A Discrete Retarded Gronwall-Bellman Type Inequality and its Applications to Difference Equations [1]

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Abstract. In this paper we present a new discrete retarded Gronwall-Bellman type inequality. As applications, the dynamics of some delay difference equations are studied. First, the asymptotic behavior of solutions for scalar difference equation $\Delta x(n) = -a(n)x(n)+B(n, x_n)$ is discussed, and some new criterion on the asymptotic stability of the zero solution are obtained under weaker assumptions. Then the dissipativity of a nonautonomous delay difference system with superlinear nonlinearities is investigated. By using the inequalities established here, it is shown that the discrete set-valued process generated by the system possesses a unique global pullback attractor.

Keywords: Retarded Gronwall-Bellman type inequality, difference equation, asymptotic stability, global pullback attractor.

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1 Introduction

In practice time lags are often inevitable. This gives birth to various kinds of mathematical models of retarded differential and difference equations. The study of the dynamical behavior and related control problems of these models naturally lead to different Gronwall-Bellman type inequalities involving time delays. A typical example is the well-known Halanay's inequality [13] which plays a crucial role in the investigation of delay differential equations. Since the Halanay's work there have appeared numerous retarded differential/integral inequalities and their discrete analogs; see e.g. [1, 4, 5, 14, 17, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 31].

In contrast to non-retarded evolution equations, generally the qualitative analysis of retarded ones often encounter more difficulties. Although the existing inequalities in the literature as mentioned above provide efficient tools in dealing with such equations, it is still a difficult task to derive needed estimates for their solutions. In fact, it is often the case that one has to fall his back on non-retarded

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Gronwall-Bellman type inequalities when studying retarded problems (particularly in the case of non-constant delays), which makes the calculations in the argument more or less involved and restrictive.

In a recent paper [18], Li et al. established a quite general retarded Gronwall-Bellman type integral inequality:

$$\begin{aligned} y(t) \le \quad & E(t,\tau)\|y_\tau\| + \textstyle\int_\tau^t K_1(t,s)\|y_s\|ds \\ & + \textstyle\int_t^\infty K_2(t,s)\|y_s\|ds + \rho, \qquad t \ge \tau \ge 0, \end{aligned} \tag{1.1}$$

where $E,\ K_1$ and K_2 are appropriate nonnegative functions on $\mathbb{R}^+ \times \mathbb{R}^+$, $\rho \ge 0$ is a constant, y is a nonnegative function on $[-r,\infty)$ for some given $r \ge 0$, and y_t denotes the lift of y in $C([-r,0])$,

$$y_t(s) = y(t+s), \qquad s \in [-r,0].$$

By developing some new techniques the authors obtained several uniform decay estimates for the functions y satisfying the inequality, which turns out to be very helpful in studying the dynamical behavior of delay differential equations (see e.g. [18, 30]). In this work we give a discrete counterpart of (1.1), namely, we extend the main results in [18] to the following discrete inequality:

$$\begin{aligned} y(n) \le \quad & f(n,m)\|y_m\| + \textstyle\sum_{k=m}^n g_1(n,k)\|y_k\| \\ & + \textstyle\sum_{k=n+1}^\infty g_2(n,k)\|y_k\| + \rho, \qquad n \ge m, m \in \mathbb{Z}^+. \end{aligned} \tag{1.2}$$

As seen in the literature, such an extension is in general not immediate because one has to overcome many technical difficulties brought by the lack of continuity. This is also the case of this present work since the argument involved in establishing the inequality (1.1) in [18] depends heavily on the continuity of the function $y(t)$ in t and the integrals with respect to the variation of the integral domains.

Applications of inequalities are always of particular interest. As an example, we first consider the asymptotic stability of scalar difference equation

$$\Delta^+ x(n) = -a(n)x(n) + B(n, x_n), \qquad n \in \mathbb{Z}^+, \tag{1.3}$$

where $\Delta^+ x(n) := x(n+1) - x(n)$, x_n denotes the *lift* of the sequence $x = x(n)$ in the space $\mathcal{S}$ consisting of mappings (sequences) from $[-r,0] \cap \mathbb{Z}$ to $\mathbb{R}^1$ for some given $r \in \mathbb{Z}^+$, and B is a function on $\mathbb{Z}^+ \times \mathcal{S}$ satisfying

$$|B(n,q)| \le b(n)\|q\|, \qquad n \in \mathbb{Z}^+,\ q \in \mathcal{S}.$$

This equation is closely related to discrete population models and financial mathematics, etc. In [21], Liz et al. considered the special case where $a(n) \equiv a$ and

$$B(n, x_n) = b \max_{0 \le k \le r}\{x(n-k)\}, \qquad n \in \mathbb{Z}^+.$$

They first established a new discrete Halanay-type inequality. Then by applying the inequality they achieved a set of exponential stability criteria for the zero solution of (1.3) under the assumption that

$$0 < b < a \le 1. \tag{1.4}$$

A slightly modified version of $B(n, x_n)$ is considered in [1, 29], where the authors proved some similar results as mentioned above. It is worth mentioning that the asymptotic behavior of (1.3) with positive varying coefficients $a(n)$ and $B(n, x_n) = b(n) \max_{0\le k\le r}\{x(n-k)\}$ was studied in [19, 25, 28] etc. Once again by using some generalized discrete Halanay-type inequalities the authors obtained some sufficient conditions for the asymptotic (or exponential) stability of the zero solution under the hypothesis $a(n) - b(n) \ge \delta > 0$ or some similar ones. Similar results can be found in [5]. Let us also mention the nice work of Liz et al. [22], in which the authors paid some special attention to the case where $a(n) \equiv a$ and $B(n, x_n) = -bf(n, x(n), \cdots, x(n-r))$. Here a, b are constants, $b > 0$, and a can be allowed to be negative.

In this paper we consider the more general case where $a(n)$ can be even allowed to change sign. We will present some new criteria concerning the asymptotic stability and exponential stability for the zero solution of the equation by applying a particular case of the inequality (1.2). Furthermore, if $a(n)$ and B satisfy some weaker hypotheses, we give some uniform estimates on the growth rate of the solutions of the equation.

Our second example concerns the dissipativity of equation

$$\Delta^- y(n) = f(n, y(n)) + g(n, y(n - r(n))), \qquad y(n) \in \mathbb{R}^d,\ n \in \mathbb{Z}^+, \tag{1.5}$$

where $\Delta^- y(n) := y(n) - y(n-1)$, f and g are mappings from $\mathbb{Z}^+ \times \mathbb{R}^d$ to $\mathbb{R}^d$, and $0 \le r(n) \le r$ ($n \in \mathbb{Z}^+$) for some fixed $r \in \mathbb{Z}^+$. When f and g are sublinear in the second variable, this problem has already been studied in [8], in which the authors studied the uniform ultimate boundedness of solutions (which is equivalent to the bounded k-dissipativity defined in [11] (see [11, page 85])) by using the Razumikhin-Lyapunov method. The corresponding continuous time versions can be found in [6, 9, 10] and references cited therein. However, in the superlinear case few results in this line can be found in the literature. This is precisely the motivation of our consideration here. We allow both f and g to be superlinear. Suppose that f and g satisfy the following dissipativity condition: there exist $p > q \ge 1$ and positive constants α, α_1, β and β_1 such that

$$\langle f(n, y), y\rangle \le -\alpha |y|^{p+1} + \alpha_1, \quad |g(n, y)| \le \beta |y|^q + \beta_1$$

for all $y \in \mathbb{R}^d$. We show that system (1.5) has a global pullback attractor.

This paper is organized as follows. Section 2 is concerned with the inequality (1.2), and Section 3 is devoted to the asymptotic stability analysis of equation (1.3). In section 4 we prove the existence of a pullback attractor for the delay difference equation (1.5).

2 A Retarded Discrete Inequality

Let $\mathbb{Z}$ be the set of integers, and $\mathbb{R}^d$ the usual Euclidean space. An *interval* J in $\mathbb{Z}$ means a subset of $\mathbb{Z}$ for which there is an interval $\tilde{J}$ in $\mathbb{R}^1$ such that

$$J = \tilde{J} \cap \mathbb{Z}.$$

Since we only concerns the discrete case here, given an interval $I \in \mathbb{R}^1$, we use the same notation I to denote the interval $J := I \cap \mathbb{Z}$ in $\mathbb{Z}$. For instance, we use the notation $[a, b]$ to denote the interval $[a, b] \cap \mathbb{Z}$ in $\mathbb{Z}$.

Let J be an interval in $\mathbb{Z}$. A *sequence* on J with values in a given set Y is a mapping from J to Y. We use the notation $\mathcal{S}\,(J;\, Y)$ to denote the set of sequences on J with values in Y. If J is finite and Y is a Banach space with norm $|\cdot|$, the set $\mathcal{S}\,(J;\, Y)$ is a Banach space as well with norm

$$\|y\| = \max_{n \in J} |y(n)|, \qquad y \in \mathcal{S}\,(J;\, Y)\,.$$

Let $r \in \mathbb{Z}^+$. The *lift* of $y \in \mathcal{S}\,(J;\, Y)$ in $\mathcal{S} := \mathcal{S}([-r, 0];\, Y)$, denoted by $\hat{y} = y_n$ $(n \in J)$, is a sequence on J taking values in $\mathcal{S}$ defined by

$$y_n(k) = \begin{cases} y(n+k), & \text{if } \ n+k \in J; \\ 0, & \text{otherwise}, \end{cases} \qquad\qquad k \in [-r, 0],\ n \in J.$$

For convenience in statement, we also make the following convention:

Convention. Let x_i $(i \in J)$ be a sequence of real numbers on interval $J \subset \mathbb{Z}$. If $m, n \in J$ and $m > n$, then we assign

$$\sum_{i=m}^{n} x_i = 0, \qquad \prod_{i=m}^{n} x_i = 1.$$

2.1 A retarded discrete inequality

Denote by $\mathscr{S}^+(Q)$ the family of nonnegative functions on $Q := \mathbb{Z}^+ \times \mathbb{Z}^+$. Set

$$\mathscr{F} = \{f \in \mathscr{S}^+(Q) : \ \lim_{n \to \infty} f(n+m, m) = 0 \ \text{ uniformly } \ \text{w.r.t. } \ m \in \mathbb{Z}^+\},$$

$$\mathscr{G}_1=\left\{g\in\mathscr{S}^+(Q):\ \sum_{k=0}^{n}g(n,k)<\infty \text{ for all } n\in\mathbb{Z}^+\right\},$$

and

$$\mathscr{G}_2=\left\{g\in\mathscr{S}^+(Q):\ \sum_{k=n+1}^{\infty}g(n,k)<\infty \text{ for all } n\in\mathbb{Z}^+\right\}.$$

For $f\in\mathscr{F}$ and $(g_1,g_2)\in\mathscr{G}_1\times\mathscr{G}_2$, we write

$$\text{ß}(f):=\sup_{n\geq m\geq 0}f(n,m),$$

and

$$\kappa(g_1,g_2):=\sup_{n\geq m\geq 0}\left(\sum_{k=m}^{n}g_1(n,k)+\sum_{k=n+1}^{\infty}g_2(n,k)\right).$$

Let $r\in\mathbb{Z}^+$, $f\in\mathscr{F}$, and $g_i\in\mathscr{G}_i$ $(i=1,2)$. Consider the retarded discrete Gronwall-Bellman type inequality

$$\begin{aligned} y(n)\leq\ & f(n,m)\|y_m\|+\textstyle\sum_{k=m}^{n}g_1(n,k)\|y_k\| \\ & +\textstyle\sum_{k=n+1}^{\infty}g_2(n,k)\|y_k\|+\rho, \qquad n\geq m, m\in\mathbb{Z}^+, \end{aligned}\tag{2.1}$$

where $\rho\geq 0$ is a constant, and y_n denotes the lift of $y=y(n)$ in $\mathcal{S}=\mathcal{S}([-r,0];\mathbb{R}^+)$. For convenience in statement, we call a sequence $y\in\mathcal{S}([-r,\infty);\mathbb{R}^+)$ satisfying (2.1) a *solution* of (2.1). Denote by $\mathscr{L}_r(f;g_1,g_2;\rho)$ the solution set of (2.1).

One of our main results is summarized in the theorem below:

Theorem 2.1 *Suppose*

$$\text{ß}(f)\leq\text{ß}<\infty,\qquad \kappa(g_1,g_2)\leq\kappa<\infty.$$

Then the following two assertions hold.

(1) *If $\kappa<1$, then for any R, $\varepsilon>0$, there exists $N>0$ such that*

$$\|y_n\|<\mu\rho+\varepsilon,\qquad n\geq N$$

for all bounded sequence $y\in\mathscr{L}_r(f;g_1,g_2;\rho)$ with $\|y_0\|\leq R$, where

$$\mu=1/(1-\kappa).\tag{2.2}$$

(2) *If $\kappa<1/(1+\text{ß})$, then there exist $M,\sigma>0$ with $\sigma<1$ such that*

$$\|y_n\|\leq M\|y_0\|\sigma^n+\gamma\rho,\qquad n\in\mathbb{Z}^+\tag{2.3}$$

for all bounded sequences $y\in\mathscr{L}_r(f;g_1,g_2;\rho)$, where

$$\gamma=(\mu+1)/(1-\kappa c),\qquad c=\max\{\text{ß}/(1-\kappa),1\}.\tag{2.4}$$

Remark 2.2 *One trivially checks that if $\kappa < 1/(1+ß)$, then $\kappa c < 1$.*

By (2.2) *we also have*

$$\kappa\mu + 1 = \mu. \tag{2.5}$$

A simpler but important case of (2.1) is that where g_2 vanishes. In such a case the inequality reads as below:

$$y(n) \le f(n,m)\|y_m\| + \sum_{k=m}^{n} g(n,k)\|y_k\| + \rho, \quad n \ge m, m \in \mathbb{Z}^+. \tag{2.6}$$

We show that if $\kappa(g_1, g_2) = \kappa(g, 0) \le \kappa < 1$, then any solution y of (2.6) is automatically bounded. Thus the boundedness requirement on y in Theorem 2.1 can be removed. Consequently one has

Theorem 2.3 *Let ß, κ, μ and γ be the constants as given in Theorem 2.1.*

(1) *If $\kappa < 1$, then for any R, $\varepsilon > 0$, there exists $N > 0$ such that*

$$\|y_n\| < \mu\rho + \varepsilon, \qquad n \in [N, \infty)$$

for all $y \in \mathscr{L}_r(f; g, 0; \rho)$ with $\|y_0\| \le R$.

(2) *If $\kappa < 1/(1+ß)$, then there exist positive constants M and σ with $\sigma < 1$ such that for all $y \in \mathscr{L}_r(f; g, 0; \rho)$,*

$$\|y_n\| \le M\|y_0\|\sigma^n + \gamma\rho, \qquad n \in \mathbb{Z}^+.$$

2.2 Proof of Theorem 2.1

To prove Theorem 2.1, we first give a boundedness estimate for solutions of (2.1).

Lemma 2.4 *Assume $\kappa < 1$. Then for any bounded sequence $y \in \mathscr{L}_r(f; g_1, g_2; \rho)$,*

$$\|y_n\| \le c\|y_0\| + \mu\rho, \qquad n \in \mathbb{Z}^+, \tag{2.7}$$

where c, μ are the constants given in Theorem 2.1.

Proof. The proof can be obtained by slightly modifying the one for [18, Lemma 2.1]. We omit the details. □

Remark 2.5 *Let $y \in \mathscr{L}_r(f; g_1, g_2; \rho)$. For $\ell \in \mathbb{Z}^+$, if we set $\tilde{y}(n) = y(n+\ell)$ and define*

$$\tilde{f}(n,m) = f(n+\ell, m+\ell), \quad \tilde{g}_i(n,m) = g_i(n+\ell, m+\ell)(i = 1,2)$$

for $n, m \in \mathbb{Z}^+$, then one easily checks that $\tilde{y} \in \mathscr{L}_r(\tilde{f}; \tilde{g}_1, \tilde{g}_2; \rho)$ with

$$\text{ß}(\tilde{f}) \leq \text{ß}(f) \leq \text{ß}, \quad \kappa(\tilde{g}_1, \tilde{g}_2) \leq \kappa(g_1, g_2) \leq \kappa < 1.$$

Therefore if y is bounded, we infer from Lemma 2.4 that

$$\|y_{n+\ell}\| \leq c\|y_n\| + \mu\rho, \qquad n, \ell \in \mathbb{Z}^+. \tag{2.8}$$

Proof of Theorem 2.1. (1) We first show that

$$\limsup_{n\to\infty} \|y_n\| \leq \mu\rho \tag{2.9}$$

for any bounded sequence $y \in \mathscr{L}_r(f; g_1, g_2; \rho)$. Let us argue by contradiction and suppose that

$$\limsup_{n\to\infty} \|y_n\| = \mu\rho + \delta$$

for some $\delta > 0$. Pick a monotone sequence $k_n \to \infty$ so that $\lim_{n\to\infty} y(k_n) = \mu\rho+\delta$. For any $\varepsilon > 0$, take a $T \in \mathbb{Z}^+$ sufficiently large such that

$$\|y_n\| < \mu\rho + \delta + \varepsilon, \qquad n \geq T.$$

Then for $k_n \geq T$, by (2.1) we obtain that

$$\begin{aligned} y(k_n) &\leq f(k_n, T)\|y_T\| + \sum_{j=T}^{k_n} g_1(k_n, j)\|y_j\| + \sum_{j=k_n+1}^{\infty} g_2(k_n, j)\|y_j\| + \rho \\ &\leq f(k_n, T)\|y_T\| + \kappa(\mu\rho + \delta + \varepsilon) + \rho. \end{aligned}$$

Setting $n \to \infty$ in the above inequality, we obtain that

$$\mu\rho + \delta \leq \kappa(\mu\rho + \delta + \varepsilon) + \rho.$$

Since ε is arbitrary, one concludes that

$$\mu\rho + \delta \leq (\kappa\mu + 1)\rho + \kappa\delta.$$

Therefore by (2.5) we have $\delta \leq \kappa\delta$, which leads to a contradiction and justifies the validity of (2.9).

We are now ready to complete the proof of assertion (1). Let $R > 0$. Set

$$\mathscr{B}_R = \{y \in \mathscr{L}_r(f; g_1, g_2; \rho) : \ y \text{ is bounded with } \|y_0\| \leq R\}.$$

By (2.7) we see that $\mathscr{B}_R$ is uniformly bounded. Thereby the envelope

$$y^*(n) = \sup_{y\in\mathscr{B}_R} y(n), \qquad n \in [-r, \infty)$$

of the family $\mathscr{B}_R$ is well-defined, which is also bounded. We infer from (2.1) that

$$y(n) \le f(n,m)\|y_m^*\| + \sum_{k=m}^{n} g_1(n,k)\|y_k^*\| + \sum_{k=n+1}^{\infty} g_2(n,k)\|y_k^*\| + \rho, \qquad \forall\, n \ge m, m \in \mathbb{Z}^+$$

for every $y \in \mathscr{B}_R$. Further, taking the supremum of the lefthand side of the inequality for $y \in \mathscr{B}_R$, it yields

$$y^*(n) \le f(n,m)\|y_m^*\| + \sum_{k=m}^{n} g_1(n,k)\|y_k^*\| + \sum_{k=n+1}^{\infty} g_2(n,k)\|y_k^*\| + \rho, \qquad \forall\, n \ge m, m \in \mathbb{Z}^+.$$

Hence $y^* \in \mathscr{L}_r(f; g_1, g_2; \rho)$. A direct consequence of (2.9) to y^* then gives us $\limsup_{n\to\infty} \|y_n^*\| \le \mu\rho$. Therefore for any $\varepsilon > 0$, there is $N > 0$ such that

$$\|y_n^*\| < \mu\rho + \varepsilon, \qquad n \ge N,$$

from which assertion (1) immediately follows.

(2) Now we assume that $\kappa < 1/(1 + ß)$. To derive the exponential decay estimate in (2.3), let us first show a temporary result:

There exists a positive number $\sigma < 1$ and a positive integer N such that if $\|y_0\| \le K + \gamma\rho$ with $K > 0$, then

$$\|y_n\| \le K\sigma^n + \gamma\rho, \qquad n \ge N. \tag{2.10}$$

For this purpose, we fix a real number

$$\lambda = (1 + \kappa c)/2.$$

By Remark 2.2 we see that $\lambda < 1$. Define

$$\eta = \min\{m \in [1, \infty) : \|y_n\| \le \lambda K + \gamma\rho \text{ for all } n \ge m\}.$$

In what follows let us give an estimate for the upper bound of η.

Since $\gamma > \mu$ (see (2.4)) and $K > 0$, by (2.9) it is clear that $\eta < \infty$. We may assume $\eta > r + 1$ (otherwise we are done). Then by the definition of η one necessarily has

$$\|y_{\eta-1}\| > \lambda K + \gamma\rho.$$

For simplicity, let us write $b(n) := f(n, 0)$. Given $n \in [(\eta - 1) - r, (\eta - 1)]$, since $\|y_0\| \le K + \gamma\rho$, by (2.1) we deduce that

$$\begin{aligned}
y(n) &\le b(n)\|y_0\| + \sum_{k=0}^{n} g_1(n,k)\|y_k\| + \sum_{k=n+1}^{\infty} g_2(n,k)\|y_k\| + \rho \\
&\le (\text{by } (2.7)) \le \|b_{\eta-1}\|\|y_0\| + \kappa(c\|y_0\| + \mu\rho) + \rho \\
&\le (\|b_{\eta-1}\| + \kappa c)\|y_0\| + (\kappa\mu + 1)\rho \\
&\le (\text{by } (2.5)) \le (\|b_{\eta-1}\| + \kappa c)(K + \gamma\rho) + \mu\rho.
\end{aligned}$$

Therefore

$$\begin{aligned}
\lambda K + \gamma\rho < \|y_{\eta-1}\| &= \max_{n\in[(\eta-1)-r,(\eta-1)]} y(n) \\
&\le (\|b_{\eta-1}\| + \kappa c)K + ((\|b_{\eta-1}\| + \kappa c)\gamma + \mu)\rho.
\end{aligned} \tag{2.11}$$

Take a number n_0 in $(0, \infty)$ such that

$$f(n+m, m)\gamma < 1, \qquad \forall\ n \ge n_0,\ m \in \mathbb{Z}^+. \tag{2.12}$$

Then for every $n \ge n_0 + r$, we have

$$\gamma\|b_n\| = \gamma \max\nolimits_{k\in[n-r,n]} f(k, 0) < 1. \tag{2.13}$$

If $\eta \le n_0 + r + 1$ then we are done. Hence we assume that $\eta > n_0 + r + 1$. Then by the definition of γ (see (2.4)) and (2.13) it can be easily seen that

$$\gamma = \kappa c\gamma + \mu + 1 \ge (\|b_{\eta-1}\| + \kappa c)\gamma + \mu.$$

Thus by (2.11) it follows that $\lambda K < (\|b_{\eta-1}\| + \kappa c)K$. Therefore

$$\|b_{\eta-1}\| > \lambda - \kappa c = (1 - \kappa c)/2 > 0. \tag{2.14}$$

Take an integer $n_1 > 0$ such that

$$f(n+m, m) \le (1 - \kappa c)/2, \qquad n \in [n_1 + 1, \infty), \quad m \in \mathbb{Z}^+. \tag{2.15}$$

Then by (2.14) we deduce that $\eta \le n_1 + r + 1$. Therefore one concludes that

$$\eta \le N := \max\{n_0, n_1\} + r + 1.$$

So far we have proved that if $\|y_0\| \le K + \gamma\rho$ $(K > 0)$ then

$$\|y_n\| \le K\lambda + \gamma\rho, \qquad n \in [N, \infty).$$

Let $\tilde{y}(n) = y(n+N)$, and set

$$\tilde{f}(n,m) = f(n+N, m+N), \quad \tilde{g}_i(n,m) = g_i(n+N, m+N)(i=1,2)$$

for $n,\ m \geq 0$. Then $\tilde{y} \in \mathscr{L}_r(\tilde{f}; \tilde{g}_1, \tilde{g}_2; \rho)$ with

$$\kappa(\tilde{g}_1, \tilde{g}_2) \leq \kappa(g_1, g_2) \leq \kappa < 1/(1+ß).$$

Since $\|\tilde{y}_0\| \leq K\lambda + \gamma\rho$, the same argument as above applies to show that

$$\|\tilde{y}_n\| \leq (K\lambda)\lambda + \gamma\rho, \qquad n \in [N, \infty),$$

that is

$$\|y_n\| \leq K\lambda^2 + \gamma\rho, \qquad n \in [2N, \infty).$$

(We emphasize that the independence of the numbers n_0 and n_1 upon $m \in \mathbb{Z}^+$ (see (2.13) and (2.15)) plays a crucial role in the argument.) Repeating the above procedure we finally obtain that

$$\|y_n\| \leq K\lambda^k + \gamma\rho, \qquad n \in [kN, \infty),\ k = 1, 2, \cdots. \tag{2.16}$$

Setting

$$\sigma = \exp\{\ln \lambda/(2N)\}, \tag{2.17}$$

one has

$$\lambda^k \leq \sigma^n, \qquad n \in [kN, (k+1)N]$$

for all $k \in [1, \infty)$. The estimate (2.10) then follows from (2.16).

We are now in a position to accomplish the proof of the theorem.

Note that (2.7) implies that if $\|y_0\| = 0$ then

$$\|y_n\| \leq \mu\rho \leq \gamma\rho, \qquad n \in \mathbb{Z}^+,$$

and hence the conclusion trivially holds true. Thus we assume $\|y_0\| > 0$.

Take $K = \|y_0\|$. Then $K > 0$, and $\|y_0\| = K \leq K + \gamma\rho$. Therefore by (2.10) we have

$$\|y_n\| \leq \|y_0\|\sigma^n + \gamma\rho, \qquad n \in [N, \infty). \tag{2.18}$$

On the other hand, by (2.8) we deduce that

$$\|y_n\| \leq c\|y_0\| + \mu\rho \leq c\|y_0\| + \gamma\rho, \qquad n \in [0, N].$$

Set

$$M = c\sigma^{-N}. \tag{2.19}$$

Then

$$\|y_n\| \leq c\|y_0\| + \gamma\rho \leq M\sigma^n\|y_0\| + \gamma\rho, \qquad n \in [0, N].$$

Combining this with (2.18) we finally arrive at the estimate

$$\|y_n\| \leq M\|y_0\|\sigma^n + \gamma\rho, \qquad n \in \mathbb{Z}^+,$$

which completes the proof of the theorem. $\square$

2.3 Proof of Theorem 2.3

Proof of Theorem 2.3. The conclusions in the theorem immediately follows from Theorem 2.1 and the following boundedness result on solutions of (2.6). $\Box$

Lemma 2.6 *Suppose $\kappa < 1$. Then*

$$y(n) \le (c+1)\|y_0\| + \mu\rho, \qquad n \in \mathbb{Z}^+ \tag{2.20}$$

for all solutions y of (2.6), where μ and c are the same constants as given in Theorem 2.1.

Proof. To prove (2.20), we show that for any $\varepsilon > 0$,

$$y(n) \le (c+1)(\|y_0\| + \varepsilon) + \mu\rho, \qquad n \in \mathbb{Z}^+.$$

For clarity, we write $\|y_0\| + \varepsilon := A_\varepsilon$. Let y be a solution of (2.6). Suppose the contrary. There would exist $m \ge 1$ such that

$$y(m) > (c+1)A_\varepsilon + \mu\rho, \tag{2.21}$$

$$y(n) \le (c+1)A_\varepsilon + \mu\rho, \qquad n \le m-1. \tag{2.22}$$

Hence we see that

$$\|y_m\| = y(m).$$

Therefore by (2.6) and (2.22) we deduce that

$$\begin{aligned} \|y_m\| = y(m) &\le f(m,0)\|y_0\| + \sum_{k=0}^{m-1} g(m,k)\|y_k\| + g(m,m)\|y_m\| + \rho \\ &\le \text{ß}A_\varepsilon + \sum_{k=0}^{m-1} g(m,k)((c+1)A_\varepsilon + \mu\rho) + g(m,m)\|y_m\| + \rho. \end{aligned}$$

Hence

$$(1 - g(m,m))\|y_m\| \le \text{ß}A_\varepsilon + \sum_{k=0}^{m-1} g(m,k)((c+1)A_\varepsilon + \mu\rho) + \rho.$$

Noticing that $1 - g(m,m) \ge 1 - \kappa > 0$, by (2.21) one concludes that

$$\begin{aligned} &(1 - g(m,m))((c+1)A_\varepsilon + \mu\rho) \\ &< (1 - g(m,m))\|y_m\| \\ &\le \text{ß}A_\varepsilon + \sum_{k=0}^{m-1} g(m,k)((c+1)A_\varepsilon + \mu\rho) + \rho. \end{aligned} \tag{2.23}$$

Thus (2.23) implies

$$(c+1)A_\varepsilon < \left(ß + \sum_{k=0}^{m} g(m,k)(c+1)\right) A_\varepsilon + \left(\sum_{k=0}^{m} g(m,k)\mu + 1 - \mu\right)\rho$$
$$\le (ß + \kappa(c+1))\, A_\varepsilon + (\kappa\mu + 1 - \mu)\rho$$
$$= (\text{by } (2.5)) = (ß + \kappa(c+1))\, A_\varepsilon.$$

(Here we have used the assumption that $\sum_{k=0}^{m} g(m,k) \le \kappa$.) Hence

$$c + 1 < ß + \kappa(c+1).$$

It follows that $c + 1 < ß/(1-\kappa) \le c$, a contradiction. □

2.4 Some remarks

We finally include some remarks to conclude this section.

Remark 2.7 *In most concrete examples from applications the function* $f(n,m)$ *in* (2.1) *takes the form:*

$$f(n,m) = M_0\theta^{(n-m)},$$

where M_0 *is a positive constant and* θ *is a real number in* $(0,1)$. *In such a case we can explicitly write out the constants* n_0, n_1, σ *and* M *in* (2.12), (2.15), (2.19) *and* (2.17) *as follows:*

$$n_0 = [-\log_\theta(M_0\gamma)] + 1, \qquad n_1 = \left[\log_\theta\left(\frac{1-\kappa c}{2M_0}\right)\right],$$

$$M = c\sqrt{2/(1+\kappa c)}, \qquad \sigma = \exp\left(\frac{\ln(1+\kappa c) - \ln 2}{2(M_1 + r + 1)}\right),$$

where $M_1 = \max\{[-\log_\theta(M_0\gamma)] + 1, [\log_\theta(\frac{1-\kappa c}{2M_0})]\}$. *Here* $[q]$ *denotes the integer part of a real number* q.

Remark 2.8 *Theorem 2.1 remains valid if we replace* (2.1) *by the following slightly modified inequality: for all* $n \ge m \ge 0$,

$$y(n) \le f(n,m)\|y_m\| + \sum_{k=m}^{n-1} g_1(n,k)\|y_k\| + \sum_{k=n}^{\infty} g_2(n,k)\|y_k\| + \rho.$$

Remark 2.9 *Theorem 2.1 can be seen as a generalization of the discrete inequality in [23, Lemma A.2] and [32, Theorem 1].*

Remark 2.10 *Lemma 2.6 may be of independent interest in its own right. For instance, by applying this result one can obtain the following growth estimate for solutions of retarded difference equations:*

Theorem 2.11 *Assume that there is a constant $\alpha \in (0,1)$ such that*

$$f(k+m,m)\alpha^k \to 0 \quad as \quad k \to +\infty$$

uniformly with respect to $m \in \mathbb{Z}^+$. Suppose that

$$\hat{\kappa} := \sup_{n \geq m \geq 0} \left(\sum_{k=m}^{n} g(n,k)\alpha^{n-k} \right) < 1.$$

Then there exist $M > 0$ such that

$$\|y_n\| \leq (M\|y_0\| + \mu\rho)\alpha^{-n}, \qquad n \in \mathbb{Z}^+$$

for all solutions y of (2.6)*.*

Proof. Multiplying both sides of inequality (2.6) with α^k and noticing that

$$\alpha^k \|y_k\| = \alpha^k \max_{-r \leq i \leq 0} y(k+i) \leq \max_{-r \leq i \leq 0} \left(y(k+i)\alpha^{k+i} \right) = \|\hat{y}_k\|, \tag{2.24}$$

we see that

$$\begin{aligned} \hat{y}(n) &\leq \hat{f}(n,m)\alpha^m \|y_m\| + \sum_{k=m}^{n} \hat{g}(n,k)\alpha^k \|y_k\| + \rho\alpha^n \\ &\leq \hat{f}(n,m)\|\hat{y}_m\| + \sum_{k=m}^{n} \hat{g}(n,k)\|\hat{y}_k\| + \rho, \qquad n \geq m \geq 0, \end{aligned}$$

where

$$\hat{y}(n) = y(n)\alpha^n, \quad \hat{f}(n,m) = f(n,m)\alpha^{n-m}, \quad \hat{g}(n,m) = g(n,m)\alpha^{n-m}.$$

By virtue of Lemma 2.6 follows that

$$\|\hat{y}_n\| \leq (c+1)\|\hat{y}_0\| + \mu\rho, \qquad n \geq 0.$$

Hence by (2.24) we conclude that

$$\|y_n\| \leq (M\|y_0\| + \mu\rho)\, \alpha^{-n}, \qquad n \geq 0,$$

where $M = (c+1)\alpha^{-r}$, which completes the proof of the theorem. □

Remark 2.12 *Theorem 2.3, Lemma 2.6 and Theorem 2.11 remain valid if we replace* (2.6) *by the following slightly modified inequality:*

$$y(n) \leq f(n,m)\|y_m\| + \sum_{k=m}^{n-1} g(n,k)\|y_k\| + \rho, \qquad \forall\, n \geq m \geq 0.$$

3 On the Asymptotic Behavior of a Scalar Delay Difference Equation

Let $r \in \mathbb{Z}^+$. Consider the scalar delay difference equation:

$$\Delta^+ x(n) = -a(n)x(n) + B(n, x_n), \qquad n \in \mathbb{Z}^+, \tag{3.1}$$

where $\Delta^+ x(n) := x(n+1) - x(n)$, x_n is the lift of the sequence $x = x(n)$ in $\mathcal{S} := \mathcal{S}([-r, 0]; \mathbb{R}^1)$, and $B(n, x_n)$ satisfies

$$|B(n, S)| \le b(n)\|S\|, \qquad n \in \mathbb{Z}^+,\ S \in \mathcal{S} \tag{3.2}$$

for some nonnegative sequence $b(n)$. It is clear that 0 is a solution of (3.1).

Since we have used forward difference in (3.1), the existence and uniqueness of solutions for the initial value problem of the equation is somewhat trivial. Denote by $x(n; m, \phi)$ the solution $x = x(n)$ of (3.1) on $[m - r, \infty)$ with initial data $x_m = \phi \in \mathcal{S}$.

The null solution 0 is called *globally asymptotically stable* (GAS in brief) if

(i) it is stable, i.e., for any $m \in \mathbb{Z}^+$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|x(n; m, \phi)| < \varepsilon$ for $n \in (m, \infty)$ and $\|\phi\| \le \delta$; and

(ii) it is globally attracting, meaning that $x(n; m, \phi) \to 0$ as $n \to \infty$ for every $(m, \phi) \in \mathbb{Z}^+ \times \mathcal{S}$.

It is called *globally exponentially asymptotically stable* (GEAS in brief), if for every $m \in \mathbb{Z}^+$, there exist positive constants M and σ with $\sigma < 1$ such that

$$|x(n; m, \phi)| \le M\|\phi\|\sigma^n, \qquad \forall\, n \in [m, \infty), \quad \phi \in \mathcal{S}.$$

Let

$$f(n, m) = \prod_{k=m}^{n-1} |1 - a(k)|, \qquad g(n, m) = f(n, m+1)b(m), \tag{3.3}$$

and set

$$ß = \sup_{n \ge m \ge 0} f(n, m), \qquad \kappa = \sup_{n \ge m \ge 0} \left(\sum_{k=m}^{n-1} g(n, k) \right).$$

Theorem 3.1 *Suppose*

$$\lim_{n \to \infty} f(m + n, m) = 0 \tag{3.4}$$

uniformly with respect to $m \in \mathbb{Z}^+$. If $\kappa < 1$, then the null solution of equation (3.1) is GAS; and if $\kappa < 1/(1 + ß)$, it is GEAS.

Proof. Let $x(n) = x(n; 0, \phi)$ and $n \geq m \geq 0$. Solving (3.1) by a standard argument via iteration and induction, one obtains that

$$x(n) = \prod_{k=m}^{n-1}(1 - a(k))x(m) + \sum_{k=m}^{n-1}\prod_{j=k+1}^{n-1}(1 - a(j))B(k, x_k), \qquad n \geq m. \quad (3.5)$$

Let $y(n) = |x(n)|$. By (3.2) and (3.5) one easily sees that

$$y(n) \leq f(n, m)\|y_m\| + \sum_{k=m}^{n-1} g(n, k)\|y_k\|, \qquad \forall\, n \geq m \geq 0.$$

The conclusions then follow immediately from Theorem 2.3 and Remark 2.12. □

Remark 3.2 *Note that we do not require that* $a(n) - b(n) \geq \delta > 0$ *for* $n \geq 0$ *and* $a(n) \leq 1$*; furthermore, the sequence* $a(n)$ *can change sign on* $\mathbb{Z}^+$*.*

In case $a(n) \equiv a$ *and* $b(n) \equiv b$*, if* a *and* b *fulfill* (1.4) *which was required in [21], one can easily verify that* f *satisfies* (3.4) *and* $\kappa < 1$*. Hence by Theorem 3.1 the null solution* 0 *of system* (3.1) *is GAS.*

Example 3.1. Consider the difference equation:

$$\Delta^+ x(n) = -\left(\sin\frac{2\pi n}{3}\right)x(n) + bx(n-1), \qquad n \in \mathbb{Z}^+, \quad (3.6)$$

where $b > 0$ is constant. Now we have

$$f(n, m) = \prod_{k=m}^{n-1}\left(1 - \sin\frac{2\pi n}{3}\right), \quad g(n, m) = b\prod_{k=m+1}^{n-1}\left(1 - \sin\frac{2\pi n}{3}\right).$$

According to the different values of m and $n - m$, one can calculate

$$ß = 1 + \frac{\sqrt{3}}{2}, \quad \kappa = \frac{12 + 2\sqrt{3}}{3}b.$$

Therefore if $b < 3/(12 + 2\sqrt{3}) \approx 0.193997$, then the null solution of (3.6) is GAS. If we further assume that $b < 3/(27 + 4\sqrt{3}) \approx 0.067688$, then it is GEAS.

Now let us do some numerical simulations via Matlab to illustrate the theoretical result. For simplicity, we set $x(-1) = x(0) = 1$. Taking $b = 0.19$ and $b = 0.0676$, our PC gives the following diagrams:

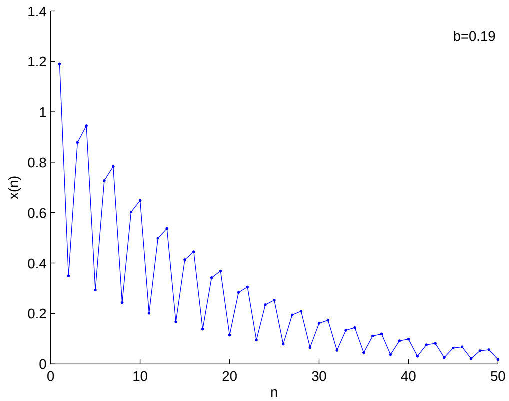

Figure 3.1: $b = 0.19$

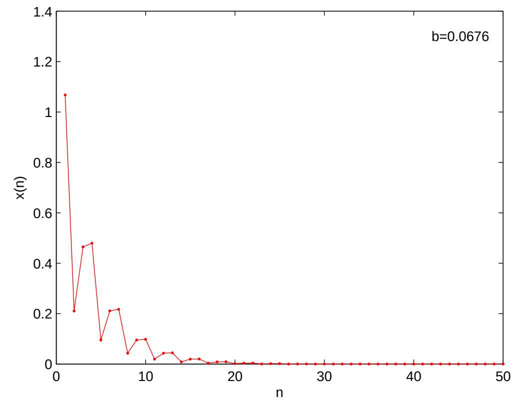

Figure 3.2: $b = 0.0676$

As a simple application of Theorem 2.11, we can also give an estimate for the growth rate of solutions of equation (3.1).

Theorem 3.3 *Let f, g be given as in* (3.3). *Suppose there is $\alpha > 0$ such that*

$$\sup_{n\geq m\geq 0}\left(f(n,m)\alpha^{n-m}\right) < \infty, \qquad \hat{\kappa} := \sup_{n\geq m\geq 0}\left(\sum_{k=m}^{n-1}\left(g(n,k)\alpha^{n-k}\right)\right) < \infty. \tag{3.7}$$

Then there exist $M > 0$ and $\beta_0 \in (0,1)$ such that

$$\|x_n\| \leq M\|x_0\|\beta_0^{-n}, \qquad n \in \mathbb{Z}^+$$

for all solutions of (3.1).

Proof. We observe that for $0 < \beta < \alpha$, one has

$$f_\beta(m+k,m) := f(m+k,m)\beta^k \leq \left(\frac{\beta}{\alpha}\right)^k \left(f(m+k,m)\alpha^k\right), \tag{3.8}$$

and

$$\begin{aligned}\kappa_\beta := \sup_{n\geq m\geq 0}\left(\sum_{k=m}^{n-1} g(n,k)\beta^{n-k}\right) &= \sup_{n\geq m\geq 0}\left(\sum_{k=m}^{n-1} g(n,k)\alpha^{n-k}\left(\frac{\beta}{\alpha}\right)^{n-k}\right)\\ &\leq \frac{\beta}{\alpha}\sup_{n\geq m\geq 0}\left(\sum_{k=m}^{n-1} g(n,k)\alpha^{n-k}\right) = \frac{\beta}{\alpha}\hat{\kappa}.\end{aligned} \tag{3.9}$$

By (3.3), (3.5) and (3.7)-(3.9) we see that if $\beta \in (0, \alpha)$ is chosen so that $\frac{\beta}{\alpha}\,\hat{\kappa} < 1$ (such a β is always available), then $f_\beta(m+k, m) \to 0$ as $k \to \infty$ uniformly with respect to $m \geq 0$, and $\kappa_\beta < 1$.

Thanks to Theorem 2.11 and Remark 2.12, one immediately concludes the validity of the result. $\square$

4 Dissipativity of Delay Difference Equations with Superlinear Nonlinearities

In this section we pay some attention to the dissipativity of the delay difference equation

$$\Delta^- y(n) = f(n, y(n)) + g(n, y(n - r(n))), \qquad y(n) \in \mathbb{R}^d,\ n \in \mathbb{Z} \tag{4.1}$$

with superlinear nonlinearities f and g, where

$$\Delta^- y(n) = y(n) - y(n-1),$$

f and g are mappings from $\mathbb{Z}\times\mathbb{R}^d$ to $\mathbb{R}^d$, and $r(n)$ ($n \in \mathbb{Z}$) are delays, $0 \leq r(n) \leq r$ for some $r \geq 0$. Specifically we show that the equation has a global pullback attractor under appropriate dissipative-type structure conditions.

4.1 Decay estimates

We will always assume that f and g satisfy the following conditions:

$$\langle f(n, y), y\rangle \leq -\alpha |y|^{p+1} + \alpha_1, \qquad y \in \mathbb{R}^d,\ n \in \mathbb{Z}, \tag{4.2}$$

$$|g(n, y)| \leq \beta |y|^q + \beta_1, \qquad y \in \mathbb{R}^d,\ n \in \mathbb{Z}, \tag{4.3}$$

where $\alpha,\ \alpha_1,\ \beta,\ \beta_1,\ p$ and q are positive constants.

Denote by $y(n; m, \phi)$ the solution of (4.1) with initial value:

$$y_m = \phi \in \mathcal{S} := \mathcal{S}\left([-r, 0]; \mathbb{R}^d\right).$$

Theorem 4.1 *Suppose that $p > q \geq 1$. Then there exist positive constants M, σ with $\sigma < 1$ and ρ such that*

$$|y(n; m, \phi)| \leq K\|\phi\|\sigma^{n-m} + \rho, \qquad \forall\,(m, \phi) \in \mathbb{Z}\times\mathcal{S},\ n \geq m.$$

Proof. Let $y = y(n) := y(n; m, \phi)$, and set $\gamma = p(q-1)/(p-q)+1$. Taking the inner product of (4.1) with $|y(n)|^{\gamma-1}y(n)$, we find that

$$\begin{aligned}|y(n)|^{\gamma+1} &= \left\langle y(n-1) + f(n, y(n)) + g(n, y(n-r(n))),\ |y(n)|^{\gamma-1}y(n)\right\rangle \\ &\le -\alpha|y(n)|^{\gamma+p} + \alpha_1|y(n)|^{\gamma-1} + |y(n)|^{\gamma}|y(n-1)| \\ &\qquad + \beta|y(n)|^{\gamma}\|y_n\|^q + \beta_1|y(n)|^{\gamma}. \end{aligned} \tag{4.4}$$

Using the Young's inequality one deduces that

$$|y(n)|^{\gamma}\|y_n\|^q \le \varepsilon\|y_n\|^{\gamma+1} + C_\varepsilon|y(n)|^{\gamma(\gamma+1)/(\gamma+1-q)} \tag{4.5}$$

for any $\varepsilon > 0$. Here (and below) C_ε denotes a general constant depending upon ε. We infer from the choice of γ that

$$\gamma(\gamma+1)/(\gamma+1-q) < \gamma + p.$$

Therefore by (4.5) one has

$$|y(n)|^{\gamma}\|y_n\|^q \le \varepsilon\|y_n\|^{\gamma+1} + \varepsilon|y(n)|^{\gamma+p} + C_\varepsilon.$$

A similar argument applies to show that

$$|y(n)|^{\gamma-1},\ |y(n)|^{\gamma} \le \varepsilon|y(n)|^{\gamma+1} + C_\varepsilon,$$

and

$$|y(n)|^{\gamma}|y(n-1)| \le \frac{\gamma}{\gamma+1}|y(n)|^{\gamma+1} + \frac{1}{\gamma+1}|y(n-1)|^{\gamma+1}.$$

Combining the above estimates with (4.4) it yields

$$\begin{aligned}|y(n)|^{\gamma+1} &\le -(\alpha - \varepsilon\beta)|y(n)|^{\gamma+p} + \varepsilon\beta\|y_n\|^{\gamma+1} + \left(\varepsilon\beta_1 + \frac{\gamma}{\gamma+1}\right)|y(n)|^{\gamma+1} \\ &\qquad + \frac{1}{\gamma+1}|y(n-1)|^{\gamma+1} + C_\varepsilon. \end{aligned} \tag{4.6}$$

Since $p > 1$, using the classical Young's inequality once again we obtain that

$$\left(\varepsilon\beta_1 + \frac{\gamma}{\gamma+1}\right)|y(n)|^{\gamma+1} \le \varepsilon|y(n)|^{\gamma+p} + C_\varepsilon.$$

Substituting the above estimate into (4.6) it yields

$$\begin{aligned}|y(n)|^{\gamma+1} \le\ & -(\alpha - \varepsilon\beta - \varepsilon)|y(n)|^{\gamma+p} + \varepsilon\beta\|y_n\|^{\gamma+1} \\ & + \tfrac{1}{\gamma+1}|y(n-1)|^{\gamma+1} + C_\varepsilon. \end{aligned} \tag{4.7}$$

In what follows we always assume that ε is chosen sufficiently small so that $\alpha - \varepsilon\beta - \varepsilon > 0$. By (4.7) we have

$$|y(n)|^{\gamma+1} \le \theta |y(n-1)|^{\gamma+1} + \varepsilon\beta \|y_n\|^{\gamma+1} + C_\varepsilon,$$

where $\theta = 1/(1+\gamma)$. Therefore, by a standard argument via iteration and induction, we deduce that

$$\begin{aligned} |y(n)|^{\gamma+1} &\le \theta^{n-l}|y(l)|^{\gamma+1} + \sum_{k=l+1}^{n} \theta^{n-k}\left(\varepsilon\beta\|y_k\|^{\gamma+1} + C_\varepsilon\right) \\ &\le \theta^{n-l}\|y_l\|^{\gamma+1} + \varepsilon\beta \sum_{k=l+1}^{n} \theta^{n-k}\|y_k\|^{\gamma+1} + C'_\varepsilon, \qquad \forall\, n \ge l \ge m. \end{aligned}$$

That is,

$$|y(n)|^{\gamma+1} \le \theta^{n-l}\|y_l\|^{\gamma+1} + \varepsilon\beta \sum_{k=l}^{n} \theta^{n-k}\|y_k\|^{\gamma+1} + C'_\varepsilon, \qquad \forall\, n \ge l \ge m.$$

We observe that ß $:= \sup_{n\ge m\ge 0} \theta^{n-m} = 1$, and

$$\kappa := \varepsilon\beta \sup_{n\ge m\ge 0} \sum_{k=m}^{n} \theta^{n-k} = \frac{\varepsilon\beta(1+\gamma)}{\gamma}.$$

Now we fix an $\varepsilon > 0$ with

$$\kappa = \frac{\varepsilon\beta(1+\gamma)}{\gamma} < 1/(1+\text{ß}) = 1/2.$$

Then the conclusion in Theorem 4.1 directly follows from Theorem 2.3. □

4.2 Existence of solutions

In general the initial value problem of an implicit difference equation may not be well-posed. This can be seen from the following easy examples of scalar equations

$$y(n+1) = \frac{1}{2}y(n) + y(n+1)^2, \qquad n \in \mathbb{Z}^+ \tag{4.8}$$

and

$$y(n+1) = a_n y(n) + y(n+1)^3, \qquad n \in \mathbb{Z}^+, \tag{4.9}$$

where $a_n \in \mathbb{R}^1$. One trivially checks that if $y(0) > 1/2$, then the first equation (4.8) has no (real) solution. The solution of the second equation (4.9) always exists for any initial value $y(0) \in \mathbb{R}^1$, whereas the uniqueness may fail to be true.

In what follows we show that the dissipativity condition ensures the existence of solutions for the initial value problem of equation (4.1).

Theorem 4.2 *Suppose the hypotheses in* (4.2)-(4.3) *are fulfilled. If* $p > q \geq 1$, *then for any* $m \in \mathbb{Z}$ *and* $\phi \in \mathcal{S}$, *equation* (4.1) *has at least one solution* $y = y(n; m, \phi)$ *on* $[m - r, \infty)$.

Proof. We rewritten equation (4.1) as

$$y(n) = f(n, y(n)) + g(n, y(n - r(n))) + y(n - 1). \tag{4.10}$$

Let $m \in \mathbb{Z}$, and $\phi \in \mathcal{S}$. To prove that equation (4.1) has at least a solution $y = y(n; m, \phi)$ on $[m - r, \infty)$, by (4.10) it can be easily seen that we only need to show that for each fixed $n \geq m$ and $u, v \in \mathbb{R}^d$, the solutions of the following two algebraic equations in $\mathbb{R}^d$ exist:

$$x = f(n, x) + g(n, x) + v, \quad x = f(n, x) + g(n, u) + v. \tag{4.11}$$

We only consider the first equation in (4.11). (The argument for the second one is similar and is thus omitted.) By virtue of the classical Schauder's fixed-point theorem (see [12, Corollary 8.1]), the existence of solutions of the equation is guaranteed as long as one can give a uniform estimates for solutions of the parameterized equation:

$$x = \eta(f(n, x) + g(n, x)) + v, \qquad \eta \in [0, 1]. \tag{4.12}$$

So let $x \in \mathbb{R}^d$ be a solution of (4.12). Taking the inner product of both sides of the equation with x, we deduce by (4.2) and (4.3) that

$$\begin{aligned} |x|^2 &= \langle \eta(f(n, x) + g(n, x)),\, x\rangle + \langle v, x\rangle \\ &\leq -\alpha|x|^{p+1} + \beta|x|^{q+1} + (\beta_1 + |v|)|x| + \alpha_1. \end{aligned}$$

Since $p > q$, one trivially deduce that $-\alpha|x|^{p+1} + \beta|x|^{q+1}$ has an upper bound M. Therefore we have

$$|x|^2 \leq (\beta_1 + |v|)|x| + \alpha_1 + M \leq \tfrac{1}{2}|x|^2 + \tfrac{1}{2}(\beta_1 + |v|)^2 + \alpha_1 + M.$$

Hence

$$|x|^2 \leq (\beta_1 + |v|)^2 + 2(\alpha_1 + M).$$

This is precisely what we desired. $\square$

Given $m \in \mathbb{Z}$ and $\phi \in \mathcal{S}$, denote by $\mathscr{S}(m, \phi)$ the solution set of (4.1) on $[m - r, \infty)$ with initial value ϕ.

Proposition 4.3 *Let $\phi_k \in \mathcal{S}$ $(k = 1, 2, \cdots)$, and $m \in \mathbb{Z}$. Suppose that $\phi_k \to \phi_0$ as $k \to \infty$. Let $y^k \in \mathscr{S}(m, \phi_k)$. Then there is a subsequence of y^k, still denoted by y^k, such that y^k converges to some $y^0 \in \mathscr{S}(m, \phi_0)$, meaning that for any $n \in [m - r, \infty)$,*

$$y^k(n) \to y^0(n) \quad \text{as } k \to \infty.$$

Proof. The proof for such results are quite standard for evolution equations by using appropriate uniform estimates for solutions as given in Theorem 4.1. We omit the details. □

4.3 Global pullback attractor

Define a discrete set-valued process $\mathcal{R}$ on $\mathcal{S}$ as below:

$$\mathcal{R}(n, m)\phi = \{y_n : \ y \in \mathscr{S}(m, \phi)\}, \qquad n \geq m > -\infty, \ \phi \in \mathcal{S}.$$

In view of Theorem 4.1 and Proposition 4.3, one can easily check that for each fixed n, m and $\phi \in \mathcal{S}$, $\mathcal{R}(n, m)\phi$ is a bounded closed subset of $\mathcal{S}$; furthermore, $\mathcal{R}(n, m)\phi$ is upper semicontinuous in ϕ. $\mathcal{R}$ possesses the following basic properties:

- $\mathcal{R}(m, m)\phi = \{\phi\}$ for all $m \in \mathbb{Z}$ and $\phi \in \mathcal{S}$;
- $\mathcal{R}(n, m)\phi = \mathcal{R}(n, k)\mathcal{R}(k, m)\phi$ for all $n \geq k \geq m$ and $\phi \in \mathcal{S}$.

These fundamental properties along with Theorem 4.1 allow us to apply the standard theory on pullback attractors (see e.g. [15, 7] and [16, Chapter 9]) to system $\mathcal{R}$ to obtain the following existence result of a global pullback attractor:

Theorem 4.4 *Suppose f and g satisfy* (4.2) *and* (4.3)*, respectively. If $p > q \geq 1$, then $\mathcal{R}$ has a pullback attractor in $\mathcal{S}$. More precisely, there exists a unique family $\mathcal{A} = \{A(n)\}_{n \in \mathbb{Z}}$ of compact sets with*

$$A(n) \subset \mathcal{B}_\rho := \{\phi \in \mathcal{S} : \ \|\phi\| < \rho\}$$

for all $n \in \mathbb{Z}$ such that

(1) *$\mathcal{R}(n, m)A(m) = A(n)$ for $n \geq m$;*

(2) *for any bounded set $B \subset \mathcal{S}$ and $n \in \mathbb{Z}$,*

$$\lim_{m \to -\infty} d_H(\mathcal{R}(n, m)B, A(n)) = 0,$$

where $d_H(\cdot,\cdot)$ denotes the Hausdorff semi-distance in $\mathcal{S}$,

$$d_H(B,A) = \sup_{\phi\in B}\inf\{\|\phi-\psi\| : \psi \in A\}, \quad \forall\, B,\, A\, \subset \mathcal{S}.$$

Remark 4.5 *We have only given some examples to illustrate the applications of the particular case of inequality* (2.1)*, namely, inequality* (2.6)*. Inequalities like the general form of* (2.1) *usually play important roles in studying the dynamical behavior of evolutions equations in case the linear parts of the equations are hyperbolic. In a forthcoming paper will report some results in this line, in which the invariant manifolds of equation*

$$y(n+1) = A_n y_n + f(n, y_{n+\delta_n}) \tag{4.13}$$

in a Banach space X are carefully investigated, where δ_n is an arbitrarily given sequence taking values in $\{0,1\}$. In case $\delta_n = 0$ with 0 being a solution of the equation, the invariant manifolds of the 0 solution of the equation was considered in a recent paper [2] by Barreira et al. Closely related works can be also found in [3] and [23], etc. We are interested in a more general case where we neither assume that the equation has a known trivial solution nor $\delta_n \equiv 0$. Note that if $\delta_n = 1$ then the equation is an implicit one. As we have seen in Section 4.2, for such an equation even if the existence of solutions can fail to be true.

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