.6369$, $\lambda_1 = 14.9600$, $P_1 = \mathrm{diag}\{10.5924, 14.5055, 6.9590\}$,

$$P_2 = \begin{pmatrix} 136.3493 & -0.0139 & -0.0135 \\ -0.0139 & 103.7976 & -0.0149 \\ -0.0135 & -0.0149 & 171.7173 \end{pmatrix}, P_3 = \begin{pmatrix} 124.0841 & 0.0052 & 0.0057 \\ 0.0052 & 98.8094 & 0.0045 \\ 0.0057 & 0.0045 & 151.8279 \end{pmatrix}.$$

$$Q_1 = \begin{pmatrix} 7.3463 & 0.0001 & 0.0001 \\ 0.0001 & 10.4292 & 0.0001 \\ 0.0001 & 0.0001 & 5.0399 \end{pmatrix}, Q_2 = \begin{pmatrix} 196.0905 & -0.0101 & -0.0082 \\ -0.0101 & 172.7337 & -0.0094 \\ -0.0082 & -0.0094 & 214.8866 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 170.7531 & 0.0090 & 0.0105 \\ 0.0090 & 153.4701 & 0.0095 \\ 0.0105 & 0.0095 & 184.6795 \end{pmatrix}.$$

Under the controller (11), Figure 3 gives the trajectories of x-state and y-layer of node 1.

To verify the theoretical result, we also give the time evolution of the x-layer and y-layer errors in Figure 4.

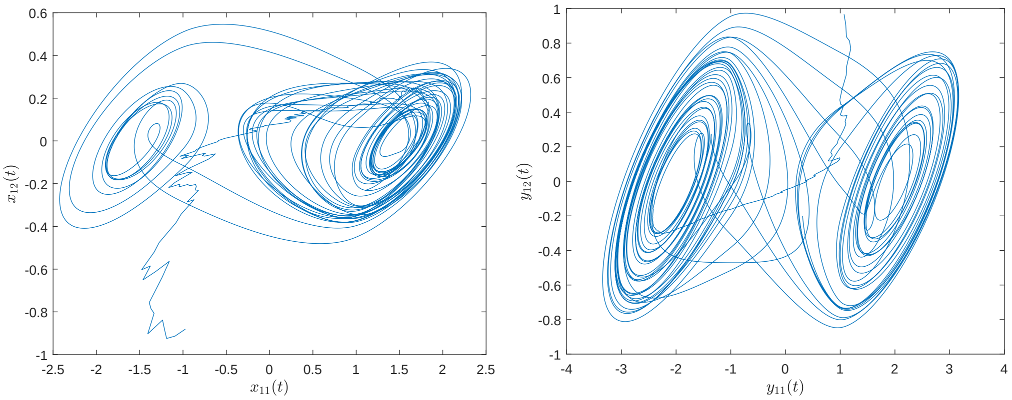
The situation under controller (16)**.** For simplicity, we set $\bar{k}_1^x = \bar{k}_2^x = \cdots = \bar{k}_N^x = \bar{k}^x$, $\bar{k}_1^y = \bar{k}_2^y = \cdots = \bar{k}_N^y = \bar{k}^y$ and thus $\bar{K}^x = \bar{k}^x I_N, \bar{K}^y = \bar{k}^y I_N$. Note that (19) is not a linear matrix inequality. Even though, we can search the solution to (18) and (19) similarly to that of controller (11) as follows.

- *Step 1*, we assign large numbers to $\bar{k}^x$ and $\bar{k}^y$ respectively, and then (19) becomes a linear matrix inequality of P_1, P_2, P_3, Q_1, Q_2, Q_3 which can be solved by the LMI toolbox of Matlab.

- *Step 2*, if (18) and (19) are feasible by the LMI toolbox, then we decrease the values of $\bar{k}^x$ and $\bar{k}^y$.

Repeat *Step 2* before the inequalities (18) and (19) become infeasible, we are going to obtain appropriate control gains $\bar{k}^x$ and $\bar{k}^y$.

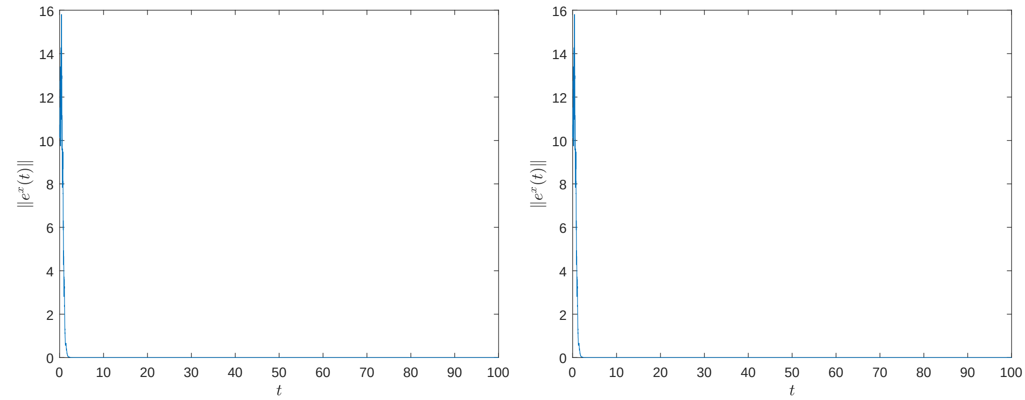
Figure 3: The dynamic behavior of the response system (4) under controller (11).



(a) Dynamic behavior x-state of node 1 of the response system under controller (11).

(b) Dynamic behavior y-state of node 1 of the response system under controller (11).

Figure 4: Time evolution of the total errors of the response system under controller (11).



(a) Time evolution of the total x-layer error of the response system under controller (11).

(b) Time evolution of the total y-layer error of the response system under controller (11).

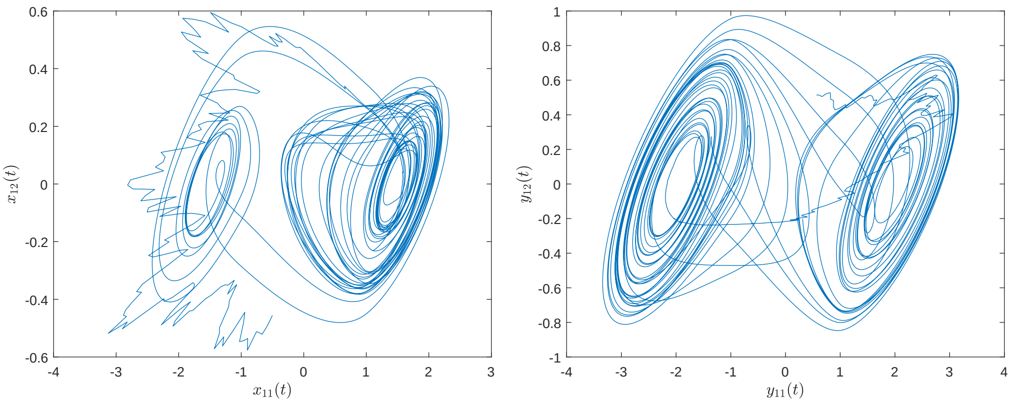
Under the preceding settings, we finally choose $\bar{k}^x = 4.5$ and $\bar{k}^y = 6.5$ and the solution to (18) and (19) is listed as follows, $\mu_1 = 166.3930$, $\lambda_1 = 43.4001$, $P_1 = 16.4969 I_3$, $Q_1 = 12.0678 I_3$,

$$P_2 = \begin{pmatrix} 136.3493 & -0.0139 & -0.0135 \\ -0.0139 & 103.7976 & -0.0149 \\ -0.0135 & -0.0149 & 171.7173 \end{pmatrix}, P_3 = \begin{pmatrix} 124.0841 & 0.0052 & 0.0057 \\ 0.0052 & 98.8094 & 0.0045 \\ 0.0057 & 0.0045 & 151.8279 \end{pmatrix}.$$

$$Q_2 = \begin{pmatrix} 521.7444 & -0.0135 & -0.0142 \\ -0.0135 & 342.7871 & -0.0138 \\ -0.0142 & -0.0138 & 656.7347 \end{pmatrix}, Q_3 = \begin{pmatrix} 440.8647 & 0.0092 & 0.0094 \\ 0.0092 & 313.9469 & 0.0093 \\ 0.0094 & 0.0093 & 536.5883 \end{pmatrix}.$$

Under the controller (16), Figure 5 gives the trajectories of x-state and y-layer of node 1.

Figure 5: The dynamic behavior of the response system (4) under controller (16).



(a) Dynamic behavior x-state of node 1 of the response system under controller (16).

(b) Dynamic behavior y-state of node 1 of the response system under controller (16).

To verify the theoretical result, we also give the time evolution of the x-layer and y-layer errors in Figure 6.

Figure 7 gives the average x-layer and y-layer control gains under controller (16), which are defined by

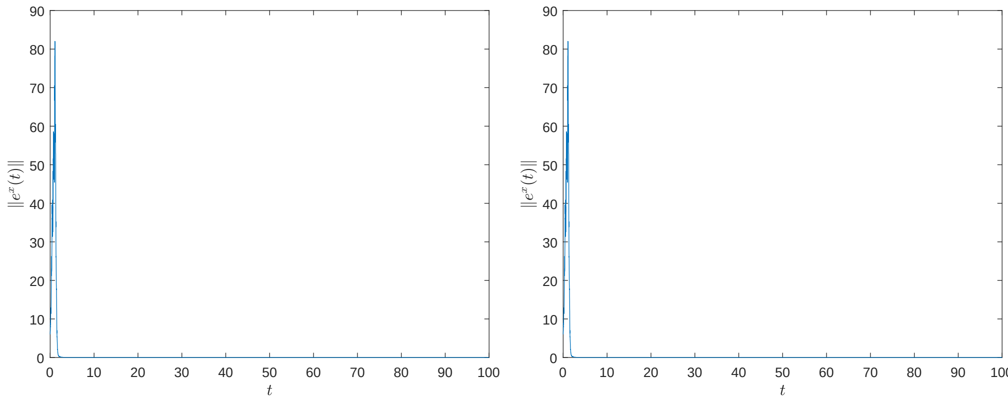
$$\bar{k}^x(t) = \frac{1}{N}\sum_{i=1}^{N} k_i^x(t) \text{ and } \bar{k}^y(t) = \frac{1}{N}\sum_{i=1}^{N} k_i^y(t),$$

respectively.

5 Conclusion

In the last few decades, a lot of synchronization schemes for the complex systems have been proposed. However, many synchronization schemes designed for a single network is not very reasonable. In this paper, we study the intra-layer synchronization of a kind of duplex networks. Different from the previous works [1, 2], Stochastic factor is introduced into the duplex network. We also incorporate both the *inter-layer* delay and the *intra-layer* delay into

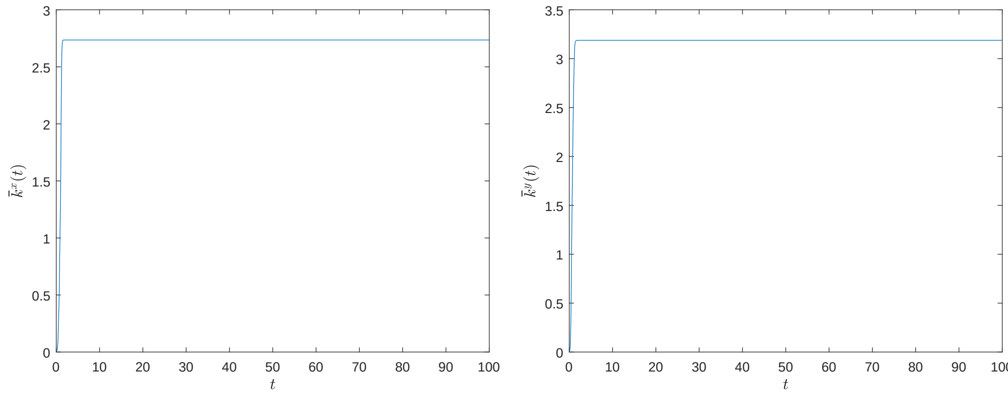
Figure 6: Time evolution of the total errors of the response system under controller (16).



(a) Time evolution of the total x-layer error of the response system under controller (16).

(b) Time evolution of the total y-layer error of the response system under controller (16).

Figure 7: Time evolution of the average control gains under controller (16).



(a) Time evolution of the average x-layer control gain under controller (16).

(b) Time evolution of the average y-layer control gain under controller (16).

the dynamical system. Both of the delays are time-varying. The paper [1] only considered the intra-layer delays and they are assumed as the constants. While the paper [2] did not consider the inter-layer delay or intra-layer delay. When the system does not achieve automatic intra-layer synchronization, we introduce two controllers: one is the state-feedback controller, the other is the adaptive state-feedback controller. Interestingly, we find that the intra-layer synchronization will achieve automatically if the inter-layer coupling strength c_1 is large enough and the intra-layer coupling strength c_2 is small enough when the time-varying inter-layer delays are absent. Finally, the simulations show the effectiveness of obtained schemes.

6 Data Availability Statement

The data used to support the findings of this study are included within the article.

7 Acknowledgement

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