# Weak solution to a Robin problem of anomalous diffusion equations: uniqueness and stable algorithm for the TPC system

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**ABSTRACT**
A *Riemann − Liouville* fractional Robin boundary-value problem is proposed to describe the fast heat transfer law both within isotropic materials and through the boundary of the materials in high temperature environment. The variational formulation of the fractional model is given, and further the energy estimation of the weak solution is deduced. The uniqueness theorem of weak solution is proved. A valid finite difference scheme is developed for the fractional model and numerical experiment is implemented. Numerical results indicate that the fractional model is applicable to discover the thermal superdiffusion in the thermal protective clothing(TPC) system and numerical algorithms are effective to improve the intelligence of TPC design.

**KEYWORDS**
Fractional diffusion equations; Fractional *Robin* boundary condition; *Riemann − Liouville* fractional derivative; Variational formulation; Weak solutions; Uniqueness; Stable difference scheme.

**AMS CLASSIFICATION**
35R11, 65M06

## 1. Introduction

In recent years, anomalous diffusion which deviates the classical Fickian diffusion has gained wide attention due mainly to its successful applications in science and engineering[1–3]. Anomalous diffusion refers to a class of non-equilibrium processes that can not be described by standard states in statistical physics. Microscopically, due to the complexity of the transmission medium, the random motion of particles is limited by the structure of irregular media, and its motion can not be described by the standard statistical method of particle motion in a uniform medium. In the view of macroscale explanation, the statistical laws of the movement of a large number of particles do not meet the standard statistical distribution, and the central limit theorem is no longer applicable.

Therefore, the relationship between the mean squared displacement and the time

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of the diffusion process under the point source conditions is described as follows:

$$\langle X^2(t) \rangle \sim \kappa_\alpha t^\alpha, \qquad \alpha > 0. \tag{1}$$

It is called a normal diffusion process when $\alpha = 1.0$; it is called an anomalous diffusion process when $\alpha \neq 1.0$. In detail, it is called subdiffusion or dispersive when $\alpha < 1.0$, it is called superdiffusion when $\alpha > 1.0$.

Practically, a diffusion process that does not conform to Fick's second diffusion law can be called anomalous diffusion[4]. Moreover the inverse problems for abnormal diffusion in sciences and engineering have attracted much more attention, we can refer to [5] for example for further detailed description.

The purpose of this paper is concerned with a kind of superdiffusion in the field of functional clothing design, such as thermal protective clothing(TPC) based on the heat transfer law. The study of heat transfer in porous media has a long history. A considerable number of fractional models have been proposed, including time fractional and spatial fractional equations. The boundary conditions are attributed to Dirichlet, Neumann or Robin boundary-value of the fractional form. Recently, research on the heat transfer within fabric materials for the TPC has gained more and more attension.

As a non-typical porous medium, the TPC is a kind of multi-layer structure products, coupled with its special working environment such as high temperature and high humidity situation, where the heat and humidity transfer process inside the TPC system is more complicated[6]. The classical Fourier's law of heat conduction or Fick's law of diffusion is no longer suitable for this situation. Inspired by the model proposed by J.T Fan et al and ourselves [7–16] and the faster transmission of superdiffusion [1,17], we proposed in this paper a class of spatial fractional heat transfer model to describe the faster transmission process instead of the classical Fourier's law.

Recalling the pioneer work for this topic, Ervin and Roop[18] proved coercivity of a Galerkin formulation and the well-posedness of the homogeneous Dirichlet boundary-value problem of a constant-coefficient conservative fractional differential equations (FDEs). Wang and Yang [19] proposed a Petrov-Galerkin formulation for the homogeneous Dirichlet boundary-value problem of variable-coefficient FDEs, and proved its coercivity and well-posedness. Most importantly, Wang and Yang [20] also proved that five out of the nine combinations of the three different forms of FDEs that are closed by three types of Neumann boundary conditions are well-posed and the remaining four kind of problems do not admit a solution.

Till now, the Robin boundary-value problems of FDEs are much more challenging than the Dirichlet or Neumann boundary-value problems.

This paper is organized as follows. In Section 1, the significance of anomalous diffusion and fractional model are introduced. In Section 2, a spatial fractional model with a fractional *Robin* boundary condition is proposed for the TPC system under high temperature situations. In Section 3, a related basic properties of the $Riemann-Liouville$ fractional derivative and some useful definitions are presented, demonstrating the energy estimation and uniqueness of the weak solution. In Section 4, the $shifted\ Grünwald\ formula$ is adopted to approximate the $Riemann-Liouville$ derivative, and a so called implicit-explicit (IMEX) method is proposed. Convergence rate of the numerical algorithm for the Robin problem with exact/noisy initial-boundary value conditions is derived. In Section 5, numerical simulation results show that the fractional model is appropriate to simulate the situation we considered. In the final section, some conclusions and prospects are given.

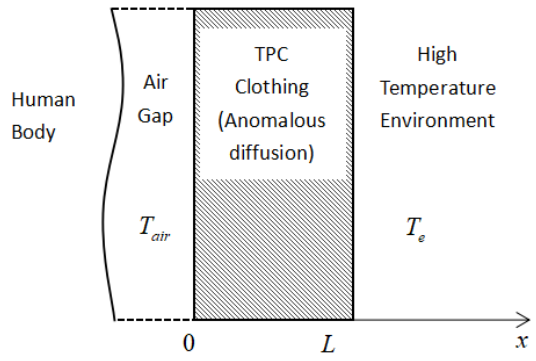



**Figure 1.** Schematic diagram of the body-clothing-environment system (TPC system).

## 2. A fractional model with a $R-L$ fractional *Robin* boundary condition

We summarily adopt some assumptions for the dynamic heat transfer process in the body-clothing-environment system[11,15], see Figure 1:

- Thermal protective clothing can be treated as a porous medium and fibrous battings are isotropic in fiber arrangement and material properties.
- The effective thermal conductivity of the fabric is constant under the condition that the external environment have a small change.
- Volume changes of the fibers due to the variation of moisture and water content are neglected and effective tortuosity is a constant.
- Heat conduction within the porous batting is $non-Fourier$ and can be described by the superdiffusion model.

In this paper we denote $L=1$, $\Omega=(0,1)\times(0,t_f), I=(0,1), \Lambda=(0,t_f)$, and adopt the following definition of $Riemann-Liouville$ fractional integrals and derivatives.

**Definition 2.1.** (Left and right $R-L$ fractional integrals)[21,22] For $\mu>0$, the left and right fractional integrals are defined by

$$I_{0+}^{\mu}f(x):=\frac{1}{\Gamma(\mu)}\int_0^x\frac{f(s)}{(x-s)^{1-\mu}}\,ds,\quad I_{1-}^{\mu}f(x):=\frac{1}{\Gamma(\mu)}\int_x^1\frac{f(s)}{(s-x)^{1-\mu}}\,ds, \tag{2}$$

where $\Gamma(\cdot)$ is the *Gamma* function, $x\in\bar{I}$.

**Definition 2.2.** (Left and right $R-L$ fractional derivatives)[21,22]For $\mu\geq 0$, the left and right fractional derivatives at $x\in I$ are defined by

$$D_{0+}^{\mu}f(x):=D^nI_{0+}^{n-\mu}f(x),\quad D_{1-}^{\mu}f(x):=(-D)^nI_{1-}^{n-\mu}f(x),\quad (n=[\mu]+1). \tag{3}$$

According to the $CTRW$($continuous\ time\ random\ walk$) scheme, the superdiffusion equation is formulated as follows:

$$\frac{\partial W}{\partial t}=K^{\mu}{}_{-\infty}D_x^{\mu}W(x,t),\quad (1<\mu<2), \tag{4}$$

where $W(x,t)$ is the pdf of being at a certain position $x$ at time $t$ (called the propagator) and ${}_{-\infty}D_x^{\mu}$ is $Weyl$ operator in one-dimension is equivalent to a $Riesz$ operator $\nabla^{\mu}$[23].

Unlike the above formula, the $Weyl$ operator ${}_{-\infty}D_x^{\mu}$ is replaced by utilizing the $Riemann-Liouville$ fractional derivative operator $D_{0+}^{\mu}, 1<\mu<2$ in the following equations because we only consider the diffusion process in a finite domain.

Based on physical and mathematical considerations, a heat transfer model with spatial fraction is proposed to describe the faster heat transfer process in high heat environment. The physical phenomenon is usually called superdiffusion in the sense of CTRW theory[1,23]. In this case the heat flux is described as the following diffusion law:

$$q(x,t)=\kappa D_{0+}^{\mu-1}T\left(x,t\right)$$

or

$$q(x,t)=\kappa D_{1-}^{\mu-1}T\left(x,t\right),$$

where $1<\mu<2$.

Let $\mu=2-\gamma$, $\gamma\in(0,1)$. So the derivation of the following space-fractional governed equation can be implemented by the same way as the classic case of heat transfer. The core process of the derivation is the element analysis method applied in arbitrary element $[x_1,x_2]\subset I$ during arbitrary time interval $[t_1,t_2]\subset\Lambda$ of one-dimensional setting:

$$\int_{t_1}^{t_2}(\kappa_\gamma D_{0+}^{1-\gamma}T(x_2,t)-\kappa_\gamma D_{0+}^{1-\gamma}T(x_1,t))dt=\int_{x_1}^{x_2}C_v(T(x,t_2)-T(x,t_1))dx,$$

that is,

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\kappa_\gamma D_{0+}^{2-\gamma}T(x,t))dxdt=\int_{t_1}^{t_2}\int_{x_1}^{x_2}C_v\frac{\partial T}{\partial t}dxdt.$$

Moreover the fractional Robin boundary conditions can be derived by the above mentioned heat flux formulaition with respect to superdiffusion:

$$\lim_{x\to0^+}k_{air}\frac{d}{dx}\frac{1}{\Gamma(\gamma)}\int\limits_0^x\frac{T(s,t)}{(x-s)^{1-\gamma}}\,ds=h_{air}\left(T\mid_{x=0}-T_{air}\right),$$

$$\lim_{x\to1^-}k_e\frac{d}{dx}\frac{1}{\Gamma(\gamma)}\int\limits_x^1\frac{T(s,t)}{(s-x)^{1-\gamma}}\,ds=h_e\left(T_e-T\mid_{x=1}\right).$$

Henthforth the the thermal superdiffusion will show that the temperature $T(x,t)$ in the TPC system satisfy the following initial/boundary value problem

$$\begin{cases} C_v T_t\left(x,t\right) = k_\gamma D_{0+}^{2-\gamma} T\left(x,t\right), & (x,t) \in \Omega, \\ T(x,0) = T_I(x), & x \in \bar{I}, \\ k_{air} D_{0+}^{1-\gamma} T\left(x,t\right) |_{x=0} = h_{air}\left(T |_{x=0} - T_{air}\right), & t \in \bar{\Lambda}, \\ k_e D_{1-}^{1-\gamma} T\left(x,t\right) |_{x=1} = h_e\left(T_e - T |_{x=1}\right), & t \in \bar{\Lambda}, \end{cases} \tag{5}$$

where $D_{0+}^{2-\gamma}T$, $D_{0+}^{1-\gamma}T$ and $D_{1-}^{1-\gamma}T$ are left and right *Riemann* – *Liouville* fractional derivatives respectively; $\gamma \in (0,1)$; $C_v$ is effective volumetric heat capacity of the fibrous batting; $k_\gamma, k_{air}, k_e$ are the thermal conductivity of textiles, air gap layer and outside environment respectively; $h_e, h_{air}$ are the heat exchange coefficient between the body and the TPC, and between the TPC and outside environent respectively, which are approximately regarded as constants; $T_{air}, T_e$ are the temperature in the air gap layer between body skin and the TPC, the outside environmental temperature respectively.

Let $c^2 = \frac{k_\gamma}{C_v}, \quad \alpha = \frac{h_{air}}{k_{air}}, \quad \beta = \frac{h_e}{k_e}$. (5) can be simplified as follows:

$$\begin{cases} T_t(x,t) = c^2 D_{0+}^{2-\gamma} T(x,t), & (x,t) \in \Omega, \\ T(x,0) = T_I(x), & x \in \bar{I}, \\ D_{0+}^{1-\gamma} T |_{x=0} - \alpha T |_{x=0} = -\alpha T_{air}, & t \in \bar{\Lambda}, \\ D_{1-}^{1-\gamma} T |_{x=1} + \beta T |_{x=1} = \beta T_e, & t \in \bar{\Lambda}. \end{cases} \tag{6}$$

The initial-boundary value problem for the space-fractional partial differential equation has been well defined according to the above process of methematical modeling. Next we will discuss its weak solution in mathematical way.

## 3. Variational formulation of the fractional model and uniqueness of the weak solution

### *3.1. Preparatory knowledge[20]*

Let $0 < \mu < 1$. Firstly we introduce a function $\kappa(\mu)$

$$\kappa \equiv \kappa(\mu) := \begin{cases} 2, & 0 < \mu < 1/2, \\ (1 - \varepsilon(\mu))/\mu, & 1/2 \le \mu < 1, \end{cases} \tag{7}$$

where $\varepsilon(\mu) < 1$ is a very small positive number. Evidently $\kappa < 2$ for any $\mu \in [1/2, 1)$.

**Definition 3.1.** (Left and right $R - L$ fractional derivatives spaces[20])

Denote left and right *Riemann* – *Liouville* fractional derivatives spaces by

$$H_l^\mu(0,1) := \left\{ v \in L^\kappa(0,1) : D_{0+}^\mu v \in L^2(0,1) \right\}, 0 < \mu < 1, \tag{8}$$

$$H_r^\mu(0,1) := \left\{ v \in L^\kappa(0,1) : D_{1-}^\mu v \in L^2(0,1) \right\}, 0 < \mu < 1, \tag{9}$$

respectively, which are equipped with the (semi) norm and seminorm respectively

$$|v|_{H_l^\mu(0,1)} := \left\| D_{0+}^\mu v \right\|_{L^2(0,1)}, \|v\|_{H_l^\mu(0,1)} := \left\{ \|v\|_{L^\kappa(0,1)}^2 + |v|_{H_l^\mu(0,1)}^2 \right\}^{\frac{1}{2}}, \tag{10}$$

$$|v|_{H_r^\mu(0,1)} := \left\| D_{1-}^\mu v \right\|_{L^2(0,1)}, \|v\|_{H_r^\mu(0,1)} := \left\{ \|v\|_{L^\kappa(0,1)}^2 + |v|_{H_r^\mu(0,1)}^2 \right\}^{\frac{1}{2}}. \tag{11}$$

These nonconventional fractional derivative spaces is defined to ensure well-posedness and regularity property of the true solution and weak solution of the IBVP (6), and also the convergence rate of its numerical solution.

We also define subspaces $H_l^{\mu,0}(0,1) \subset H_l^\mu(0,1)$ and $H_r^{\mu,0}(0,1) \subset H_r^\mu(0,1)$

$$H_l^{\mu,0}(0,1) := \left\{ v \in H_l^\mu(0,1) : \int_0^1 I_{0+}^{1-\mu} v(x)\, dx = 0 \right\}, 0 < \mu < 1, \tag{12}$$

$$H_r^{\mu,0}(0,1) := \left\{ v \in H_r^\mu(0,1) : \int_0^1 I_{1-}^{1-\mu} v(x)\, dx = 0 \right\}, 0 < \mu < 1, \tag{13}$$

or

$$H_l^{\mu,0}(0,1) := \left\{ v - \frac{\int_0^1 I_{0+}^{1-\mu} v(x) dx}{\Gamma(\mu)} x^{\mu-1} | \forall v \in H_l^\mu(0,1) \right\}, 0 < \mu < 1, \tag{14}$$

$$H_r^{\mu,0}(0,1) := \left\{ v - \frac{\int_0^1 I_{1-}^{1-\mu} v(x) dx}{\Gamma(\mu)} (1-x)^{\mu-1} | \forall v \in H_r^\mu(0,1) \right\}, 0 < \mu < 1. \tag{15}$$

**Remark 1.** The subspaces $H_l^{\mu,0}(0,1)$ and $H_r^{\mu,0}(0,1)$ mean the subsets of the spaces $H_l^\mu(0,1), H_r^\mu(0,1)$ with homogeneous boundaries respectively. For example, we see that $x^{\mu-1} \in H_l^\mu(0,1)$, and $v(x) = x^{\mu-1} - \Gamma(\mu) \in H_l^{\mu,0}(0,1)$.

Due to the Riemann-Liouville fractional Friediches inequality, for example Corollary 3.2 in [20], corresponding norms are defined for the subspaces $H_l^{\mu,0}(0,1) \subset H_l^\mu(0,1), H_r^{\mu,0}(0,1) \subset H_r^\mu(0,1)$ respectively

$$\|v\|_{H_l^{\mu,0}(0,1)} := |v|_{H_l^\mu(0,1)}, \|v\|_{H_r^{\mu,0}(0,1)} := |v|_{H_r^\mu(0,1)}. \tag{16}$$

**Remark 2.** (Adjoint property of fractional integral operators) The left and right R-L fractional integral operators are adjoints in the $L^2$ sense, i.e., for all $\mu > 0$,

$$(I_{0+}^\mu w, v)_{L^2(0,1)} = (w, I_{1-}^\mu v)_{L^2(0,1)}, \quad \forall w, v \in L^2(0,1). \tag{17}$$

The detailed analysis of the above-mentioned fractional spaces can be found in [20].

### 3.2. Variational formulation and definition of weak solution

To derive the variational formulation by the similar way as the classical case, homogenizing the boundary value conditions yields

$$\begin{cases} u_t = c^2 D_{0+}^{2-\gamma} u + \Theta(x,t), & (x,t) \in \Omega, \\ u(x,0) = u_0(x), & x \in \bar{I}, \\ D_{0+}^{1-\gamma} u \mid_{x=0} - \alpha u \mid_{x=0} = 0, & t \in \bar{\Lambda}, \\ D_{1-}^{1-\gamma} u \mid_{x=1} + \beta u \mid_{x=1} = 0, & t \in \bar{\Lambda}. \end{cases} \tag{18}$$

Here

$$u(x,t) = T(x,t) - h(x,t), \tag{19}$$

$$\Theta(x,t) = \frac{c^2 A(t)}{\Gamma(\gamma)} x^{\gamma-1} + \frac{c^2 B(t)}{\Gamma(\gamma-1)} x^{\gamma-2} - A'(t)x - B'(t), \tag{20}$$

$$u_0(x) = T_I(x) - h(x,0), \tag{21}$$

$$h(x,t) = A(t)x + B(t). \tag{22}$$

$$A(t) = \frac{-(\beta + \frac{1}{\Gamma(\gamma)})T_{air}(t) + \beta T_e(t)}{\frac{1}{\Gamma(1-\gamma)} + \beta}, \tag{23}$$

$$B(t) = T_{air}(t). \tag{24}$$

If $u(x,t) \in L^2(\Lambda; H^{2-\gamma,0}(I)) \subset L^2(\Lambda; H^{1-\gamma,0}(I))$ satisfies the governed equation and prescribed initial/boundary conditions in (18), then $u(x,t)$ is called a classic solution of (18).

Multiplying the equation in (18) by $I_{0+}^{\gamma} v$ and integration on $\bar{\Omega}$ yields

$$(u_t,\ I_{0+}^{\gamma} v) = c^2 (D_{0+}^{2-\gamma} u\ , I_{0+}^{\gamma} v) + (\Theta(x,t),\ I_{0+}^{\gamma} v), \quad \forall v \in L^2(\Lambda; H^{1-\gamma,0}(I)). \tag{25}$$

By the partial integral formula, we have

$$\begin{aligned} (D_{0+}^{2-\gamma} u\ , I_{0+}^{\gamma} v) &= -\left(D_{0+}^{1-\gamma} u,\ D_{0+}^{1-\gamma} v\right) + \int_0^{t_f} \alpha u|_{x=0}\, I_{0+}^{\gamma} v|_{x=0}\, dt + \int_0^{t_f} \beta u|_{x=1}\, I_{0+}^{\gamma} v|_{x=1}\, dt \\ &= -\left(D_{0+}^{1-\gamma} u,\ D_{0+}^{1-\gamma} v\right). \end{aligned} \tag{26}$$

So the variational formulation of (18) is concluded as follows: find $u \in L^2(\Lambda; H^{1-\gamma,0}(I))$ such that

$$(u_t,\ I_{0+}^{\gamma} v) = -c^2(D_{0+}^{1-\gamma} u,\ D_{0+}^{1-\gamma} v) + (\Theta(x,t),\ I_{0+}^{\gamma} v), \quad \forall v \in L^2(\Lambda; H_l^{1-\gamma,0}(I)). \tag{27}$$

We observe that the inner product $(\Theta(x,t),\ I_{0+}^{\gamma} v)$ does not make sense due to the less regularity of the nonlinear source term $\Theta(x,t)$ at the point $x = 0$. Therefore we turn to make reformulation of the weak solution since it is not necessary to assume that the selection of the test function can offset the singularity. One can check that $x^{2-\gamma}u \in L^2(\Lambda; H_l^{1-\gamma,0}(I))$ when $u \in L^2(\Lambda; H_l^{1-\gamma,0}(I))$.

Motivating by above observation, we thus deduce the following reformulation of weak solution in variational formulation.

**Definition 3.2.** (Weak solution) We call $u(x,t)$ a weak solution of (18) if there exist $\tilde{u} = x^{2-\gamma}u \in L^2(\Lambda; H_l^{1-\gamma,0}(I))$ such that

$$\begin{cases} (\tilde{u}_t,\ I_{0+}^{\gamma} v) = -c^2(D_{0+}^{1-\gamma} \tilde{u},\ D_{0+}^{1-\gamma} v) + (x^{2-\gamma}\Theta(x,t),\ I_{0+}^{\gamma} v), \forall v \in L^2(\Lambda; H_l^{1-\gamma,0}(I)), \\ \tilde{u}(x,0) = \tilde{u}_0(x) \end{cases} \tag{28}$$

where $\tilde{u}_0(x) = x^{2-\gamma}u_0(x)$.

The detailed analysi of reformulation of the weak solution can be found in [15].

### *3.3. Energy estimation of weak solutions*

According to (28), we have

$$(\tilde{u}_t,\ I_{0+}^{\gamma} v) + c^2(D_{0+}^{1-\gamma} \tilde{u},\ D_{0+}^{1-\gamma} v) = (x^{2-\gamma}\Theta(x,t),\ I_{0+}^{\gamma} v), \forall v \in L^2(\Lambda; H_l^{1-\gamma,0}(I)). \tag{29}$$

Let $v = \tilde{u}$, we have

$$(\tilde{u}_t,\ I_{0+}^{\gamma} \tilde{u})_{L^2(0,1)} = \frac{1}{2} \cdot \frac{d}{dt} \left\| I_{0+}^{\gamma} (\tilde{u}^2) \right\|_{L^1(0,1)}, \tag{30}$$

$$c^2(D_{0+}^{1-\gamma} \tilde{u},\ D_{0+}^{1-\gamma} \tilde{u})_{L^2(0,1)} = c^2 \left\| D_{0+}^{1-\gamma} \tilde{u} \right\|_{L^2(0,1)}^2 = c^2 \left\| \tilde{u} \right\|_{H_l^{1-\gamma,0}(0,1)}^2, \tag{31}$$

$$\left| (x^{2-\gamma}\Theta(x,t),\ I_{0+}^{\gamma} \tilde{u}) \right|_{L^2(0,1)} \le \left\| x^{2-\gamma}\Theta(x,t) \right\|_{L^2(0,1)} \left\| I_{0+}^{\gamma} \tilde{u} \right\|_{L^2(0,1)} \le \frac{1}{2} \left\| x^{2-\gamma}\Theta(x,t) \right\|_{L^2(0,1)}^2 + \frac{1}{2} \left\| I_{0+}^{\gamma} . \tilde{u} \right\|_{L^2(0,1)}^2 \tag{32}$$

Then one can easily obtain that

$$\frac{d}{dt} \left\| I_{0+}^{\gamma} (\tilde{u}^2) \right\|_{L^1(0,1)} + 2c^2 \left\| \tilde{u} \right\|_{H_l^{1-\gamma,0}(0,1)}^2 \le \left\| x^{2-\gamma}\Theta(x,t) \right\|_{L^2(0,1)}^2 + \left\| I_{0+}^{\gamma} \tilde{u} \right\|_{L^2(0,1)}^2 . \tag{33}$$

Integrating by parts form 0 to $t_f$, one can obtain

$$\begin{aligned}&\left\|I_{0+}^{\gamma}(\tilde{u}(\cdot,t_f)^2)\right\|_{L^1(0,1)}+2c^2\int_0^{t_f}\|\tilde{u}\|^2_{H_l^{1-\gamma,0}(0,1)}\ dt\\ \leq&\left\|I_{0+}^{\gamma}(\tilde{u}_0^2)\right\|_{L^1(0,1)}+\int_0^{t_f}\left\|x^{2-\gamma}\Theta(x,t)\right\|^2_{L^2(0,1)}\ dt+\int_0^{t_f}\left\|I_{0+}^{\gamma}\tilde{u}\right\|^2_{L^2(0,1)}\ dt.\end{aligned}\tag{34}$$

Since there is a positive constant $C$ independent of $u\in H_l^{1-\gamma}(0,1)$, we have $||I_{0+}^{\gamma}u||_{L^2(0,1)}\leq C||u||_{H_l^{1-\gamma}(0,1)}$[20]. Therefore, the above formula can be simplified as

$$\begin{aligned}&\left\|I_{0+}^{\gamma}(\tilde{u}(\cdot,t_f)^2)\right\|_{L^1(0,1)}+C\int_0^{t_f}\|\tilde{u}\|^2_{H_l^{1-\gamma,0}(0,1)}\ dt\\ \leq&\left\|I_{0+}^{\gamma}(\tilde{u}_0^2)\right\|_{L^1(0,1)}+\int_0^{t_f}\left\|x^{2-\gamma}\Theta(x,t)\right\|^2_{L^2(0,1)}\ dt.\end{aligned}\tag{35}$$

So we can get the following theorem:

**Theorem 3.3.** *(Energy estimation of weak solution) For any $\gamma\in(0,1)$, if $u(x,t)$ is a weak solution to the problem (18), then we can get the following energy estimate*

$$\begin{aligned}&\left\|I_{0+}^{\gamma}(x^{4-2\gamma}u^2(x,t_f))\right\|_{L^1(0,1)}+C\int_0^{t_f}\left\|x^{2-\gamma}u(x,t)\right\|^2_{H_l^{1-\gamma,0}(0,1)}\ dt\\ \leq&\left\|I_{0+}^{\gamma}(x^{4-2\gamma}u_0^2(x))\right\|_{L^1(0,1)}+\int_0^{t_f}\left\|x^{2-\gamma}\Theta(x,t)\right\|^2_{L^2(0,1)}\ dt.\end{aligned}\tag{36}$$

### *3.4. Uniqueness of the weak solution*

According to the energy estimate of the weak solution in Theorem 3.3, we can derive the uniqueness of the weak solution of the problem (18).

**Lemma 3.4.** *(Uniqueness of the weak solution) For any $\gamma\in(0,1)$, if $u_1,u_2$ are two weak solutions to the problem(18), then $u_1=u_2$ in $L^2(\Lambda;H_l^{1-\gamma,0}(I))$.*

***Proof.*** If $u_1,u_2$ are two weak solutionsto the problem (18), then we have $\tilde{u}_1=x^{2-\gamma}u_1,\tilde{u}_2=x^{2-\gamma}u_2$ respectively satisfying

$$\begin{cases}((\tilde{u}_1)_t,\ I_{0+}^{\gamma}v)=-c^2(D_{0+}^{1-\gamma}\tilde{u}_1,\ D_{0+}^{1-\gamma}v)+(x^{2-\gamma}\Theta(x,t),\ I_{0+}^{\gamma}v), & \forall v\in L^2(\Lambda;H_l^{1-\gamma,0}(I)),\\ \tilde{u}_1(x,0)=\tilde{u}_0(x).\end{cases}\tag{37}$$

$$\begin{cases}((\tilde{u}_2)_t,\ I_{0+}^{\gamma}v)=-c^2(D_{0+}^{1-\gamma}\tilde{u}_2,\ D_{0+}^{1-\gamma}v)+(x^{2-\gamma}\Theta(x,t),\ I_{0+}^{\gamma}v), & \forall v\in L^2(\Lambda;H_l^{1-\gamma,0}(I)),\\ \tilde{u}_2(x,0)=\tilde{u}_0(x).\end{cases}\tag{38}$$

Let $u = u_1 - u_2$, $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, then $\tilde{u}$ satisfied

$$\begin{cases} (\tilde{u}_t,\ I_{0+}^{\gamma} v) = -c^2 (D_{0+}^{1-\gamma}\tilde{u},\ D_{0+}^{1-\gamma} v), \qquad \forall v \in L^2(\Lambda; H_l^{1-\gamma,0}(I)), \\ \tilde{u}(x,0) = 0. \end{cases} \tag{39}$$

According to the variational formulation and the definition of the weak solution, $u$ that (39) satisfies $\tilde{u} = x^{2-\gamma} u$ is a weak solution of (40).

$$\begin{cases} u_t = c^2 D_{0+}^{2-\gamma} u, & (x,t) \in \Omega, \\ u(x,0) = 0, & x \in \bar{I}, \\ D_{0+}^{1-\gamma} u \mid_{x=0} -\alpha u \mid_{x=0} = 0, & t \in \bar{\Lambda}, \\ D_{1-}^{1-\gamma} u \mid_{x=1} +\beta u \mid_{x=1} = 0, & t \in \bar{\Lambda}. \end{cases} \tag{40}$$

From Theorem 3.3, we therefore derive

$$\left\| I_{0+}^{\gamma} (x^{4-2\gamma} u^2(x,t_f)) \right\|_{L^1(0,1)} + C \int_0^{t_f} \|\tilde{u}\|^2_{H_l^{1-\gamma,0}(0,1)}\ dt \le 0. \tag{41}$$

Which implies that $\tilde{u} = 0$, i.e., $u_1 = u_2$ in $L^2(\Lambda; H_l^{1-\gamma,0}(I \setminus E_0))$, where $E_0$ is set of measure zero. □

From Lemma 3.4 and the relationship $T = u + h$, we conclude the following theorem.

**Theorem 3.5.** *For any $\gamma \in (0,1)$, if $u(x,t)$ is a weak solution to the problem (18), then the weak solution $T = u + h \in L^2(\Lambda; H_l^{1-\gamma,0}(I \setminus (0,\varepsilon)))$ of the problem (6) is unique.*

## 4. Numerical algorithm

### *4.1. Equation discretization*

We adopt the *shifted Grünwald formula* at all time levels for approximating the fractional derivative[24,25]

$$D_{0+}^{2-\gamma} T(x_i, t_{n+1}) = \frac{1}{h^{2-\gamma}} \sum_{j=0}^{i+1} g_j T(x_i - (j-1)h, t_{n+1}) + \mathcal{O}(h). \tag{42}$$

Here the normalized *Grünwald* weights are defined by

$$g_0 = 1, \quad g_j = (-1)^j \frac{(2-\gamma)(1-\gamma)\cdots(3-\gamma-j)}{j!}, j = 1,2,3,\cdots. \tag{43}$$

Particularly, $g_0 = 1,\ g_1 = -(2-\gamma),\ g_2 = \frac{(2-\gamma)(1-\gamma)}{2}$. Thus we have

$$\begin{aligned} C_v \frac{T_i^{n+1} - T_i^n}{\tau} &= \frac{k_\gamma}{h^{2-\gamma}} \sum_{j=0}^{i+1} g_j T(x_i - (j-1)h, t_{n+1}) \\ &= \frac{k_\gamma}{h^{2-\gamma}} \sum_{j=0}^{i+1} g_j T_{i+1-j}^{n+1}, \quad i = 1, 2, \cdots, M-1, n = 0, 1, \cdots, N-1. \end{aligned} \tag{44}$$

Let $s = \frac{k_r \tau}{h^{2-\gamma} C_v}$, we have

$$\begin{cases} (1 - sg_1)T_1^{n+1} - sg_2 T_2^{n+1} = T_1^n + sg_2 T_0^{n+1}, \\ -sg_i T_1^{n+1} - sg_{i-1} T_2^{n+1} - \cdots - sg_2 T_{i-1}^{n+1} + (1 - sg_1)T_i^{n+1} - sg_0 T_{i+1}^{n+1} \\ = T_i^n + sg_{i+1} T_0^{n+1}, \quad i = 2, 3, \cdots, M-1, n = 0, 1, \cdots, N-1. \end{cases} \tag{45}$$

### 4.2. Boundary discretization

We also adopt the *shifted Grünwald formula* at all time levels for approximating the fractional derivative[24,25]

$$D_{0+}^{1-\gamma} T(x_i, t_{n+1}) = \frac{1}{h^{1-\gamma}} \sum_{k=0}^{i+1} \omega_k T(x_i - (k-1)h, t_{n+1}) + \mathcal{O}(h). \tag{46}$$

$$D_{1-}^{1-\gamma} T(x_i, t_{n+1}) = \frac{1}{h^{1-\gamma}} \sum_{k=0}^{M-i+1} \omega_k T(x_i + (k-1)h, t_{n+1}) + \mathcal{O}(h), \tag{47}$$

where

$$\omega_0 = 1, \quad \omega_k = (-1)^k \frac{(1-\gamma)(-\gamma)\cdots(2-\gamma-k)}{k!}, k = 1, 2, 3, \cdots.$$

Hence when $x = 0$, we have

$$\frac{k_{air}}{h^{1-\gamma}} \sum_{k=0}^{1} \omega_k T_0^{n+1} = \frac{k_{air}}{h^{1-\gamma}} \left( \omega_0 T_1^{n+1} + \omega_1 T_0^{n+1} \right) = h_{air}(T_0^{n+1} - T_{air}^{n+1}), \quad n = 0, 1, \cdots, N-1. \tag{48}$$

When $x = 1$, we have

$$\frac{k_e}{h^{1-\gamma}} \sum_{k=0}^{1} \omega_k T_M^{n+1} = \frac{k_e}{h^{1-\gamma}} \left( \omega_0 T_{M-1}^{n+1} + \omega_1 T_M^{n+1} \right) = h_e(T_e^{n+1} - T_M^{n+1}), \quad n = 0, 1, \cdots, N-1. \tag{49}$$

### 4.3. Discretization scheme and convergence analysis

The above equations are expressed in the matrix form

$$\begin{pmatrix} 1-\frac{sg_2}{\alpha\sigma h^{1-\gamma}}-sg_1 & -sg_0 & & & & \\ -\left(\frac{sg_3}{\alpha\sigma h^{1-\gamma}}+sg_2\right) & 1-sg_1 & -sg_0 & & & \\ -\left(\frac{sg_4}{\alpha\sigma h^{1-\gamma}}+sg_4\right) & -sg_2 & 1-sg_1 & -sg_0 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ -\left(\frac{sg_{M-1}}{\alpha\sigma h^{1-\gamma}}+sg_{M-2}\right) & -sg_{M-3} & \cdots & -sg_2 & 1-sg_1 & -sg_0 \\ -\left(\frac{sg_M}{\alpha\sigma h^{1-\gamma}}+sg_{M-1}\right) & -sg_{M-2} & \cdots & -sg_3 & -sg_2 & 1+\frac{sg_0}{\beta\eta h^{1-\gamma}}-sg_1 \end{pmatrix}\begin{pmatrix} T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \\ \vdots \\ T_{M-2}^{n+1} \\ T_{M-1}^{n+1} \end{pmatrix}$$

$$=\begin{pmatrix} T_1^{n} \\ T_2^{n} \\ T_3^{n} \\ \vdots \\ T_{M-2}^{n} \\ T_{M-1}^{n} \end{pmatrix}+\frac{s}{\sigma}T_{air}^{n+1}\begin{pmatrix} g_2 \\ g_3 \\ g_4 \\ \vdots \\ g_{M-1} \\ g_M \end{pmatrix}+\frac{s}{\eta}T_{e}^{n+1}\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad n=0,1,\cdots,N-1, \tag{50}$$

where

$$\sigma = 1-\frac{\gamma-1}{\alpha h^{1-\gamma}}, \qquad \eta=\frac{\gamma-1}{\beta h^{1-\gamma}}+1. \tag{51}$$

**Remark 3.** (Convergence rate of the FDM algorithm) Clearly, (50) is called an implicit-explicit (IMEX) scheme. One can prove that the above scheme is unconditionally stable and has convergence rate $\mathcal{O}(h)+\mathcal{O}(\tau)$, since the inverse of the coefficient Matrix, that is, the spectral radius of the iterative matrix is less than one[23,26].

In the problem (6), the initial-boundary measurements are usually noised with a noise level $\delta$, henceforth, the problem (6) can be rewritten as

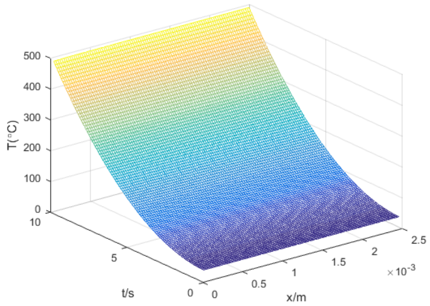
$$\begin{cases} C_vT_t^{\delta}(x,t)=k_\gamma D_{0+}^{2-\gamma}T^{\delta}(x,t), & (x,t)\in\Omega, \\ T^{\delta}(x,0)=T_I^{\delta}(x), & x\in\bar{I}, \\ k_{air}D_{0+}^{1-\gamma}T^{\delta}(x,t)\mid_{x=0}=h_{air}\left(T^{\delta}\mid_{x=0}-T_{air}^{\delta}\right), & t\in\bar{\Lambda}, \\ k_eD_{1-}^{1-\gamma}T^{\delta}(x,t)\mid_{x=1}=h_e\left(T_e^{\delta}-T^{\delta}\mid_{x=1}\right), & t\in\bar{\Lambda}. \end{cases} \tag{52}$$

From Theorem 3.3 and the relationship $T=h+u$, we can derive the following estimate.

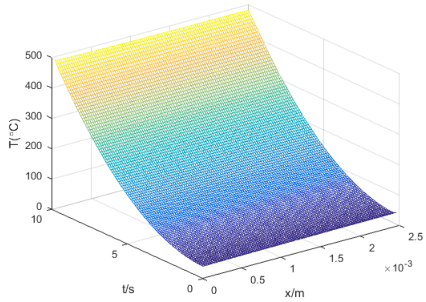
**Remark 4.** Assume that the boundary conditions $T_{air}(x,t), T_e(x,t)$ and $T_I(x)$ are noised by $T_{air}^{\delta}(x,t), T_e^{\delta}(x,t)$ and $T_I^{\delta}(x)$ such that $||T_{air}^{\delta}(t)-T_{air}(t)||\le\delta, ||T_e^{\delta}(t)-T_e(t)||\le\delta, ||T_I^{\delta}(x)-T_I(x)||\le\delta$. Then the perturbed solution with noisy data $T^{\delta}(x,t)$ satisfies

$$\left\|I_{0+}^{\gamma}(x^{4-2\gamma}(T^{\delta}-T)(\cdot,t_f))^2\right\|_{L^1(0,1)}+C\int_0^{t_f}\left\|x^{2-\gamma}(T^{\delta}-T)\right\|^2_{H_l^{1-\gamma,0}(0,1)}dt=\mathcal{O}(\delta^2) \tag{53}$$

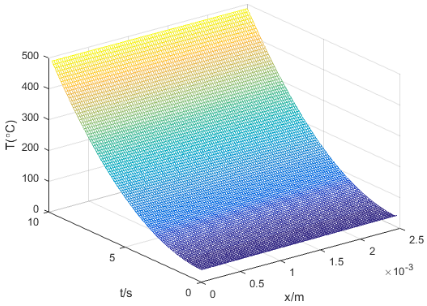
as $\delta\to0$.



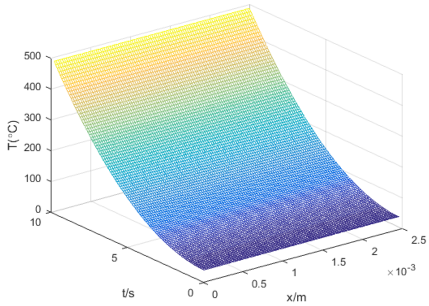

(a) $\gamma = 0$.




(b) $\gamma = 0.2$.




(c) $\gamma = 0.5$.




(d) $\gamma = 0.7$.

**Figure 2.** Temperature distribution within the domain $[0, L] \times [0, t_f]$.

## 5. Numerical simulation for the TPC system

### *5.1. Parameters and conditions in the body-clothing-environment system*

The fractional thermal conductivity $k_\gamma$ of textiles will be approximated by $k_\gamma = \varepsilon k_{air} + (1-\varepsilon)k_f$.

In the numerical simulation, we set thermal conductivities: $k_f = 0.084W \cdot m^{-1} \cdot K^{-1}$, $k_{air} = 0.024W \cdot m^{-1} \cdot K^{-1}$, $k_e = 0.1W \cdot m^{-1} \cdot K^{-1}$; thickness: $L = 2.5 \times 10^{-3}m$, terminal time $t_f \in [0, 10s]$; left boundary value condition: $T_{air} = 37°C$; right boundary value condition: $T_e = 500°C$; initial condition: $T_I(x) = -\frac{T_{air}-T_e}{L^2}x^2 + T_{air}$.

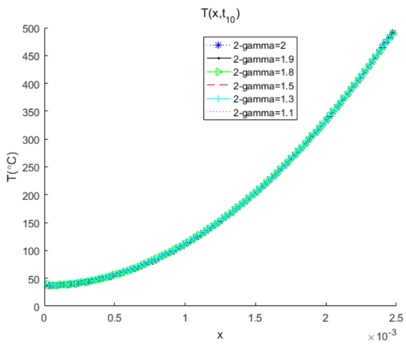
The physical parameters in equations are given as follows[15]: $\varepsilon = 0.88$, $C_v = 1715.0kJ \cdot m^{-3} \cdot K^{-1}$, $h_{air} = 0.021W \cdot m^{-2} \cdot K^{-1}$, $h_e = 40W \cdot m^{-2} \cdot K^{-1}$.

### *5.2. Numerical examples for exact measurements*

In this section, we present some numerical examples where the initial-boundary measurements are exact, and show the temperature distribution within the TPC system.

In the numerical experiments we show that in high temperature situation, the fractional model gives more reasonable results than the classical heat transfer model.

From Figure 2, it can be seen that the fractional model of different $\gamma \in (0, 1)$ shows faster heat transfer than the classic case $\gamma = 0$. And for different fractional order, the heat transfer changes with different speed. This fact also conforms to real-world phenomena.



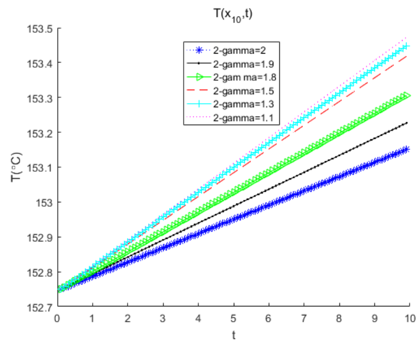



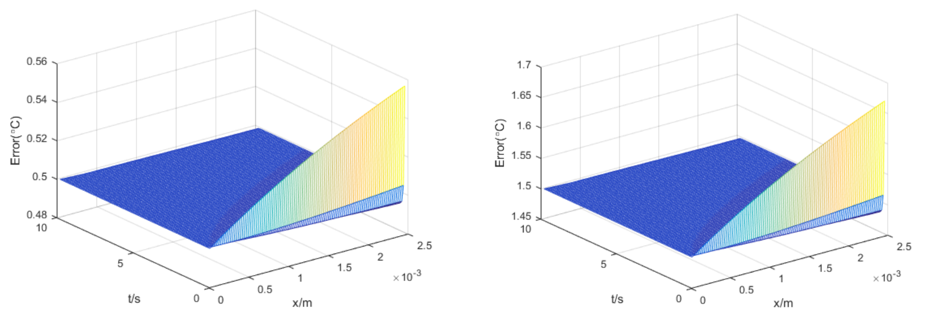

**Figure 3.** Temperature distribution when $x$ is fixed. **Figure 4.** Temperature distribution when $t$ is fixed.

To clarify these results, we plot several curves in Figure 3 and Figure 4 with $2-\gamma = 1.1, 1.3, 1.5, 1.8, 1.9, 2.0$ respectively. The temperature $T(x,t)$ for a fixed $x = x_{10}(M = 100, N = 10)$ decreases faster and faster as $\gamma$ decreases. For a fixed $t = t_{10}$, the temperature of $T(x,t)$ has same decreasing tendency as $\gamma$ decreases.

**Remark 5.** The result of the fractional model are coincident with the conclude of [15] at high temperature. On the other hand, the model describes the faster propagation as we expected, which is also applicable to the TPC design[6].

### *5.3. Numerical examples for noisy measurements*

In this section, we present two numerical examples where the initial-boundary measurements are noised, and show the error distribution $T_i^{n,\delta} - T_i^n$ in the case of $\delta$ is known.




(a) $\gamma = 0.5, \delta = 0.5$.

(b) $\gamma = 0.5, \delta = 1.5$.

**Figure 5.** Error distribution within the domain $[0, L] \times [0, t_f]$.

In the numerical experiments we shows that in the high temperature situation, the error level of the approximate solution is related to the data error level $\delta$. From Figure 5, the error of the approximate solution increases with the increase of the data error level, and we can calculate $E_N^2 := \Sigma_{i=1}^M (T_i^{N,\delta} - T_i^N)^2 = 4.9572$, when $\gamma = 0.5, \delta = 0.5$; $E_N^2 = 14.8716$, when $\gamma = 0.5, \delta = 1.5$. All results confirm our judgement numerically.

## 6. Conclusions and prospects

In this paper, a spatial fractional model involving fractional equation and fraction Robin boundary conditions is reformulated to describe the law of superdiffusion within the TPC system in high temperature environment. The weak solution to the definite solution problem uniquely exists and owns weighted stability estimation. Numerical algorithms are developed to perform the distribution of the temperature. Numerical simulation exerimentation shows that the proposed fractional model is reasonable and the developed algorithm is efficient.

There are some challenging problems need future discussing. The future research topics include:

- *More Applicable Models.* Some other factors should be considered, for example, the heat radiation and moisture transfer have to be included in the TPC system because the phenomenon of coupled heat and moisture transfer in porous media usually occurs. Moreover, the proposed model can be generalised to much more practical case, for example in many kinds of anomalous diffusion phenomenon in the involved domain or on the boundary.
- *More Efficiency Algorithms.* The finite element approximation for the variational formulation reported in this paper also deserves further consideration. We will develop some novel numerical algorithms, which owns faster convergence rate and higher efficiency.
- *New Inverse Problems for Fractional Models.* Based on the proposed fractional model, some kinds of inverse problems of parameters determination for the TPC design will be discussed in the future, for example thickness-porosity-conductivity determination problems are the promising topics in the functional clothing industry[11,27]. Meanwhile the optimal choice of the fractional order $\gamma \in (0, 1)$ in the proposed model should be much more reasonably determined whenever the practical measurements are obtained in different heat-moisture transfer situation.

The related theories and methodologies for above interesting topics will be presented in the forthcoming papers.

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