EXISTENCE OF EXPONENTIAL ATTRACTOR TO $p(x)$-LAPLACIAN VIA THE l-TRAJECTORIES METHOD

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ABSTRACT. This article is devoted to the study of the existence of an exponential attractor for a family of problems, in which diffusion d_λ blows up in localized regions inside the domain

$$\begin{cases} u_t^\lambda - \operatorname{div}(d_\lambda(x)(|\nabla u^\lambda|^{p(x)-2} + \eta)\nabla u^\lambda) + |u^\lambda|^{p(x)-2}u^\lambda = B(u^\lambda), & \text{in } \Omega \\ u^\lambda = 0, & \text{on } \partial\Omega \\ u^\lambda(0) = u_0^\lambda \in L^2(\Omega), \end{cases}$$

and their limit problem via the l-trajectory method.

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CONTENTS

1. INTRODUCTION

The existence of exponential attractors is an important feature for nonlinear systems of differential equations, thanks to the exponential rate of exponential attraction, attractors are more robust under perturbations than the global attractor. Several authors have studied the existence of an exponential attractor, see [6, 7, 13]. The following definition was proposed in [14].

Let $(\mathrm{M}, \mathrm{d_M})$ *be a metric space. A subset* $\mathcal{E} \subset \mathrm{M}$ *in an exponential attractor for a semigroup* $\{S(t);\, t \geqslant 0\}$ *if* $\mathcal{E} \neq \emptyset$ *is compact, has finite fractal dimension* $\dim_{\mathrm{f}}(\mathcal{E}) < \infty$*, is semi-invariant, that is,* $S(t)\mathcal{E} \subset \mathcal{E}$ *for all* $t \geqslant 0$*, and for all limit subset* $D \subset \mathrm{M}$ *there exist constants* $c_1, c_2 > 0$ *such that*

$$\mathrm{dist_H}(\mathrm{S(t)D}, \mathcal{E}) \leqslant \mathrm{c_1 e^{-c_2 t}}, \textit{ for all } \mathrm{t} \geqslant 0.$$

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Here $\dim_{\mathrm{f}}(\mathrm{A}) = \limsup_{\varepsilon\to 0} \dfrac{\log \mathrm{N}^{\mathrm{M}}_{\varepsilon}(\mathrm{A})}{\log\left(\frac{1}{\varepsilon}\right)}$, *and* $\mathrm{N}^{\mathrm{M}}_{\varepsilon}(\mathrm{A})$ *denotes the minimum number of* ε*-balls in space* M *with centers in* A *necessary to cover the subset* $A \subset \mathrm{M}$.

Unlike the global attractor, the exponential attractor has no unity, and, therefore, the algorithm used for its construction assumes an important role for its understanding. The existence of the exponential attractor can guarantee through the squeezing property, see [14], or through softness properties, see [28].

This paper concerns the existence of the exponential attractor for a family of problems dominated by a perturbation of $p(x)$-Laplacian with great localized diffusion and its limit problem, which will be described next.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected and smooth subset, with $n \geqslant 1$. Consider the following family of problems

$$
\begin{cases}
u^{\lambda}_t - \operatorname{div}(d_{\lambda}(x)(|\nabla u^{\lambda}|^{p(x)-2} + \eta)\nabla u^{\lambda}) + |u^{\lambda}|^{p(x)-2}u^{\lambda} = B(u^{\lambda}), & \text{in } \Omega \\
u^{\lambda} = 0, & \text{on } \partial\Omega \\
u^{\lambda}(0) = u^{\lambda}_0 \in L^2(\Omega), &
\end{cases}
\tag{1.1}
$$

for $\lambda \in (0,1]$, where $p \in C(\Omega)$ satisfies

$$2 < p^- := \inf\ \operatorname{ess} p \leqslant p(x) \leqslant \sup\ \operatorname{ess} p := p^+ < +\infty,$$

$B : L^2(\Omega) \to L^2(\Omega)$ is globally Lipschitz and $\eta > 0$.

Let Ω_0 be an open subset smooth of Ω with $\overline{\Omega}_0 \subset \Omega$ and $\Omega_0 = \bigcup_{i=1}^{m} \Omega_{0,i}$ where m is a positive integer and $\Omega_{0,i}$ are smooth subdomains of Ω satisfying $\overline{\Omega}_{0,i} \cap \overline{\Omega}_{0,j} = \emptyset$, for $i \neq j$. Define $\Omega_1 = \Omega \setminus \overline{\Omega}_0$, $\Gamma_{0,i} = \partial\Omega_{0,i}$ and $\Gamma_0 = \bigcup_{i=1}^{m} \Gamma_{0,i}$ as the boundaries of $\Omega_{0,i}$ and Ω_0, respectively. Notice that $\partial\Omega_1 = \Gamma \cup \Gamma_0$.

In addition, the diffusion coefficients $d_{\lambda} : \Omega \subset \mathbb{R}^n \to (0,\infty)$ are bounded and smooth functions in Ω, satisfying

$$0 < m_0 \leqslant d_{\lambda}(x) \leqslant M_{\lambda},$$

for all $x \in \Omega$ and $0 < \lambda \leqslant 1$. We also assume that the diffusion is large in Ω_0 as $\lambda \to 0$, or more precisely,

$$
d_{\lambda}(x) \xrightarrow{\lambda\to 0} \begin{cases} d_0(x), & \text{uniformly on } \Omega_1; \\ \infty, & \text{uniformly on compact subsets of } \Omega_0, \end{cases}
\tag{1.2}
$$

where $d_0 : \Omega_1 \to (0,\infty)$ is a smooth function with $0 < m_0 \leqslant d_0(x) \leqslant M_0$ for all $x \in \Omega_1$.

If, in a reaction-diffusion process the diffusion coefficient behaves as expressed in (1.2), we expect that the solutions of (1.1) will become approximately constant on Ω_0. For this reason, suppose that u^{λ} converges to u as $\lambda \to 0$, in some sense, and that u takes, on Ω_0, a time dependent spatially constant value, which we will denote by $u_{\Omega_0}(t)$.

In this context we will obtain the equation that describes the limit problem. Notice that, since the limit function u is in $W^{1,p(x)}(\Omega)$ and its constant value in Ω_0, $u_{\Omega_0}(t)$ cannot be arbitrary. Also, in the boundary $\Gamma_0 = \partial\Omega_0$ we must have $u_{|_{\Gamma_0}} = u_{\Omega_0}(t)$. In Ω_1,we have

$$u_t^\lambda - \operatorname{div}(d_\lambda(x)(|\nabla u^\lambda|^{p(x)-2} + \eta)\nabla u^\lambda) + |u^\lambda|^{p(x)-2}u^\lambda = B(u^\lambda). \tag{1.3}$$

From properties of convergence of the function $d_\lambda(x)$ in Ω_1, when $\lambda \to 0$, we get

$$u_t - \operatorname{div}(d_0(x)(|\nabla u|^{p(x)-2} + \eta)\nabla u) + |u|^{p(x)-2}u = B(u), \text{ for } u \in W^{1,p(x)}(\Omega).$$

Integrating (1.3) on Ω_0, from Gauss's Divergence Theorem, it follows that

$$\int\limits_{\Omega_0} u_t^\lambda \,\mathrm{d}x + \int\limits_{\Gamma_0} d_\lambda(x)(|\nabla u^\lambda|^{p(x)-2} + \eta)\frac{\partial u^\lambda}{\partial \vec{n}} \,\mathrm{d}x + \int\limits_{\Omega_0} |u^\lambda|^{p(x)-2}u^\lambda \,\mathrm{d}x = \int\limits_{\Omega_0} B(u^\lambda)\,\mathrm{d}x,$$

where $\vec{n}$ denotes the unit inward normal to Ω_0 in the surface integral. Taking the limit as $\lambda \to 0$, we get the following ordinary differential equation

$$\dot{u}_{\Omega_0}(t) + \frac{1}{|\Omega_0|}\left(\int\limits_{\Gamma_0} d_0(x)(|\nabla u|^{p-2} + \eta)\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x + \int_{\Omega_0} |u_{\Omega_0}(t)|^{p-2}u_{\Omega_0}(t)\,\mathrm{d}x\right) = B(u_{\Omega_0}(t)).$$

With these considerations we can write the limiting problem in the following way

(1.4)
$$\begin{cases} u_t - \operatorname{div}(d_0(x)(|\nabla u|^{p(x)-2} + \eta)\nabla u) + |u|^{p(x)-2}u = B(u), & \text{in } \Omega_1 \\ u_{|_{\Omega_{0,i}}} := u_{\Omega_{0,i}}, & \text{in } \Omega_{0,i} \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|}\left[\displaystyle\int\limits_{\Gamma_{0,i}} d_0(x)(|\nabla u|^{p(x)-2} + \eta)\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x + \int\limits_{\Omega_{0,i}} |u_{\Omega_{0,i}}|^{p(x)-2}u_{\Omega_{0,i}}\,\mathrm{d}x\right] = B(u_{\Omega_{0,i}}) & \\ u = 0, & \text{on } \partial\Omega \\ u(0) = u_0. & \end{cases}$$

Several authors have studied the asymptotic behavior of problems with large diffusion located in some regions of the domain. In physics, this situation can be found in composite materials where the heat distribution of the material differ from one part to another. In [2] the authors obtained the upper semicontinuity of the family of attractors associated with nonlinear reaction-diffusion equations (1.1) with principal part governed by a degenerate p-Laplacian, where p is constant and $\eta = 0$.

Another work that assumes similar hypotheses about the diffusion is [5], which the authors analyze perturbations in elliptic equations, subjected to various boundary conditions

$$\begin{cases} -\operatorname{div}(d_\epsilon(x)\nabla u^\epsilon) + (\lambda + V_\epsilon(x))u^\epsilon = f(u^\epsilon), & \text{in } \Omega \\ \dfrac{\partial u^\epsilon}{\partial \vec{n}_\epsilon} + b_\epsilon(x)u^\epsilon = g^\epsilon, & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

where $0 < \epsilon \leqslant \epsilon_0$, $\Omega \subset \mathbb{R}^n$ is a bounded regular open connected set. Here $\frac{\partial u}{\partial \vec{n}_\epsilon}$ denotes the conormal derivative relative to the diffusion operator $-\operatorname{div}(d_\epsilon(x)\nabla u)$, i. e., $\frac{\partial u}{\partial \vec{n}_\epsilon} = d_\epsilon(x)\langle \nabla u, \vec{n}\rangle$. Also, $\lambda \in \mathbb{R}$ and the potencials $V_\epsilon(x)$ and $b_\epsilon(x)$ are given functions on Ω and $\partial\Omega$, respectively.

The diffusion is going to infinity in localized regions inside the domain and therefore solutions undergo a localized spatial homogenization. The limiting elliptic operators is analyzed as well as convegence of solutions, eigenvalues, and eigenfuctions.

The existence of an exponential attractor for a dynamic system provides a good understanding of the asymptotic behavior of the solutions, because of the robust behavior of the exponential attractor. Initially, in this work, we tried to guarantee the existence of an exponential attractor for the problem (1.1) as presented in [2], with p constant and $\eta = 0$. Successful results were obtained, even considering the general case where p is a function, by adding the term $\eta > 0$ in the main part of the problem as (1.1). In physical terms, this change could indicate, for example, a viscosity of the material as mentioned in [6].

To guarantee the existence of a family of exponential attractors, $\{\mathcal{E}_\lambda\}_{\lambda\in[0,1]}$, associated with problems (1.1)- (1.4), we will make an adaptation of the method known as the $\ell-$trajectories method, suggested by Málek and Pražák in [1]. In this work the authors proved the existence of a finite dimensional fractal global attractor and the existence of an exponential attractor, through the $\ell-$trajectories method, for the problems of form

$$\begin{cases} u'(t) = F(u(t)), \quad t > 0, \text{ in } X, \\ u(0) = u_0, \end{cases} \tag{1.6}$$

where X is an infinite dimensional Banach space, $F : X \to X$ is a nonlinear operator and $u_0 \in X$.

Let $\ell > 0$ be a constant. Briefly, the $\ell-$trajectories method comes from the observation that there is an equivalent dynamic system, defined in a space of trajectories with amplitude $\ell > 0$, in which we can obtain conclusions about asymptotic behavior more easily and transfer these conclusions, through an application with good properties, to the original dynamical system defined in the phase space.

References [6] and [11] use the $\ell-$trajectories method to guarantee the existence of the exponential attractor. In [6], the author studies the generalized logistic equation

$$u_t - \operatorname{div}(\nu\nabla u + \widetilde{\mu}|\nabla u|^{p-2}\nabla u) = \kappa u(1 - \int_0^\tau u(x, t-s)d\mu(s)), \tag{1.7}$$

in $\Omega\times(0,\infty) \subset \mathbb{R}^2\times(0,\infty)$, where the delay is captured by the convolution time with non-negative μ Borel measure, with $\mu([0,\tau]) = 1$, and constants $\nu > 0$, $\widetilde{\mu} \geqslant 0$ and $p \geqslant 2$, demonstrating the existence of the exponential attractor, once proven that the solutions are asymptotically bounded. Therefore, like in problems (1.1)-(1.4), considering $p(x)$ constant equal to p, in (1.7), the diffusion is in the Laplacian plus the p-Laplacian.

In [11], the authors used the $\ell-$trajectories method to build an exponential attractor for the dynamic system associated with the equation

$$\begin{cases} u_t - \operatorname{div}(a(x,u,\nabla u)) + f(u) = g(x), & (x,t) \in \Omega \times (0,+\infty) \\ u(\cdot,t)|_{\partial\Omega} = 0, & t \in (0,+\infty) \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

where $a\colon \Omega \times \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ satify some hypothesis, among them

$$|a(x,u,\nabla u) - a(x,v,\nabla v)|_{\mathbb{R}^n} \leqslant \beta_0|\nabla u - \nabla v|_{\mathbb{R}^n} + \beta_1|u-v|. \tag{1.8}$$

A typical example, mentioned in [11], is

$$u_t - \mathrm{div}(((|\nabla u|^2 + \varepsilon)^{\frac{p-2}{2}} + \eta)\nabla u) + |u|^q u - |u|^r u = g(x),$$

where $p \in]1, 2[$, $\varepsilon > 0$, $q > r \geqslant 0$ and $\eta > 0$.

Property (1.8), essential in [11], is not satisfied for

$$a(x, u, \nabla u) = d(x)(|\nabla u|^{p(x)-2} + \eta)\nabla u,$$

since $p(x) > 2$, therefore Lemma 2.2, in [27], allows us to estimate

$$||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v|$$

in function of $|\nabla u - \nabla v|$ and $|\nabla u| + |\nabla v|$, and still use the $\ell-$trajectories method.

In the case of $\eta = 0$, we did not obtain conclusions about the global attractor fractal dimension $\mathcal{A}_\lambda$ for each $\lambda \in (0, 1]$. In [21], for example, the authors concluded that the global attractor in $L^2(\Omega)$, associated with problems of the form

$$\begin{cases} u_t - \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) + f(x, u) = g, & \text{in } \Omega \times \mathbb{R}^+ \\ u = 0, & \text{in } \partial\Omega \times \mathbb{R}^+ \\ u(x, 0) = 0, & \text{in } \Omega, \end{cases} \tag{1.9}$$

where $g, u_0 \in L^2(\Omega)$, $p \in C(\overline{\Omega})$ with $2 \leqslant p(x) < \infty$, for all $x \in \overline{\Omega}$, and f satisfying some hypothesis; possesses infinite fractal dimension. Now, in [24], supposing in (1.9), $p(x)$ constant equal to p and $f(x, u) = f(u)$, the authors showed that the global attractor associated with (1.9) admits a finite fractal dimension in $L^{q+\delta}(\Omega)$, where q is the conjugate exponent of p and $\delta \in [0, +\infty)$.

This paper is organized as follows. In Section 2, we define the operators A_λ and A_0, from the main part of the equations, we also show some properties and we present the results of strong solution to (1.1) and (1.4). In Section 3, we verify some estimates for the solutions, seeking to the existence of a compact set positively invariant for the dynamics in phase space associated with (1.1) and (1.4), as well as, for the equivalent dynamics defined in the space of the $\ell-$trajectories.

In Section 4 we present the dynamical of $1-$trajectories associated with (1.1) and (1.4) and guarantee the finite fractal dimension of global attractors in Lemma 4.3. Finally, we prove the main result of this article, Theorem 4.1, which guarantee the existence of an exponential attractor for the family of problems(1.1)-(1.4).

2. EXISTENCE OF SOLUTIONS

In this section we present the operators associated with our problems and establish some of their properties. In addition we guarantee the existence of a unique solution for (1.1)-(1.4).

We will consider the following spaces and notations.

$$\begin{aligned} V &:= W_0^{1,p(x)}(\Omega), & V_0 &:= W_{\Omega_0,0}^{1,p(x)}(\Omega) := \{u \in W_0^{1,p(x)}(\Omega) : u \text{ is constant in } \Omega_0\}, \\ H &:= L^2(\Omega), & H_0 &:= L^2_{\Omega_0}(\Omega) := \{u \in L^2(\Omega) : u \text{ is constant in } \Omega_0\}. \end{aligned}$$

The space V_0 is equipped with the norm in V

$$\|v\|_V := \|v\|_{L^{p(x)}(\Omega)} + \|\nabla v\|_{L^{p(x)}(\Omega)},$$

where $L^{p(x)}(\Omega) = \{v : \Omega \to \mathbb{R}; v \text{ is measurable and } \int_\Omega |v(x)|^{p(x)}\,dx < \infty\}$.

Note that V and V_0 are reflexive Banach spaces, V dense in Hilbert space H and V_0 dense in H_0. Moreover $V \hookrightarrow\hookrightarrow H \hookrightarrow V'$ which implies that $V_0 \hookrightarrow\hookrightarrow H_0 \hookrightarrow V_0'$.

We denote by $\rho_{p(\cdot)}(v)$, or simply for $\rho(v)$, $\int_\Omega |v(x)|^{p(x)}\,dx$, and we have

$$\|v\|_{L^{p(x)}(\Omega)} = \|v\|_{p(x)} := \inf\left\{\lambda > 0 : \rho\left(\frac{v}{\lambda}\right) \leqslant 1\right\}.$$

Next, we present a result that makes many estimates involving spaces $L^{p(x)}(\Omega)$ more flexible, for more details check [26].

Proposition 2.1. *Let $u \in L^{p(x)}(\Omega)$.*

(i) *If $\|u\|_{p(x)} \geqslant 1$, so $\|u\|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant \|u\|_{p(x)}^{p^+}$.*

(ii) *If $\|u\|_{p(x)} \leqslant 1$, so $\|u\|_{p(x)}^{p^+} \leqslant \rho(u) \leqslant \|u\|_{p(x)}^{p^-}$.*

From this Proposition it is possible to obtain the following estimates for $v \in L^{p(x)}(\Omega)$,

$$\min\{\rho(v)^{\frac{1}{p^-}}, \rho(v)^{\frac{1}{p^+}}\} \leqslant \|v\|_{p(x)} \leqslant \max\{\rho(v)^{\frac{1}{p^-}}, \rho(v)^{\frac{1}{p^+}}\}. \tag{2.1}$$

and

$$\min\{\|v\|_{p(x)}^{p^-}, \|v\|_{p(x)}^{p^+}\} \leqslant \rho_p(v) \leqslant \max\{\|v\|_{p(x)}^{p^-}, \|v\|_{p(x)}^{p^+}\}. \tag{2.2}$$

For completeness we enunciate the well-known Aubin-Lions Lemma. For more details see [12].

Lemma 2.1. *Let $p_1 \in (1, \infty]$ and $p_2 \in [1, \infty)$. Let X be a Banach space and Y, Z separable and reflexive Banach spaces in such a way that $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$. So, for all $T \in (0, \infty)$,*

$$\{u \in L^{p_1}(0, T; Y); u' \in L^{p_2}(0, T; Z)\} \hookrightarrow\hookrightarrow L^{p_1}(0, T; X).$$

For $\lambda \in (0, 1]$ we consider

$$\mathcal{D}(A_\lambda) = \{u \in V : -\mathrm{div}(d_\lambda(x)(|\nabla u|^{p(x)-2} + \eta)\nabla u) \in L^2(\Omega)\}$$

and for $u \in \mathcal{D}(A_\lambda)$,

$$A_\lambda(u) = -\mathrm{div}(d_\lambda(x)(|\nabla u|^{p(x)-2} + \eta)\nabla u) + |u|^{p(x)-2}u.$$

If $\lambda = 0$,

$$\mathcal{D}(A_\lambda) = \{u \in V_0 : -\mathrm{div}(d_0(x)(|\nabla u|^{p(x)-2} + \eta)\nabla u) \in L^2(\Omega_1)\}$$

and for $u \in \mathcal{D}(A_0)$,

$$\begin{aligned} A_0(u) &= (-\mathrm{div}(d_0(x)(|\nabla u|^{p(x)-2} + \eta)\nabla u) + |u|^{p(x)-2}u)\chi_{\Omega_1} \\ &\quad + \sum_{i=1}^{n} \frac{1}{|\Omega_{0,i}|}\left(\int_{\Gamma_{0,i}} d_0(x)(|\nabla u|^{p(x)-2} + \eta)\frac{\partial u}{\partial \vec{n}}\,\mathrm{d}x + \int_{\Omega_{0,i}} |u_{\Omega_{0,i}}|^{p(x)-2}u_{\Omega_{0,i}}\,\mathrm{d}x\right)\chi_{\Omega_{0,i}} \end{aligned}$$

where χ_E is the characteristic function of the set E.

We denote by $\langle\cdot,\cdot\rangle$ the inner product in H and by $\langle\cdot,\cdot\rangle_{V_\lambda{}',V_\lambda}$ the duality between V_λ' and V_λ, $\lambda\in[0,1]$, where $V_\lambda = V$, for $\lambda\in(0,1]$.

The next results deal with some important properties of operators A_λ, $\lambda\in[0,1]$.

Lemma 2.2. *For all* $\lambda\in[0,1]$, *we have*

$$\langle A_\lambda u,u\rangle_{V_\lambda',V_\lambda}\geqslant\begin{cases}\dfrac{\min\{m_0,1\}}{2^{p^+}}\|u\|_{V_\lambda}^{p^+}, & \textit{if } \|u\|_{V_\lambda}\leqslant 1,\\ \dfrac{\min\{m_0,1\}}{2^{p^+}}\|u\|_{V_\lambda}^{p^-}, & \textit{if } \|u\|_{V_\lambda}\geqslant 1.\end{cases}$$

Proof. We will demonstrate case $\lambda=0$, case $\lambda\in(0,1]$ can be similarly demonstrated. Let $u\in V_0$ be arbitrary, by the Divergence Theorem, we have

$$\langle A_0u,u\rangle_{V_0',V_0}=\int\limits_{\Omega_1}d_0(x)|\nabla u|^{p(x)}\,dx+\int\limits_{\Omega_1}d_0(x)\eta|\nabla u|^2\,dx+\int\limits_{\Omega}|u|^{p(x)}\,dx. \tag{2.3}$$

So,

$$\langle A_0u,u\rangle_{V_0',V_0}\geqslant m_0\int\limits_{\Omega_1}|\nabla u|^{p(x)}\,dx+\int\limits_{\Omega}|u|^{p(x)}\,dx\geqslant\min\{m_0,1\}(\rho(|\nabla u|)+\rho(u)). \tag{2.4}$$

Suppose that $\|u\|_{V_0}\leqslant 1$, so necessarily $\|u\|_{p(x)}\leqslant 1$ and $\|\nabla u\|_{p(x)}\leqslant 1$. By Proposition 2.1,

$$\rho(u)+\rho(|\nabla u|)\geqslant\|u\|_{p(x)}^{p^+}+\|\nabla u\|_{p(x)}^{p^+}\geqslant\frac{1}{2^{p^+}}(\|u\|_{p(x)}+\|\nabla u\|_{p(x)})^{p^+}=\frac{1}{2^{p^+}}\|u\|_{V_0}^{p^+}. \tag{2.5}$$

Hence, from (2.4), it follows that

$$\langle A_0u,u\rangle_{V_0',V_0}\geqslant\min\{m_0,1\}\frac{1}{2^{p^+}}\|u\|_{V_0}^{p^+},\quad\text{if}\quad\|u\|_{V_0}\leqslant 1. \tag{2.6}$$

When $\|u\|_{V_0}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)}\geqslant 1$, we shall analyze four possibilities.

(i) If $\|u\|_{p(x)}\leqslant 1$ and $\|\nabla u\|_{p(x)}\leqslant 1$. It follows from (2.5) that

$$\rho(u)+\rho(|\nabla u|)\geqslant\frac{1}{2^{p^+}}\|u\|_{V_0}^{p^-}.$$

(ii) If $\|u\|_{p(x)}\leqslant 1$ and $\|\nabla u\|_{p(x)}\geqslant 1$. From Proposition 2.1 we conclude that

$$\begin{aligned}\rho(u)+\rho(|\nabla u|)&\geqslant\|u\|_{p(x)}^{p^+}+\|\nabla u\|_{p(x)}^{p^-}\\&\geqslant\|\nabla u\|_{p(x)}^{p^-}=\frac{1}{2^{p^-}}(2\|\nabla u\|_{p(x)})^{p^-}\geqslant\frac{1}{2^{p^+}}\|u\|_{V_0}^{p^-}.\end{aligned}$$

(iii) If $\|u\|_{p(x)}\geqslant 1$ and $\|\nabla u\|_{p(x)}\leqslant 1$. Following the same arguments as case (ii), we have

$$\rho(u)+\rho(|\nabla u|)\geqslant\|u\|_{p(x)}^{p^-}\geqslant\frac{1}{2^{p^+}}\|u\|_{V_0}^{p^-}.$$

(iv) If $\|u\|_{p(x)}\geqslant 1$ and $\|\nabla u\|_{p(x)}\geqslant 1$. From Proposition 2.1 we have

$$\rho(u)+\rho(|\nabla u|)\geqslant\|u\|_{p(x)}^{p^-}+\|\nabla u\|_{p(x)}^{p^-}\geqslant\frac{1}{2^{p^-}}(\|u\|_{p(x)}+\|\nabla u\|_{p(x)})^{p^-}\geqslant\frac{1}{2^{p^+}}\|u\|_{V_0}^{p^-}.$$

Therefore, from (2.4), it follows that

$$\langle A_0 u, u\rangle_{V_0', V_0} \geqslant \min\{m_0, 1\}\frac{1}{2^{p^+}}\|u\|_{V_0}^{p^-}, \quad \text{if} \quad \|u\|_{V_0} \geqslant 1.$$

Together with (2.6) we conclude the demonstration. □

Theorem 2.1. *The operators $A_\lambda : \mathcal{D}(A_\lambda) \subset V \to V'$, $\lambda \in (0,1]$ and $A_0 : \mathcal{D}(A_0) \subset V_0 \to V_0'$ are monotone, hemicontinuous and coercive.*

Proof. Let $u, v \in V_0$, it is possible to conclude, using the Divergence Theorem and the Tartar inequality, that

$$\begin{aligned}
&\langle A_0 u - A_0 v, u - v\rangle_{V_0', V_0}\\
&\quad\geqslant m_0 \int_{\Omega_1} \frac{2^{3-p(x)}}{p(x)} |\nabla(u-v)|^{p(x)}\, dx + m_0\eta \int_{\Omega_1} |\nabla(u-v)|^2\, dx + \int_{\Omega} \frac{2^{3-p(x)}}{p(x)} |u-v|^{p(x)}\, dx\\
&\quad\geqslant m_0 \frac{2^{3-p^+}}{p^+} \rho(|\nabla(u-v)|) + m_0\eta \|\nabla(u-v)\|_{H_0}^2 + \frac{2^{3-p^+}}{p^+}\rho(u-v).
\end{aligned}$$

Then, A_0 is a monotone operator. Besides, let $w \in V_0$ and $0 < t < 1$ be arbitrary, we have

$$\begin{aligned}
&|\langle A_0(u+tv) - A_0(u), w\rangle_{V_0', V_0}|\\
&\leqslant \left| \int_{\Omega_1} d_0(x)[(|\nabla(u+tv)|^{p(x)-2} + \eta)\nabla(u+tv) - (|\nabla u|^{p(x)-2} + \eta)\nabla u]\nabla w\ dx \right|\\
&\qquad\qquad + \left| \int_{\Omega} (|u+tv|^{p(x)-2}(u+tv) - |u|^{p(x)-2}u) w\ dx \right|.
\end{aligned}$$

From the Dominated Convergence Theorem, we have the hemicontinuity of A_0. Finally, the coercivity is obtained from Lemma 2.2. The case $\lambda \in (0,1]$ can be demonstrated analogously. □

It follows from the Example 2.3.7, p.26 in [8] that the operators A_λ, $\lambda \in [0,1]$ are monotone maximals.

We define the sets

$$\begin{aligned}
&D(A_\lambda^H) := \{v \in V : A_\lambda v \in H\}, \quad \text{for} \quad \lambda \in (0,1],\\
&D(A_0^{H_0}) := \{v \in V_0 : A_0 v \in H_0\},
\end{aligned}$$

and consider the operators $A_\lambda^H : D(A_\lambda^H) \subset H \to H$ given by

$$\begin{aligned}
&A_\lambda^H(u) = A_\lambda u, \quad \forall u \in D(A_\lambda^H), \quad \text{for} \quad \lambda \in (0,1] \text{ and}\\
&A_0^{H_0}(u) = A_0 u, \quad \forall u \in D(A_0^{H_0}).
\end{aligned}$$

Hence, operators A_λ^H and $A_0^{H_0}$ are maximal monotones. In addition, these operators can also be seen as subdifferential type, meaning that, $A_\lambda^H = \partial\varphi^\lambda$, where $\varphi^\lambda : H \to (-\infty, \infty]$ are lower semicontinuous convex functions defined by

$$\varphi^{\lambda}(u)=\begin{cases}\displaystyle\int_{\Omega}\frac{d_{\lambda}(x)}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x+\int_{\Omega}\frac{d_{\lambda}(x)\eta}{2}|\nabla u|^{2}\,\mathrm{d}x+\int_{\Omega}\frac{1}{p(x)}|u|^{p(x)}\,\mathrm{d}x, & \text{if } u\in V\\ \infty, & \text{otherwise}\end{cases}$$

for $\lambda\in(0,1]$.

For $\lambda=0$, $A_0^{H_0}=\partial\varphi$ where $\varphi: H_0\to(-\infty,\infty]$ is a lower semicontinuous convex function defined by

(2.7)

$$\varphi(u)=\begin{cases}\displaystyle\int_{\Omega_1}\frac{d_0(x)}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x+\int_{\Omega_1}\frac{d_0(x)\eta}{2}|\nabla u|^{2}\,\mathrm{d}x+\int_{\Omega}\frac{1}{p(x)}|u|^{p(x)}\,\mathrm{d}x, & \text{if } u\in V_0\\ \infty, & \text{otherwise.}\end{cases}$$

Problems (1.1) and (1.4) can be written abstractly as

$$\begin{cases}u_t^{\lambda}+A_{\lambda}u^{\lambda}=B(u^{\lambda})\\ u^{\lambda}(0)=u_0^{\lambda}, \qquad\qquad \text{for all} \quad \lambda\in(0,1],\end{cases}\tag{2.8}$$

and

$$\begin{cases}u_t+A_0u=B(u)\\ u(0)=u_0.\end{cases}\tag{2.9}$$

The next lemma guarantees the density of the sets $D(A_0^{H_0})$ and $D(A_{\lambda}^{H})$ for each $\lambda\in(0,1]$.

Lemma 2.3. *The set $D(A_0^{H_0})$ is dense in H_0. For each $\lambda\in(0,1]$, $D(A_{\lambda}^{H})$ is dense in H.*

Proof. Consider $C_c^{\infty}(\Omega)$ the space of functions with compact support in Ω which admits infinite continuous derivatives. We define

$$C_{c,0}^{\infty}(\Omega):=\{f\in C_c^{\infty}(\Omega); f \text{ is constant is } \Omega_0\}$$

and

$$L_{\Omega_0}^{\infty}:=\{f\in L^{\infty}(\Omega); f \text{ is constant is } \Omega_0\}.$$

Let $u\in C_{c,0}^{\infty}(\Omega)$, we will show that $u\in D(A_0^{H_0})$. First of all, $u\in V_0$, since

$$C_{c,0}^{\infty}(\Omega)\subset\overline{C_{c,0}^{\infty}(\Omega)}^{W^{1,p(x)}(\Omega)}=W_{\Omega_0,0}^{1,p(x)}(\Omega)=V_0.$$

Besides, denoting by χ_E the characteristic function of the set E, consider

$$\begin{aligned}\alpha_u:=&(-\mathrm{div}(d_0(\cdot)(|\nabla u|^{p(\cdot)-2}+\eta)\nabla u)+|u|^{p(\cdot)-2}u)\chi_{\Omega_1}\\ &+\sum_{i=1}^{m}\frac{1}{|\Omega_{0,i}|}\left(\int\limits_{\Gamma_{0,i}}d_0(x)(|\nabla u|^{p(x)-2}+\eta)\frac{\partial u}{\partial\vec{n}}\,dx\;+\int\limits_{\Omega_{0,i}}|u_{\Omega_{0,i}}|^{p(x)-2}u_{\Omega_{0,i}}\,dx\right)\chi_{\Omega_{0,i}}.\end{aligned}$$

Note that, if $u=0$ then $\alpha_u=0$. As $u\in C_{c,0}^{\infty}(\Omega)$, then the support of α_u is bounded. That is, $\alpha_u\in L_{\Omega_0}^{\infty}(\Omega)\subset H_0$.

Let $w \in V_0$ be arbitrary. Such as $w_{\Omega_{0,i}} = w_{|_{\Gamma_{0,i}}}$ we have that

$$
\begin{aligned}
(\alpha_u, w)_{H_0} = & \int_{\Omega_1} - \operatorname{div}(d_0(x)(|\nabla u|^{p(x)-2} + \eta))\nabla u \, w \, dx + \int_{\Omega_1} |u|^{p(x)-2} u w \, dx \\
& + \sum_{i=1}^{m} \int_{\Omega_{0,i}} \frac{1}{|\Omega_{0,i}|} \left(\int_{\Gamma_{0,i}} d_0(y)(|\nabla u|^{p(y)-2} + \eta) \frac{\partial u}{\partial \vec{n}} \, dy \right) w \, dx \\
& + \sum_{i=1}^{m} \int_{\Omega_{0,i}} \frac{1}{|\Omega_{0,i}|} \left(\int_{\Omega_{0,i}} |u_{\Omega_{0,i}}|^{p(y)-2} u_{\Omega_{0,i}} \, dy \right) w \, dx \\
= & \int_{\Omega_1} - \operatorname{div}(d_0(x)(|\nabla u|^{p(x)-2} + \eta))\nabla u \, w \, dx + \int_{\Omega_1} |u|^{p(x)-2} u w \, dx \\
& + \sum_{i=1}^{m} \int_{\Gamma_{0,i}} d_0(y)(|\nabla u|^{p(y)-2} + \eta) \frac{\partial u}{\partial \vec{n}} w \, dy + \sum_{i=1}^{m} \int_{\Omega_{0,i}} |u_{\Omega_{0,i}}|^{p(y)-2} u_{\Omega_{0,i}} w \, dy \\
= & \langle A_0 u, w \rangle_{V_0', V_0}.
\end{aligned}
$$

Hence, $\alpha_u = A_0^{H_0} u$ and $u \in D(A_0^{H_0})$. In other words, $C_{c,0}^{\infty}(\Omega) \subset D(A_0^{H_0})$. Therefore, $H_0 \subset \overline{D(A_0^{H_0})}^{H_0}$.

In the case where $\lambda \in (0, 1]$, it follows in an analogous way. □

Next we will present the strong and weak solution concept for (2.8) and (2.9).

Definition 2.1. (1) *Let $T > 0$. We say that $u^{\lambda} \in C([0, T]; H)$ is a strong solution of* (2.8)*, if*

(i) u^{λ} *is absolutely continuous in any compact subinterval of* $(0, T)$*;*

(ii) $u^{\lambda}(t) \in D(A_{\lambda}^{H})$ *almost always in* $(0, T)$*, with* $u^{\lambda}(0) = u_0^{\lambda}$*;*

(iii) $\dfrac{du^{\lambda}}{dt}(t) + A_{\lambda}^{H}(u^{\lambda}(t)) = B(u^{\lambda}(t))$*, almost always occurs in* $(0, T)$*.*

We say that $u^{\lambda} \in C([0, T]; H)$ is a weak solution of (2.8)*, if there exists a sequence of strong solutions, of* (2.8)*, that converges to u^{λ} in $C([0, T]; H)$.*

(2) *Let $T > 0$. We say that $u \in C([0, T]; H_0)$ is a strong solution of* (2.9)*, if*

(i) u *is absolutely continuous in any compact subinterval of* $(0, T)$*;*

(ii) $u(t) \in D(A_0^{H_0})$ *almost always in* $(0, T)$*, with* $u(0) = u_0$*;*

(iii) $\dfrac{du}{dt}(t) + A_0^{H_0}(u(t)) = B(u(t))$*, almost always occurs in* $(0, T)$*.*

We say that $u \in C([0, T]; H_0)$ is a weak solution of (2.9)*, if there exists a sequence of strong solutions, of* (2.9)*, that converges to u in $C([0, T]; H_0)$.*

It follows from Theorem 3.17 and Remark 3.14 in [8] that (2.8) has a global weak solution $u^{\lambda}(\cdot, u_0^{\lambda})$ starting in $u^{\lambda}(0) = u_0^{\lambda} \in \overline{D(A_{\lambda}^{H})}^{H} = H$. If $u_0^{\lambda} \in D(A_{\lambda}^{H})$ then the function $u^{\lambda}(\cdot, u_0^{\lambda})$ is a strong solution of (2.8) Lipschitz continuous, for each $\lambda \in (0, 1]$. Analogously for (2.9).

For $\lambda \in (0,1]$, we can define in H a semigroup $\{T_\lambda(t)\}_{t\geqslant 0}$ of nonlinear operators, associated with (2.8) by $T_\lambda(t)u_0^\lambda = u^\lambda(t,u_0^\lambda)$, $t \geqslant 0$.

To simplify, we will denote the solution $u^0(t,u_0)$ of (2.9) just by $u(t,u_0)$. Thus, if $\lambda = 0$, we can define in $\overline{D(A_0^{H_0})}^{H_0} = H_0$ a semigroup $\{T(t)\}_{t\geqslant 0}$ of nonlinear operators, associated with (2.9) by $T(t)u_0 = u(t,u_0)$, $t \geqslant 0$.

Furthermore, we have that the applications $\mathbb{R}^+ \times H \ni (t,u_0^\lambda) \mapsto T_\lambda(t)u_0^\lambda \in H$, for all $\lambda \in (0,1]$, and $\mathbb{R}^+ \times H_0 \ni (t,u_0) \mapsto T_0(t)u_0 \in H_0$ are continuous.

3. Estimates involving the solution

One of the purposes of this section is to ensure that there is an absorbent ball in H for the dynamical systems $(T_\lambda(t), H)$, for all $\lambda \in (0,1]$, and $(T_0(t), H_0)$.

Lemma 3.1. *Let u be the strong solution of* (2.9)*. Then,*

(i) *There exist positive constants t_0 and r_0 such that $\|u(t)\|_{H_0} \leqslant r_0$, for all $t \geqslant t_0$.*
(ii) *There exist positive constants t_1 and $r_1$such thay $\|u(t)\|_{V_0} \leqslant r_1$, for all $t \geqslant t_1$.*

We obtain the same estimative if u^λ is the strong solution of (2.8)*, uniformly for λ in* $(0,1]$.

Proof. Let u be a strong solution of (2.9). Taking the scalar product with $u(t)$ in (2.9), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H_0}^2 + \langle A_0 u(t), u(t)\rangle_{V_0',V_0} &= \left(\frac{du}{dt}(t), u(t)\right)_{H_0} + (A_0^{H_0}u(t), u(t))_{H_0}\\ &\leqslant \|B(u(t)) - B(0)\|_{H_0}\|u(t)\|_{H_0} + \|B(0)\|_{H_0}\|u(t)\|_{H_0}.\end{aligned} \tag{3.1}$$

We will consider the cases $\|u(t)\|_{V_0} \geqslant 1$ and $\|u(t)\|_{V_0} < 1$ separately as in [16].

If $\|u(t)\|_{V_0} \geqslant 1$, it follows by Lemma 2.2 that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H_0}^2 + \frac{\min\{m_0,1\}}{2^{p^+}}\|u(t)\|_{V_0}^{p^-} &\leqslant \frac{1}{2}\frac{d}{dt}\|u(t)\|_{H_0}^2 + \langle A_0u(t),u(t)\rangle_{V_0',V_0}\\ &\leqslant L_B\|u(t)\|_{H_0}^2 + \|B(0)\|_{H_0}\|u(t)\|_{H_0}.\end{aligned}$$

Since $V_0 \hookrightarrow H_0$ then $\|u(t)\|_{H_0} \leqslant \mu\|u(t)\|_{V_0}$, where $\mu = |\Omega| + 1$, hence

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H_0}^2 + \frac{c}{2^{p^+}}\|u(t)\|_{V_0}^{p^-} \leqslant c_1\|u(t)\|_{V_0}^2 + c_2\|u(t)\|_{V_0}, \tag{3.2}$$

where $c = \min\{m_0, 1\}$, $c_1 = L_B\mu^2$ and $c_2 = \mu\|B(0)\|_{H_0}$.

Consider $\theta = \dfrac{p^-}{2}$ and $\varepsilon > 0$, chosen in a way that $\dfrac{c}{2^{p^+}} - \dfrac{1}{\theta}\varepsilon^\theta - \dfrac{1}{p^-}\varepsilon^{p^-} > 0$. It follows from Young inequality that

$$\begin{aligned}c_1\|u(t)\|_{V_0}^2 + c_2\|u(t)\|_{V_0} &= \varepsilon\|u(t)\|_{V_0}^2\frac{c_1}{\varepsilon} + \varepsilon\|u(t)\|_{V_0}\frac{c_2}{\varepsilon}\\ &\leqslant \frac{1}{\theta}\varepsilon^\theta\|u(t)\|_{V_0}^{p^-} + \frac{1}{\theta'}\left(\frac{c_1}{\varepsilon}\right)^{\theta'} + \frac{1}{p^-}\varepsilon^{p^-}\|u(t)\|_{V_0}^{p^-} + \frac{1}{q}\left(\frac{c_2}{\varepsilon}\right)^q,\end{aligned}$$

where $q=(p^-)'$. Let $\gamma=\dfrac{c}{2^{p^+}}-\dfrac{1}{\theta}\varepsilon^{\theta}-\dfrac{1}{p^-}\varepsilon^{p^-}>0$, then from (3.2) we have that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H_0}^2+\gamma\|u(t)\|_{V_0}^{p^-}\leqslant\frac{1}{\theta'}\left(\frac{c_1}{\varepsilon}\right)^{\theta'}+\frac{1}{q}\left(\frac{c_2}{\varepsilon}\right)^{q}.$$

Again, by $V_0\hookrightarrow H_0$, we can conclude that

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{H_0}^2+\frac{2\gamma}{\mu^{p^-}}\|u(t)\|_{H_0}^{p^-}&\leqslant\frac{d}{dt}\|u(t)\|_{H_0}^2+2\gamma\|u(t)\|_{V_0}^{p^-}\\&\leqslant\frac{2}{\theta'}\left(\frac{c_1}{\varepsilon}\right)^{\theta'}+\frac{2}{q}\left(\frac{c_2}{\varepsilon}\right)^{q},\quad\forall t\geqslant 0.\end{aligned}$$

Taking $\delta=\dfrac{2}{\theta'}\left(\dfrac{c_1}{\varepsilon}\right)^{\theta'}+\dfrac{2}{q}\left(\dfrac{c_2}{\varepsilon}\right)^{q}$, $\tilde{\gamma}=\dfrac{2\gamma}{\mu^{p^-}}$ e $y(t)=\|u(t)\|_{H_0}^2$ we have

$$\frac{d}{dt}y(t)+\tilde{\gamma}y(t)^{\frac{p^-}{2}}\leqslant\delta,\quad\forall t\geqslant 0.$$

By Lemma 5.1 in [13] we conclude that

$$y(t)\leqslant\left(\frac{\delta}{\tilde{\gamma}}\right)^{\frac{2}{p^-}}+\left(\tilde{\gamma}\left(\frac{p^--2}{2}\right)t\right)^{-\frac{2}{p^--2}},\ \ \forall t\geqslant 0.$$

That is,

$$\begin{aligned}\|u(t)\|_{H_0}&\leqslant\left(\left(\frac{\delta}{\tilde{\gamma}}\right)^{\frac{2}{p^-}}+\left(\tilde{\gamma}\left(\frac{p^--2}{2}\right)t\right)^{-\frac{2}{p^--2}}\right)^{\frac{1}{2}}\\&\leqslant\left(\frac{\delta}{\tilde{\gamma}}\right)^{\frac{1}{p^-}}+\left(\tilde{\gamma}\left(\frac{p^--2}{2}\right)t\right)^{-\frac{1}{p^--2}},\ \ \forall t\geqslant 0.\end{aligned}$$

Fixed $t_0>0$, it follows that

$$\|u(t)\|_{H_0}\leqslant k_1,\quad \text{for all}\quad t\geqslant t_0,$$

where $k_1=\left(\frac{\delta}{\tilde{\gamma}}\right)^{\frac{1}{p^-}}+\left(\tilde{\gamma}\left(\frac{p^--2}{2}\right)t_0\right)^{-\frac{1}{p^--2}}$.
If $\|u(t)\|_{V_0}<1$ we have that $\|u(t)\|_{H_0}\leqslant\mu\|u(t)\|_{V_0}<\mu$.
Let $r_0=\max\{k_1,\mu\}$, so

$$\|u(t)\|_{H_0}\leqslant r_0,\ \ \forall t\geqslant t_0,$$

which concludes the demonstration of item *(i)*.
For item *(ii)*, using Young inequality, we have

$$
\begin{aligned}
\frac{d}{dt}\varphi(u) &= (\partial\varphi(u), u_t)_{H_0} = (A_0^{H_0}u, u_t)_{H_0} = (B(u) - u_t, u_t)_{H_0} \\
&= (B(u) - u_t, u_t - B(u))_{H_0} + (B(u) - u_t, B(u))_{H_0} \\
&\leqslant -\|B(u) - u_t\|_{H_0}^2 + \|B(u) - u_t\|_{H_0}\|B(u)\|_{H_0} \\
&\leqslant -\frac{1}{2}\|B(u) - u_t\|_{H_0}^2 + \frac{1}{2}\|B(u)\|_{H_0}^2 \\
&\leqslant \frac{1}{2}\|B(u)\|_{H_0}^2 \leqslant \frac{1}{2}(L_B\|u\|_{H_0} + \|B(0)\|_{H_0})^2.
\end{aligned}
$$

Which guarantee, through item *(i)*, that

$$
\frac{d}{dt}\varphi(u) \leqslant \frac{1}{2}\|B(u)\|_{H_0}^2 \leqslant \frac{1}{2}k_2^2, \quad \forall t \geqslant t_0,
$$

where $k_2 := L_B r_0 + \|B(0)\|_{H_0}$. By the subdifferential definition

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{H_0}^2 + \varphi(u) &= (u_t, u)_{H_0} + \varphi(u) \\
&\leqslant (u_t, u)_{H_0} + (\partial\varphi(u), u)_{H_0} = (B(u), u)_{H_0} \\
&\leqslant \|B(u)\|_{H_0}\|u\|_{H_0} \leqslant k_2 r_0, \ \forall t \geqslant t_0. \qquad (3.3)
\end{aligned}
$$

Fixed $k > 0$ and integrating estimative (3.3) in $[t, t+k]$, with $t \geqslant t_0$, we get

$$
\int_t^{t+k} k_2 r_0 \ ds \geqslant \frac{1}{2}\|u(t+k)\|_{H_0}^2 - \frac{1}{2}\|u(t)\|_{H_0}^2 + \int_t^{t+k} \varphi(u) \ ds.
$$

Therefore,

$$
\begin{aligned}
\int_t^{t+k} \varphi(u) \ ds &\leqslant \frac{1}{2}\|u(t+k)\|_{H_0}^2 + \int_t^{t+k} \varphi(u) \ ds \\
&\leqslant \frac{1}{2}\|u(t)\|_{H_0}^2 + \int_t^{t+k} k_2 r_0 \ ds \\
&\leqslant \frac{1}{2}r_0^2 + k k_2 r_0 := k_3.
\end{aligned}
$$

Using the Uniform Grönwall-Belman Lemma, Lemma 1.1 in [13], for $y = \varphi(u)$, $g = 0$ and $h = \frac{1}{2}k_2^2$ we conclude that

$$
\varphi(u(t+k)) \leqslant \frac{k_3}{k} + \frac{1}{2}k_2^2 k := k_4, \quad \forall t \geqslant t_0. \qquad (3.4)
$$

Then, from (2.7) and (3.4), we concluded that

$$
\begin{aligned}
\frac{1}{p^+}\min\{m_0, 1\}(\rho(u) + \rho(|\nabla u|)) &\leqslant \int_\Omega \frac{1}{p^+}|u(t,x)|^{p(x)} \, dx + \int_{\Omega_1} m_0 \frac{1}{p^+}|\nabla u(t,x)|^{p(x)} \, dx \\
&\leqslant \int_\Omega \frac{1}{p(x)}|u(t,x)|^{p(x)} \, dx + \int_{\Omega_1} d_0(x)\frac{1}{p(x)}|\nabla u(t,x)|^{p(x)} \, dx \\
&\leqslant \varphi(u(t)) \leqslant k_4, \quad \forall t \geqslant t_0 + k. \qquad (3.5)
\end{aligned}
$$

In the case where $\|u(t)\|_{V_0} \leqslant 1$, there is nothing to demonstrate. If $\|u(t)\|_{V_0} \geqslant 1$, then we will analyze the following cases:

(a) If $\|u\|_{p(x)} \leqslant 1$ and $\|\nabla u\|_{p(x)} \geqslant 1$. It follows from Proposition 2.1, that

$$\|u\|_{p(x)}^{p^+} \leqslant \rho(u) \leqslant \|u\|_{p(x)}^{p^-} \quad \text{and} \quad \|\nabla u\|_{p(x)}^{p^-} \leqslant \rho(|\nabla u|) \leqslant \|\nabla u\|_{p(x)}^{p^+}.$$

so

$$\|u\|_{p(x)}^{p^+} \leqslant \rho(u) + \rho(|\nabla u|) \quad \text{and} \quad \|\nabla u\|_{p(x)}^{p^-} \leqslant \rho(u) + \rho(|\nabla u|).$$

It follows from (3.5) that

$$\begin{aligned} \|u\|_{V_0} =& \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \\ \leqslant& (\rho(u) + \rho(|\nabla u|))^{\frac{1}{p^+}} + (\rho(u) + \rho(|\nabla u|))^{\frac{1}{p^-}} \\ \leqslant& \left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^+}} + \left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^-}}, \forall t \geqslant t_0 + k. \end{aligned}$$

(b) If $\|u\|_{p(x)} \geqslant 1$ and $\|\nabla u\|_{p(x)} \leqslant 1$.
It follows analogously from item (a), that is,

$$\begin{aligned} \|u\|_{V_0} \leqslant& (\rho(u) + \rho(|\nabla u|))^{\frac{1}{p^-}} + (\rho(u) + \rho(|\nabla u|))^{\frac{1}{p^+}} \\ \leqslant& \left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^-}} + \left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^+}}, \forall t \geqslant t_0 + k. \end{aligned}$$

(c) If $\|u\|_{p(x)} \geqslant 1$ and $\|\nabla u\|_{p(x)} \geqslant 1$. Again, by Proposition 2.1, we have that

$$\|u\|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant \|u\|_{p(x)}^{p^+} \quad \text{and} \quad \|\nabla u\|_{p(x)}^{p^-} \leqslant \rho(|\nabla u|) \leqslant \|\nabla u\|_{p(x)}^{p^+}.$$

Then,

$$\|u\|_{V_0}^{p^-} \leqslant 2^{p^-}(\|u\|_{p(x)}^{p^-} + \|\nabla u\|_{p(x)}^{p^-}) \leqslant 2^{p^-}(\rho(u) + \rho(|\nabla u|)).$$

It follows from (3.5) that

$$\|u\|_{V_0} \leqslant 2\left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^-}}, \quad \forall t \geqslant t_0 + k.$$

(d) If $\|u\|_{p(x)} \leqslant 1$ and $\|\nabla u\|_{p(x)} \leqslant 1$.
It follows analogously from item (c), just change p^- for p^+. In this case, we have

$$\|u\|_{V_0} \leqslant 2\left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^+}}, \quad \forall t \geqslant t_0 + k.$$

Therefore, for all $t \geqslant t_1 := t_0 + k$, we conclude that

$$\|u(t)\|_{V_0} \leqslant \max\left\{2\left[\left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^+}} + \left(\frac{p^+ k_4}{\min\{m_0, 1\}}\right)^{\frac{1}{p^-}}\right], 1\right\} =: r_1.$$

Where u^λ is the strong solution of (2.8), we can estimate by the same constants, that is, $r_0^\lambda = r_0$ and $r^\lambda = r_1$, for all $\lambda \in (0, 1]$. That is, we obtain uniform estimates in $\lambda \in (0, 1]$.

$\square$

Lemma 3.2. *Let u be the strong solution of* (2.9)*, with $u(0) = u_0 \in V_0$, and $T > 0$. There exists constant $R > 0$, depending on $\|u_0\|_{V_0}$, such that $\|u(t)\|_{V_0} \leqslant R$, for all $0 \leqslant t \leqslant T$.*

Besides, if u^λ is the strong solution of (2.8)*, with $u^\lambda(0) = u_0^\lambda \in V$, there exists $R_\lambda > 0$, depending on $\|u_0^\lambda\|_V$, such that $\|u(t)\|_V \leqslant R_\lambda$, for all $0 \leqslant t \leqslant T$.*

Proof. Initially, we will show that there exists $R_0 > 0$, depending on $\|u_0\|_{H_0}$, such that $\|u(t)\|_{H_0} \leqslant R_0$, for all $t \geqslant 0$. We note that for initial data in bounded subsets of H_0, we have R_0 uniformly defined.

Through Lemma 3.1, there exist $t_0 > 0$ and $r_0 > 0$ such that $\|u(t)\|_{H_0} \leqslant r_0$, for all $t \geqslant t_0$. Let $0 < t \leqslant t_0$ and $s \in (0, t)$, proceeding as in (3.1) and using Young inequality, we can conclude

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(s)\|_{H_0}^2 &\leqslant L_B\|u(s)\|_{H_0}^2 + \|B(0)\|_{H_0}\|u(s)\|_{H_0} \\
&\leqslant \left(L_B + \frac{1}{2}\right)\|u(s)\|_{H_0}^2 + \frac{1}{2}\|B(0)\|_{H_0}^2 \\
&\leqslant \max\left\{L_B + \frac{1}{2}, \frac{1}{2}\|B(0)\|_{H_0}^2\right\}(\|u(s)\|_{H_0}^2 + 1).
\end{aligned}
$$

Let $c_1 = \max\left\{L_B + \frac{1}{2}, \frac{1}{2}\|B(0)\|_{H_0}^2\right\}$. Integrating this last inequality to s varying in interval $[0, t]$, we have

$$
\|u(t)\|_{H_0}^2 - \|u(0)\|_{H_0}^2 \leqslant 2c_1\int_0^t(\|u(s)\|_{H_0}^2 + 1)ds, \quad \forall t \in [0, t_0].
$$

Applying the Grönwall-Bellman Lemma for $\phi(t) = \|u(t)\|_{H_0}^2 + 1$, we conclude that

$$
\|u(t)\|_{H_0} \leqslant \sqrt{(\|u(0)\|_{H_0}^2 + 1)e^{2c_1t_0} - 1}, \quad \forall t \in [0, t_0].
$$

Therefore,

$$
\|u(t)\|_{H_0} \leqslant R_0, \quad \forall t \geqslant 0, \tag{3.6}
$$

where $R_0 = \max\left\{\sqrt{(\|u(0)\|_{H_0}^2 + 1)e^{2c_1t_0} - 1}, r_0\right\}$. More precisely,

$$
R_0 = \max\left\{\sqrt{(\|u(0)\|_{H_0}^2 + 1)e^{2c_1t_0} - 1}, \left(\frac{\delta}{\tilde{\gamma}}\right)^{\frac{1}{p^-}} + \left(\tilde{\gamma}\left(\frac{p^- - 2}{2}\right)t_0\right)^{-\frac{1}{p^- - 2}}, \mu\right\},
$$

where t_0 is a real fixed positive and $\mu = |\Omega| + 1$.

Again, through Lemma 3.1, using (3.6), we have that

$$
\frac{d}{dt}\varphi(u) \leqslant \frac{1}{2}(L_B\|u\|_{H_0} + \|B(0)\|_{H_0})^2 \leqslant \frac{1}{2}K_2^2, \quad \forall t \geqslant 0,
$$

where $K_2 := L_BR_0 + \|B(0)\|_{H_0}$.

Let $t > 0$. Integrating the previous inequality in $(0, t)$, we obtain

$$
\varphi(u(t)) \leqslant \varphi(u_0) + \frac{T}{2}K_2^2, \quad \forall\, 0 \leqslant t \leqslant T. \tag{3.7}
$$

In turn,

$$\begin{aligned}\varphi(u_0) &= \int_{\Omega_1} \frac{d_0(x)}{p(x)}|\nabla u_0|^{p(x)}\,dx + \int_{\Omega_1} \frac{d_0(x)\eta}{2}|\nabla u_0|^2\,dx + \int_{\Omega} \frac{1}{p(x)}|u_0|^{p(x)}\,dx \\ &\leqslant \int_{\Omega_1} \frac{M_0}{p^-}|\nabla u_0|^{p(x)}\,dx + \int_{\Omega_1} \frac{M_0\eta}{2}|\nabla u_0|^2\,dx + \int_{\Omega} \frac{1}{p^-}|u_0|^{p(x)}\,dx \\ &= \frac{M_0}{p^-}\rho_p(|\nabla u_0|) + \frac{M_0\eta}{2}\|\nabla u_0\|^2_{L^2(\Omega_1)} + \frac{1}{p^-}\rho_p(u_0).\end{aligned}$$

We will analyze four cases.

(i) If $\|\nabla u_0\|_{L^{p(x)}(\Omega_1)} \leqslant 1$ and $\|u_0\|_{L^{p(x)}(\Omega_1)} \leqslant 1$, since $L^{p(x)}(\Omega_1) \hookrightarrow L^2(\Omega_1)$, we have

$$\|\nabla u_0\|_{L^2(\Omega_1)} \leqslant \mu\|\nabla u_0\|_{L^{p(x)}(\Omega_1)} \leqslant \mu.$$

Thus,

$$\varphi(u_0) \leqslant \frac{M_0}{p^-}\|\nabla u_0\|^{p^-}_{L^{p(x)}(\Omega_1)} + \frac{M_0\eta}{2}\mu^2 + \frac{1}{p^-}\|u_0\|^{p^-}_{L^{p(x)}(\Omega_1)} \leqslant \frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-}.$$

Since $\|u_0\|_{V_0} \leqslant 2$, it follows that

$$1 \leqslant \frac{\|u_0\|^{p^+}_{V_0}}{2^{p^+}} \leqslant \frac{\|u_0\|^{p^+}_{V_0}}{2} \leqslant 2\|u_0\|^{p^+}_{V_0} \leqslant 2(\|u_0\|^{p^+}_{V_0} + 1).$$

Therefore,

$$\varphi(u_0) \leqslant \frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-} \leqslant 2\left(\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-}\right)(\|u_0\|^{p^+}_{V_0} + 1).$$

(ii) If $\|\nabla u_0\|_{L^{p(x)}(\Omega_1)} \leqslant 1$ and $\|u_0\|_{L^{p(x)}(\Omega_1)} \geqslant 1$, since $L^{p(x)}(\Omega_1) \hookrightarrow L^2(\Omega_1)$, we have

$$\|\nabla u_0\|_{L^2(\Omega_1)} \leqslant \mu\|\nabla u_0\|_{L^{p(x)}(\Omega_1)} \leqslant \mu.$$

Thus,

$$\begin{aligned}\varphi(u_0) &\leqslant \frac{M_0}{p^-}\|\nabla u_0\|^{p^-}_{L^{p(x)}(\Omega_1)} + \frac{M_0\eta}{2}\mu^2 + \frac{1}{p^-}\|u_0\|^{p^+}_{L^{p(x)}(\Omega_1)} \\ &\leqslant \frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-}\|u_0\|^{p^+}_{L^{p(x)}(\Omega_1)} \\ &\leqslant 2\left(\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-}\right)(\|u_0\|^{p^+}_{V_0} + 1).\end{aligned}$$

(iii) If $\|\nabla u_0\|_{L^{p(x)}(\Omega_1)} \geqslant 1$ and $\|u_0\|_{L^{p(x)}(\Omega_1)} \leqslant 1$, since $L^{p(x)}(\Omega_1) \hookrightarrow L^2(\Omega_1)$, we have

$$\|\nabla u_0\|_{L^2(\Omega_1)} \leqslant \mu\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}.$$

Thus,

$$
\begin{aligned}
\varphi(u_0) &\leqslant \frac{M_0}{p^-}\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} + \frac{M_0\eta}{2}\|\nabla u_0\|_{L^2(\Omega_1)}^2 + \frac{1}{p^-}\|u_0\|_{L^{p(x)}(\Omega_1)}^{p^-} \\
&\leqslant \left(\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2}\right)\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} + \frac{1}{p^-} \\
&\leqslant \left(\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2}\right)\|u_0\|_{V_0}^{p^+} + \frac{1}{p^-} \\
&\leqslant 2\left(\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-}\right)(\|u_0\|_{V_0}^{p^+} + 1).
\end{aligned}
$$

(iv) If $\|\nabla u_0\|_{L^{p(x)}(\Omega_1)} \geqslant 1$ and $\|u_0\|_{L^{p(x)}(\Omega_1)} \geqslant 1$, by inclusion $L^{p(x)}(\Omega_1) \hookrightarrow L^2(\Omega_1)$, we have

$$
\begin{aligned}
\varphi(u_0) &\leqslant \frac{M_0}{p^-}\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} + \frac{M_0\eta}{2}\|\nabla u_0\|_{L^2(\Omega_1)}^2 + \frac{1}{p^-}\|u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} \\
&\leqslant \frac{M_0}{p^-}\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} + \frac{M_0\eta\mu^2}{2}\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^2 + \frac{1}{p^-}\|u_0\|_{L^{p(x)}(\Omega_1)}^{p^+}.
\end{aligned}
$$

Since $\|\nabla u_0\|_{p(x)} \geqslant 1$ then $\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} \geqslant \|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^2$.
Therefore,

$$
\begin{aligned}
\varphi(u_0) &\leqslant \frac{M_0}{p^-}\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} + \frac{M_0\eta\mu^2}{2}\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} + \frac{1}{p^-}\|u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} \\
&\leqslant \max\left\{\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2}, \frac{1}{p^-}\right\}(\|\nabla u_0\|_{L^{p(x)}(\Omega_1)}^{p^+} + \|u_0\|_{L^{p(x)}(\Omega_1)}^{p^+}) \\
&\leqslant \max\left\{\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2}, \frac{1}{p^-}\right\} 2(\|\nabla u_0\|_{L^{p(x)}(\Omega_1)} + \|u_0\|_{L^{p(x)}(\Omega_1)})^{p^+} \\
&\leqslant 2\left(\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-}\right)(\|u_0\|_{V_0}^{p^+} + 1).
\end{aligned}
$$

We obtain the same conclusion in all cases

$$
\varphi(u_0) \leqslant 2\left(\frac{M_0}{p^-} + \frac{M_0\eta\mu^2}{2} + \frac{1}{p^-}\right)(\|u_0\|_{V_0}^{p^+} + 1) := K_3. \tag{3.8}
$$

Returning to (3.7), we have

$$
\varphi(u(t)) \leqslant K_3 + \frac{T}{2}K_2^2 := K_4, \quad \forall\, t \in [0, T].
$$

The same way that it was made in (3.5), we can conclude that

$$
\frac{1}{p^+}\min\{m_0, 1\}(\rho(u) + \rho(|\nabla u|)) \leqslant \varphi(u(t)) \leqslant K_4, \quad \forall\, t \in [0, T].
$$

Proceeding as in the analysis of cases (a)-(b)-(c)-(d) of Lemma 3.1, having $t_0 = 0$ and replacing k_4 for K_4, we obtain

$$\|u(t)\|_{V_0} \leqslant \max\left\{2\left[\left(\frac{p^+K_4}{\min\{m_0,1\}}\right)^{\frac{1}{p^+}} + \left(\frac{p^+K_4}{\min\{m_0,1\}}\right)^{\frac{1}{p^-}}\right], 1\right\} := R, \quad 0 \leqslant t \leqslant T.$$

The demonstration for u^λ strong solution of (2.8) can be made in an analogous way. In that case, besides de dependency on $\|u_0^\lambda\|_V$, the constant $R > 0$ is written in terms of constant M_λ, that is, the estimative is not uniform in λ varying in interval $(0, 1]$. □

This last Lemma allows us to conclude that a given $T > 0$ and a bounded $D \subset V_0$, the set $\bigcup\limits_{t\in[0,T]} T_0(t)D$ is bounded in V_0. Analogously, for each $\lambda \in (0, 1]$, $\bigcup\limits_{t\in[0,T]} T_\lambda(t)D$ is bounded in V, being $D \subset V$ bounded.

Lemma 3.3. *Let u be the strong solution of* (2.9) *and $T > 0$. There exists constant $C_1 > 0$, depending on T and on $\|u_0\|_{H_0}$, such that*

$$\int_0^T \int_{\Omega_1} |\nabla u(t,x)|^{p(x)} dxdt \leqslant C_1.$$

We obtain the same conclusion if u^λ is the strong solution of (2.8)*, where $\lambda \in (0, 1]$. More precisely, for each $\lambda \in (0, 1]$, there exists $C_1^\lambda > 0$, depending on T and on $\|u_0^\lambda\|_H$, such that*

$$\int_0^T \int_{\Omega} |\nabla u^\lambda(t,x)|^{p(x)} dxdt \leqslant C_1^\lambda.$$

Proof. Let u be the strong solution of (2.9). Using (3.1), (2.3) and the Young inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|_{H_0}^2 + \int_{\Omega_1} d_0(x)|\nabla u|^{p(x)}\,dx + \int_{\Omega_1} d_0(x)\eta|\nabla u|^2\,dx + \int_{\Omega} |u|^{p(x)}\,dx& \\ \leqslant (\|B(u) - B(0)\|_{H_0} + \|B(0)\|_{H_0})\|u\|_{H_0}& \\ \leqslant L_B\|u\|_{H_0}^2 + \frac{1}{2}\|B(0)\|_{H_0}^2 + \frac{1}{2}\|u\|_{H_0}^2.& \qquad (3.9)\end{aligned}$$

Since $\int_{\Omega_1} d_0(x)\eta|\nabla u|^2\,dx \geqslant m_0\eta\|\nabla u\|_{L^2(\Omega_1)}^2 \geqslant 0$ and $\rho(u) = \int_{\Omega} |u|^{p(x)}\,dx \geqslant 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H_0}^2 + m_0\int_{\Omega_1} |\nabla u|^{p(x)}\,dx \leqslant \left(L_B + \frac{1}{2}\right)\|u\|_{H_0}^2 + \frac{1}{2}\|B(0)\|_{H_0}^2. \qquad (3.10)$$

Integrating (3.10) with respect to $t \in [0, T]$, it follows that

$$\begin{aligned}\|u(T)\|_{H_0}^2 + 2m_0\int_0^T \int_{\Omega_1} |\nabla u(t)|^{p(x)}\,dx\,dt& \\ \leqslant 2\left(L_B + \frac{1}{2}\right)\int_0^T \|u(t)\|_{H_0}^2\,dt + \|B(0)\|_{H_0}^2 T + \|u_0\|_{H_0}^2,&\end{aligned}$$

that is,

$$2m_0\int_0^T\int_{\Omega_1}|\nabla u(t)|^{p(x)}\,dx\,dt \leqslant (2L_B+1)\int_0^T \|u(t)\|_{H_0}^2\,dt + \|B(0)\|_{H_0}^2 T + \|u_0\|_{H_0}^2. \tag{3.11}$$

Neglecting the second term on the left side of (3.10), once $\rho(|\nabla u|)\geqslant 0$, and integrating under $[0,t]$, where $0<t<T$, we obtain

$$\|u(t)\|_{H_0}^2 \leqslant \|u_0\|_{H_0}^2 + \|B(0)\|_{H_0}^2 T + (2L_B+1)\int_0^t \|u(\theta)\|_{H_0}^2\,d\theta.$$

Using the Grönwal-Belmann Lemma

$$\|u(t)\|_{H_0}^2 \leqslant (\|u_0\|_{H_0}^2 + \|B(0)\|_{H_0}^2 T)e^{(2L_B+1)t}, \text{ for all } t\in[0,T]. \tag{3.12}$$

Integrating this last inequality with respect to $t\in[0,T]$, we conclude

$$\int_0^T \|u(t)\|_{H_0}^2\,dt \leqslant \frac{1}{(2L_B+1)}(\|u_0\|_{H_0}^2 + \|B(0)\|_{H_0}^2 T)(e^{(2L_B+1)T}-1).$$

Therefore, returning to equation (3.11),

$$2m_0\int_0^T\int_{\Omega_1}|\nabla u(t)|^{p(x)}\,dx\,dt \leqslant (\|u_0\|_{H_0}^2 + \|B(0)\|_{H_0}^2 T)e^{(2L_B+1)T},$$

which concludes the demonstration. □

Lemma 3.4. *Let $T>0$. If u is the weak solution of* (2.9) *in* $[0,T]$ *then there exists constant* $k_0 = k_0(\|u(0)\|_{H_0}, T) > 0$, *such that*

$$\int_0^T \|B(u(t))\|_{H_0}^2\,dt \leqslant k_0.$$

We obtain the same conclusion if u^λ is the weak solution of (2.8) *in* $[0,T]$, *where* $\lambda\in[0,1)$. *More precisely, for each $\lambda\in(0,1]$, there exists* $k_0^\lambda = k_0^\lambda(\|u^\lambda(0)\|_H, T) > 0$, *such that*

$$\int_0^T \|B(u^\lambda(t))\|_H^2\,dt \leqslant k_0^\lambda.$$

Proof. Being u the weak solution of (2.9), there exists a sequence $(u_i)_{i\in\mathbb{N}}$ of strong solution of (2.9) such that

$$\|u-u_i\|_{C([0,T];H_0)} \xrightarrow{i\to\infty} 0.$$

For each $i\in\mathbb{N}$, proceeding as in Lemma 3.3 to conclude (3.12), we obtain

$$\|u_i(t)\|_{H_0} \leqslant \sqrt{\|u_i(0)\|_{H_0}^2 + T\|B(0)\|_{H_0}^2}\; e^{(L_B+\frac{1}{2})T}, \quad \forall t\in[0,T],\ \forall i\in\mathbb{N}.$$

Consequently,

$$\|u\|_{C([0,T];H_0)} \leqslant \|u-u_i\|_{C([0,T];H_0)} + \|u_i(0)\|_{H_0}e^{(L_B+\frac{1}{2})T} + \sqrt{T}\|B(0)\|_{H_0}e^{(L_B+\frac{1}{2})T},$$

making $i\to\infty$, we have

$$\|u\|_{C([0,T];H_0)} \leqslant \|u(0)\|_{H_0}e^{(L_B+\frac{1}{2})T} + \sqrt{T}\|B(0)\|_{H_0}e^{(L_B+\frac{1}{2})T} := \mathrm{const}(\|u(0)\|_{H_0}, T). \tag{3.13}$$

Defining $k_0 := 2L_B^2 T(\text{const}(\|u(0)\|_{H_0}, T))^2 + 2T\|B(0)\|_{H_0}^2$ and using (3.13), we obtain

$$\begin{aligned}\int_0^T \|B(u(t))\|_{H_0}^2\, dt &\leqslant \int_0^T (\|B(u(t)) - B(0)\|_{H_0} + \|B(0)\|_{H_0})^2\, dt \\ &\leqslant 2L_B^2 \int_0^T \|u(t)\|_{H_0}^2\, dt + 2\|B(0)\|_{H_0}^2 T \\ &\leqslant 2L_B^2 T\|u\|_{C([0,T];H_0)}^2 + 2\|B(0)\|_{H_0}^2 T \\ &\leqslant k_0.\end{aligned}$$

□

4. Existence of exponential attractor via L-trajectory method

The aim of this section is to demonstrate that $(T_\lambda(t), H)$ has an exponential attractor, for all $\lambda \in [0,1]$. In particular, this implies that $(T_\lambda(t), H)$ has a global attractor with finite fractal dimension, for all $\lambda \in [0,1]$.

Let r be a positive constant obtained in item (ii) of Lemma 3.1, associated with $t_0 = 1$ and $k = 1$. We consider the set

$$B_1 = \{u \in V_0; \|u\|_{p(x)} \leqslant r \text{ and } \|\nabla u\|_{p(x)} \leqslant r\}. \tag{4.1}$$

Let u be a weak solution of (2.9), with $u(0) \in V_0$. From (2.7), we have

$$D(\varphi) := \{u \in H_0; \varphi(u) < +\infty\} = V_0.$$

Using Lemma 3.4, we can apply Theorem 3.6 from [8] for $H = H_0$, $\phi = \varphi$ and $f = B(u)$. Since $u(0) \in D(\varphi)$, we conclude that $t \mapsto \varphi(u(t))$ is absolutely continuous in $[0,T]$, where $T > 0$ is arbitrary fixed. In particular $\varphi(u(t))$ is bounded, for all $t \in [0,T]$, that is, $u(t) \in D(\varphi) = V_0$, for all $t \in [0,T]$. Therefore $T_0(t)u_0 = u(t) \in V_0$, for all $t \geqslant 0$. Then,

$$T_0(t)V_0 \subset V_0, \quad \forall t \geqslant 0. \tag{4.2}$$

It follows from Theorem 3.6 from [8] that u is the strong solution of (2.9). Thus, through the demonstration of Lemma 3.1, fixing positive values $t_0 = 1$ and $k = 1$, we have that

$$\|u(t)\|_{V_0} \leqslant r, \quad \forall t \geqslant 2.$$

So, $\|u(t)\|_{p(x)} \leqslant r$ and $\|\nabla u(t)\|_{p(x)} \leqslant r$, for all $t \geqslant 2$. Therefore, since $B_1 \subset V_0$, we have

$$T_0(t)B_1 \subset T_0(t)V_0 \subset B_1, \quad \text{for all } t \geqslant 2. \tag{4.3}$$

We define

$$B_0 = \bigcup_{t\in[0,2]} T_0(t)B_1. \tag{4.4}$$

We observe that $B_0 \subset V_0$. Indeed, given $u \in B_0$ arbitrary, there exist $\tilde{t} \in [0,2]$ and $b_1 \in B_1$ such that $u = T_0(\tilde{t})b_1$. Since $B_1 \subset V_0$, then $u = T_0(\tilde{t})b_1 \in T_0(\tilde{t})V_0$. For (4.2), we have that $u \in V_0$.

For $\lambda \in (0,1]$, we consider $B_1^\lambda = \{u \in V; \|u\|_{p(x)} \leqslant r \text{ and } \|\nabla u\|_{p(x)} \leqslant r\}$ and

$$B_0^\lambda = \bigcup_{t\in[0,2]} T_\lambda(t)B_1^\lambda. \tag{4.5}$$

We note that $B_1 \subset \overline{B^{V_0}(0,2r)}$ and $B_1^\lambda \subset \overline{B^V(0,2r)}$, for each $\lambda \in (0,1]$, then B_1 and B_1^λ are bounded sets in V_0. Therefore, by Lemma 3.2, B_0 is bounded in V_0 and, for each $\lambda \in (0,1]$, B_0^λ is bounded in V.

Lemma 4.1. *Set B_0, defined in* (4.4)*, is a compact subset of H_0. Moreover, B_0 is positively invariant with respect to $\{T_0(t)\}_{t\geqslant 0}$.*

For each $\lambda \in (0,1]$, set B_0^λ, defined in (4.5)*, is a compact subset of H. Moreover, B_0^λ is positively invariant with respect to $\{T_\lambda(t)\}_{t\geqslant 0}$.*

Proof. First of all, we will verify that B_0 is positively invariant with respect to $\{T_0(t)\}_{t\geqslant 0}$. Let $\tau \geqslant 0$ be arbitrary, we have that

$$T_0(\tau)B_0 = \bigcup_{t\in[0,2]} T_0(t)T_0(\tau)B_1.$$

If $\tau \geqslant 2$, from (4.3), we have

$$T_0(\tau)B_0 \subset \bigcup_{t\in[0,2]} T_0(t)B_1 = B_0, \quad \forall \tau \geqslant 2. \tag{4.6}$$

Now, if $0 \leqslant \tau < 2$, then $0 < 2-\tau \leqslant 2$. Hence, we can write

$$T_0(\tau)B_0 = \left(\bigcup_{t\in[0,2-\tau]} T_0(t+\tau)B_1\right) \cup \left(\bigcup_{t\in[2-\tau,2]} T_0(t+\tau)B_1\right), \quad \forall\, 0 \leqslant \tau < 2. \tag{4.7}$$

For $t \in [0, 2-\tau]$ we have $\tau \leqslant t+\tau \leqslant 2$, that is, $0 \leqslant t+\tau \leqslant 2$. Therefore,

$$\bigcup_{t\in[0,2-\tau]} T_0(t+\tau)B_1 \subset \bigcup_{t+\tau\in[0,2]} T_0(t+\tau)B_1 = B_0. \tag{4.8}$$

For $t \in [2-\tau, 2]$ we have $t+\tau \geqslant 2$, then from (4.3) we have

$$\bigcup_{t\in[2-\tau,2]} T_0(t+\tau)B_1 \subset B_1 \subset \bigcup_{t\in[0,2]} T_0(t)B_1 = B_0. \tag{4.9}$$

Substituting (4.8) and (4.9) in (4.7), we conclude that $T_0(\tau)B_0 \subset B_0$, for all $0 \leqslant \tau < 2$. Together with (4.6), we have $T_0(\tau)B_0 \subset B_0$, for all $\tau \geqslant 0$.

Now, we will demonstrate the compactness of B_0. Clearly B_1 is a bounded subset of V_0, since $V_0 \hookrightarrow\hookrightarrow H_0$, we have $\overline{B_1}^{H_0}$ compact in H_0.

In turn, B_1 is a closed subset in H_0. In fact, consider $(u_i)_{i\in\mathbb{N}}$ a sequence in B_1 and $u \in H_0$, such that

$$\|u_i - u\|_{H_0} \xrightarrow{i\to\infty} 0. \tag{4.10}$$

Since $(u_i)_{i\in\mathbb{N}}$is a bounded sequence in the reflexive space V_0, then $(u_i)_{i\in\mathbb{N}}$ admits subsequence, $(u_{i_k})_{k\in\mathbb{N}}$, that weakly converges to V_0, that is, there exists $\tilde{u} \in V_0$ such that

$$(f, u_{i_k} - \tilde{u})_{H_0} = \langle f, u_{i_k} - \tilde{u}\rangle_{V_0', V_0} \xrightarrow{k\to\infty} 0, \quad \forall f \in V_0'.$$

Therefore,

$$u_{i_k} \xrightarrow{k\to\infty} \tilde{u}, \quad \text{in } H_0.$$

Now, by convergence in (4.10), we have that $u_{i_k} \xrightarrow{k\to\infty} u$ in H_0, so $\tilde{u} = u$. Then, $u \in V_0$.

We must still show that $\|u\|_{p(x)} \leqslant r$ and $\|\nabla u\|_{p(x)} \leqslant r$. It follows from Theorem 4.9 in [23], restricting to a subsequence if necessary, which we will still denote by $(u_i)_{i\in\mathbb{N}}$, that

$$u_i(x) \xrightarrow{i\to\infty} u(x), \text{ almost everywhere in } \Omega.$$

Thus, for $0 < \dfrac{\varepsilon}{|\Omega|} < 1$ there exists $i_0 \in \mathbb{N}$ such that

$$|u_i(x) - u(x)| < \frac{\varepsilon}{|\Omega|}, \quad \text{ for all } i \geqslant i_0 \text{ and for almost every } x \in \Omega.$$

Since $p(x) \geqslant p^- > 2 > 1$, it follows that

$$\rho(u_i - u) = \int_\Omega |u_i(x) - u(x)|^{p(x)}\, dx \leqslant \int_\Omega \left(\frac{\varepsilon}{|\Omega|}\right)^{p(x)} dx \leqslant \int_\Omega \frac{\varepsilon}{|\Omega|}\, dx = \varepsilon, \quad \forall i \geqslant i_0.$$

Therefore, for all $i \geqslant i_0$, we have $\rho(u_i - u)^{\frac{1}{p^+}} \leqslant \varepsilon^{\frac{1}{p^+}}$. For all $i \geqslant i_0$, it follows from (2.1), that

$$\|u\|_{p(x)} \leqslant \|u_i - u\|_{p(x)} + \|u_i\|_{p(x)} \leqslant \max\{\varepsilon^{\frac{1}{p^-}}, \varepsilon^{\frac{1}{p^+}}\} + r,$$

as ε is arbitrarily small we concluded that $\|u\|_{p(x)} \leqslant r$.

We observe that $(\nabla u_i)_{i\in\mathbb{N}}$ is a bounded sequence in $L^{p(x)}(\Omega, \mathbb{R}^n)$, since $L^{p(x)}(\Omega, \mathbb{R}^n)$ is a reflexive space, passing through a subsequence if necessary, which we will denote by $(\nabla u_i)_{i\in\mathbb{N}}$, there exists $v \in L^{p(x)}(\Omega, \mathbb{R}^n)$ such that

$$\nabla u_i \xrightarrow{i\to\infty} v.$$

Thus, $\|v\|_{p(x)} \leqslant \liminf_{i\to\infty} \|\nabla u_i\|_{p(x)} \leqslant r$. We must just conclude that $v = \nabla u$, for which, consider $\phi \in C_c^\infty(\Omega)$ arbitrary, it is possible to conclude through the inequality of Hölder and by (4.10), that

$$\int_\Omega u_i \frac{\partial \phi}{\partial x_j}\, dx \xrightarrow{i\to\infty} \int_\Omega u \frac{\partial \phi}{\partial x_j}\, dx. \tag{4.11}$$

Besides this, since $\nabla u_i \xrightarrow{i\to\infty} v$, we have

$$\int_\Omega \frac{\partial u_i}{\partial x_j} \phi\, dx \xrightarrow{i\to\infty} \int_\Omega v_j \phi\, dx. \tag{4.12}$$

So, it follows from (4.11) and (4.12), that

$$\int_\Omega u \frac{\partial \phi}{\partial x_j}\, dx = -\int_\Omega v_j \phi\, dx.$$

That is, $\dfrac{\partial u}{\partial x_j} = v_j$ for all $j = 1, \ldots, n$.

Therefore, $u \in B_1$ and B_1 is closed in H_0. With that, B_1 is compact in H_0 and, consequently, $[0, 2] \times B_1$ is compact in $\mathbb{R}^+ \times H_0$. For operator continuity $\mathbb{R}^+ \times H_0 \ni (t, u_0) \mapsto T_0(t)u_0 \in H_0$, we conclude that $B_0 = T_0([0, 2])B_1$ is compact.

The conclusion for B_0^λ, with $\lambda \in (0, 1]$, follows analogously. □

For each $\lambda \in (0,1]$, we will denote by $\mathfrak{X}_\lambda$ the set of 1-trajectories associated with (2.8), that is, the set of all weak solutions of (2.8) defined in interval $[0,1]$, equipped with space topology $L^2(0,1;H)$.

Let $\lambda \in [0,1]$. If $\lambda \in (0,1]$, we will consider $\{L_\lambda(t)\}_{t\geqslant 0}$ the shift semigroup of 1-trajectories, that is,

$$\begin{aligned} L_\lambda(t)\colon \mathfrak{X}_\lambda \to\ & \mathfrak{X}_\lambda \\ \chi \mapsto\ & L_\lambda(t)\chi\colon [0,1] \to H \\ & \qquad\qquad \theta \mapsto u^\lambda(t+\theta), \end{aligned}$$

where u^λ is the only weak solution of (2.8) with $\chi = u^\lambda|_{[0,1]}$. If $\lambda = 0$, $\mathfrak{X}_0 = \mathfrak{X}$ denote the set of 1-trajectories, that is, the set of all weak solutions of (2.9) defined in interval $[0,1]$, equipped with space topology $L^2(0,1;H_0)$ and $\{L_0(t) = L(t)\}_{t\geqslant 0}$ the shift semigroup of 1-trajectories, $L(t)\chi\colon [0,1] \to H_0$, $\theta \mapsto u(t+\theta)$ where u is the only weak solution of (2.9) with $\chi = u|_{[0,1]}$.

We define

$$\mathcal{B}_0 = \{\chi \in \mathfrak{X}; \chi(0) \in B_0\} \tag{4.13}$$

and

$$\mathcal{B}_0^\lambda = \{\chi \in \mathfrak{X}_\lambda; \chi(0) \in B_0^\lambda\}, \tag{4.14}$$

for all $\lambda \in (0,1]$.

Lemma 4.2. *Set $\mathcal{B}_0$, defined in* (4.13)*, is compact in $L^2(0,1;H_0)$ and positively invariant with respect to $\{L(t)\}_{t\geqslant 0}$. For each $\lambda \in (0,1]$, set $\mathcal{B}_0^\lambda$, defined in* (4.14)*, is compact in $L^2(0,1;H)$ and positively invariant with respect to $\{L_\lambda(t)\}_{t\geqslant 0}$.*

Proof. Initially, we will verify that $\mathcal{B}_0$ is positively invariant with respect to $\{L(t)\}_{t\geqslant 0}$. Let $\chi \in \mathcal{B}_0$ and $\tau \geqslant 0$, we have that

$$(L(\tau)\chi)(s) = u(\tau+s), \quad \forall s \in [0,1],$$

where u is the only weak solution of (2.9) in $[0,\tau+1]$ such that $\chi = u|_{[0,1]}$.

From Lemma 4.1, $(L(\tau)\chi)(0) = u(\tau) \in B_0$, for each $\tau \geqslant 0$, implying that $L(\tau)\chi \in \mathcal{B}_0$, for $\tau \geqslant 0$. Hence, $L(\tau)(\mathcal{B}_0) \subset \mathcal{B}_0$, for all $\tau \geqslant 0$.

To check compactness, we first show that $\mathcal{B}_0$ is bounded in

$$\{u \in L^2(0,1;V_0);\ u_t \in L^2(0,1;H_0)\}.$$

In fact, given $\chi \in \mathcal{B}_0$ arbitrary, there exists u solution of (2.9), such that $\chi(t) = u(t)$, for all $t \in [0,1]$, where $u_0 = \chi(0) \in B_0 \subset V_0 = D(\varphi)$. By Theorem 3.6 from [8],

$$\|\chi_t\|^2_{L^2(0,1;H_0)} \leqslant \left[\left(\int_0^1 \|B(u(t))\|^2_{H_0}\, dt \right)^{\frac{1}{2}} + \sqrt{\varphi(u(0))} \right]^2.$$

By consequence of Lemma 3.4 and from the inequality (3.8), there exists a constant $k_5 > 0$ such that

$$\|\chi_t\|^2_{L^2(0,1;H_0)} \leqslant \left[\left(\int_0^1 \|B(u(t))\|^2_{H_0}\, dt \right)^{\frac{1}{2}} + \sqrt{\varphi(u(0))} \right]^2 \leqslant k_5, \quad \forall \chi \in \mathcal{B}_0,$$

it suffices to consider $k_5 := 2(k_0 + K_3)$. We note that k_5 is uniform with respect to initial data $u_0 = \chi(0) \in B_0$, since from Lemma 3.2, B_0 is a bounded subset V_0.

Moreover, from Poincaré inequality, there exist $C_2 > 0$ such that

$$\|\chi\|^2_{L^2(0,1;V_0)} = \int_0^1 (\|\chi(t)\|_{p(x)} + \|\nabla\chi(t)\|_{p(x)})^2\,dt \leqslant C_2 \int_0^1 \|\nabla u(t)\|^2_{p(x)}\,dt. \tag{4.15}$$

We will consider the case where $\|\nabla u(t)\|_{p(x)} \geqslant 1$. Since $2 < p^- \leqslant p(x)$ it follows from Proposition 2.1 that

$$\|\nabla u(t)\|^2_{p(x)} \leqslant \|\nabla u(t)\|^{p^-}_{p(x)} \leqslant \rho_p(\nabla u(t)).$$

Thus, returning to (4.15) and applying Lemma 3.3, we have that there exists $C_1 > 0$ such that

$$\|\chi\|^2_{L^2(0,1;V_0)} \leqslant C_2 \int_0^1 \int_\Omega |\nabla u(t)|^{p(x)}\,dxdt \leqslant C_2C_1.$$

We note that the constant $C_1 = C_1(\|u_0\|_{H_0})$ is uniform with respect to initial data $u_0 = \chi(0) \in B_0$, since from Lemma 4.1, B_0 is a bounded subset of H_0.

The case where $\|\nabla u(t)\|_{p(x)} \leqslant 1$, we have from (4.15) that $\|\chi\|^2_{L^2(0,1;V_0)} \leqslant C_2$. Therefore,

$$\|\chi\|^2_{L^2(0,1;V_0)} \leqslant \min\{C_2, C_1C_2\}, \quad \forall \chi \in \mathcal{B}_0.$$

Since $V_0 \hookrightarrow\hookrightarrow H_0$, from Lemma 2.1 we have that

$$\{u \in L^2(0,1;V_0); u_t \in L^2(0,1;H_0)\} \hookrightarrow\hookrightarrow L^2(0,1;H_0).$$

So, $\overline{\mathcal{B}_0}^{L^2(0,1;H_0)}$ is compact in $L^2(0,1;H_0)$.

In turn, $\mathcal{B}_0$ is closed $L^2(0,1;H_0)$. Indeed, let $\chi \in \overline{\mathcal{B}_0}^{L^2(0,1;H_0)}$ arbitrary, so there exists $(\chi_i)_{i\in\mathbb{N}}$ in $\mathcal{B}_0$ such that

$$\|\chi_i - \chi\|_{L^2(0,1;H_0)} \xrightarrow{i\to\infty} 0. \tag{4.16}$$

For all $i \in \mathbb{N}$, we have that $\chi_i \in \mathcal{B}_0$, that is, χ_i is the weak solution of (2.9), in $[0,1]$, with $\chi_i(0) \in B_0$. In particular $\chi_i \in C([0,1];H_0)$, so $B(\chi_i) \in C([0,1];H_0)$, for all $i \in \mathbb{N}$.

Let $i, j \in \mathbb{N}$, $i \neq j$, applying Lemma 3.1 from [8] for $u = \chi_j$, $v = \chi_i$, $A = A_0^{H_0}$, $f = B(\chi_j)$ and $g = B(\chi_i)$, we have

$$\|\chi_j(t) - \chi_i(t)\|_{H_0} \leqslant \|\chi_j(0) - \chi_i(0)\|_{H_0} + L_B \int_0^t \|\chi_j(s) - \chi_i(s)\|_{H_0}\,ds,$$

for all $t \in [0,1]$. Through Grönwall-Bellman Lemma, we concluded that

$$\|\chi_j(t) - \chi_i(t)\|_{H_0} \leqslant \|\chi_j(0) - \chi_i(0)\|_{H_0} e^{L_B},\ \forall t \in [0,1],\ \forall i,j \in \mathbb{N}. \tag{4.17}$$

In turn, $(\chi_i(0))_{i\in\mathbb{N}}$ is a sequence in B_0 that from Lemma 4.1, is a compact set of H_0. Then, $(\chi_i(0))_{i\in\mathbb{N}}$ admits subsequence, $(\chi_{i_k}(0))_{k\in\mathbb{N}}$, converging in H_0, in particular $(\chi_{i_k}(0))_{k\in\mathbb{N}}$ is a Cauchy sequence in H_0. Therefore, in (4.17), we have

$$\sup_{t\in[0,1]} \|\chi_{i_{k+1}}(t) - \chi_{i_k}(t)\|_{H_0} \leqslant e^{L_B}\|\chi_{i_{k+1}}(0) - \chi_{i_k}(0)\|_{H_0} \xrightarrow{k\to\infty} 0.$$

That is, $(\chi_{i_k})_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([0,1];H_0)$. Since $C([0,1];H_0)$ is a complete space, restricting to a subsequence if necessary, there exists $\tilde{\chi} \in C([0,1];H_0)$ such that

$$\|\chi_{i_k} - \tilde{\chi}\|_{C([0,1];H_0)} \xrightarrow{k\to\infty} 0. \tag{4.18}$$

Besides,

$$\|\chi_{i_k} - \tilde{\chi}\|_{L^2(0,1;H_0)} \leqslant \|\chi_{i_k} - \tilde{\chi}\|_{C([0,1];H_0)} \xrightarrow{k\to\infty} 0.$$

It follows from (4.16) that $\tilde{\chi} = \chi$, then $\chi \in C([0,1]; H_0)$.

Since $(\chi_{i_k})_{k\in\mathbb{N}}$ is a sequence of weak solutions of (2.9) in $[0,1]$, it is possible to conclude that there is a sequence of strong solutions that converge to χ in $C([0,1]; H_0)$. So, χ is a weak solution of (2.9) in $[0,1]$. We still have, from (4.18), that $\chi(0) \in \overline{B_0}^{H_0} = B_0$. Therefore, $\chi \in \mathcal{B}_0$ and with that $\mathcal{B}_0$ is closed in $L^2(0,1;H_0)$, therefore we have the compactness of $\mathcal{B}_0$.

The demonstration can be made analogously for $\mathcal{B}_0^\lambda$, where $\lambda \in (0,1]$.

□

Let $W^{1,2}_{\Omega_0,0}(\Omega) = \{f \in W^{1,2}_0(\Omega); f \text{ is constant in } \Omega_0\}$. We will consider the sets

$$Y_0 = \left\{\chi \in L^2(0,1; W^{1,2}_{\Omega_0,0}(\Omega)); \frac{d\chi}{dt} \in L^2(0,1; V_0^{'})\right\}$$

and

$$Y = \left\{\chi \in L^2(0,1; W^{1,2}_0(\Omega)); \frac{d\chi}{dt} \in L^2(0,1; V')\right\}.$$

From Lemma 2.1, we obtain the following compact inclusions, $Y_0 \hookrightarrow\hookrightarrow L^2(0,1;H_0)$ and $Y \hookrightarrow\hookrightarrow L^2(0,1;H)$. We provide Y, and Y_0, with the following norm

$$\|u\|_Y = \|\nabla u\|_{L^2(0,1;H)} + \|u_t\|_{L^2(0,1;V')}.$$

In particular, if $u \in Y_0$ we have

$$\|u\|_{Y_0} = \|\nabla u\|_{L^2(0,1;H_0)} + \|u_t\|_{L^2(0,1;V_0')}.$$

In the next result we will demonstrate the Lipschitz property for operators $L(1)\colon L^2(0,1;H_0) \to Y_0$, in $\mathcal{B}_0$, and $L_\lambda(1)\colon L^2(0,1;H) \to Y$, in $\mathcal{B}_0^\lambda$, for all $\lambda \in (0,1]$.

Lemma 4.3. *There exist constants $\omega_1 > 0$ and $\omega_1^\lambda > 0$ such that*

$$\begin{aligned}
&\|L(1)\chi_1 - L(1)\chi_2\|_{Y_0} \leqslant \omega_1 \|\chi_1 - \chi_2\|_{L^2(0,1;H_0)},\ \forall \chi_1,\chi_2 \in \mathcal{B}_0,\\
&\|L_\lambda(1)\chi_1 - L_\lambda(1)\chi_2\|_{Y} \leqslant \omega_1^\lambda \|\chi_1 - \chi_2\|_{L^2(0,1;H)},\ \forall \chi_1,\chi_2 \in \mathcal{B}_0^\lambda.
\end{aligned}$$

Proof. We will prove the Lipschitz property for $L(1)\colon L^2(0,1;H_0) \to Y_0$, in $\mathcal{B}_0$. The demonstration for operators $L_\lambda(1)\colon L^2(0,1;H) \to Y$, where $\lambda \in (0,1]$, can be made analogously.

Let $\chi_1, \chi_2 \in \mathcal{B}_0$ be arbitraries, so there exist only u and v strong solutions of (2.9) such that $u|_{[0,1]} = \chi_1$ and $v|_{[0,1]} = \chi_2$. We have that

$$u_t + A_0^{H_0} u = Bu \quad \text{and} \quad v_t + A_0^{H_0} v = Bv,$$

making the difference of the equations and denoting $w = u - v$, we can write

$$w_t + A_0^{H_0} u - A_0^{H_0} v = Bu - Bv. \tag{4.19}$$

Consider $\psi \in V_0 = W^{1,p(x)}_{\Omega_0,0}(\Omega)$, we have

$$
\begin{aligned}
|(w_t,\psi)_{H_0}| &\leqslant |(A_0^{H_0}u - A_0^{H_0}v,\psi)_{H_0}| + |(Bu-Bv,\psi)_{H_0}|\\
&\leqslant \left| \int_{\Omega_1} d_0(x)\Big[(|\nabla u|^{p(x)-2}+\eta)\nabla u - (|\nabla v|^{p(x)-2}+\eta)\nabla v\Big]\nabla\psi\,dx \right|\\
&\qquad + \left|\int_\Omega (|u|^{p(x)-2}u - |v|^{p(x)-2}v)\psi\,dx\right| + \|Bu-Bv\|_{H_0}\|\psi\|_{H_0}\\
&\leqslant M_0\int_{\Omega_1} ||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v||\nabla\psi|dx + M_0\eta\int_{\Omega_1}|\nabla w||\nabla\psi|dx\\
&\qquad + \int_\Omega ||u|^{p(x)-2}u - |v|^{p(x)-2}v||\psi|dx + L_B\|w\|_{H_0}\|\psi\|_{H_0}.
\end{aligned}
\tag{4.20}
$$

Let $\widetilde{\Omega}_1 := \{x\in\Omega_1 \colon \nabla w(t,x)\neq\vec{0}\}$. It follows from Hölder inequality and from Lemma 2.1 in [27], with $\delta = 0$, that

$$
\begin{aligned}
&\int_{\Omega_1} ||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v||\nabla\psi|dx\\
&\quad = \int_{\widetilde{\Omega}_1} ||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v||\nabla\psi|dx\\
&\quad \leqslant \left(\int_{\widetilde{\Omega}_1} ||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v||\nabla w|dx\right)^{\frac12}\\
&\qquad\qquad \left(\int_{\widetilde{\Omega}_1} \frac{||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v|}{|\nabla w|}|\nabla\psi|^2dx\right)^{\frac12}\\
&\quad = \left(\int_{\Omega_1} ||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v||\nabla w|dx\right)^{\frac12}\\
&\qquad\qquad \left(\int_{\widetilde{\Omega}_1} \frac{||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v|}{|\nabla w|}|\nabla\psi|^2dx\right)^{\frac12}\\
&\quad \leqslant \sqrt[4]{n}\sqrt{p^+-1}\left(\int_{\Omega_1} ||\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v||\nabla w|dx\right)^{\frac12}\\
&\qquad\qquad \left(\int_{\widetilde{\Omega}_1} (|\nabla u|+|\nabla v|)^{p(x)-2}|\nabla\psi|^2dx\right)^{\frac12}.
\end{aligned}
\tag{4.21}
$$

In turn, from Hölder inequality, we have

$$
\int_{\widetilde{\Omega}_1} (|\nabla u|+|\nabla v|)^{p(x)-2}|\nabla\psi|^2dx \leqslant \left(\frac{1}{r^-}+\frac{1}{s^-}\right)\||\nabla\psi|^2\|_{L^{r(x)}(\widetilde{\Omega}_1)}\|(|\nabla u|+|\nabla v|)^{p(x)-2}\|_{L^{s(x)}(\widetilde{\Omega}_1)},
\tag{4.22}
$$

where $r(x)=\frac{p(x)}{2}$ and $s(x)$ is obtained through $\frac{1}{s(x)}+\frac{1}{r(x)}=1$, that is, $s(x)=\frac{p(x)}{p(x)-2}$. We observe that $|\nabla\psi|^2\in L^{r(x)}(\widetilde{\Omega}_1)$ and $(|\nabla u|+|\nabla v|)^{p(x)-2}\in L^{s(x)}(\widetilde{\Omega}_1)$, since $\psi,u,v\in V_0$,

$$\rho_r(|\nabla\psi|^2)=\int_{\widetilde{\Omega}_1}|\nabla\psi|^{p(x)}dx\leqslant\int_{\Omega_1}|\nabla\psi|^{p(x)}dx \tag{4.23}$$

and

$$\begin{aligned}\rho_s((|\nabla u|+|\nabla v|)^{p(x)-2})&\leqslant\int_{\Omega_1}(|\nabla u|+|\nabla v|)^{p(x)}dx\\&\leqslant 2^{p^+}\left(\int_{\Omega_1}|\nabla u|^{p(x)}dx+\int_{\Omega_1}|\nabla v|^{p(x)}dx\right).\end{aligned}\tag{4.24}$$

Thus, from (2.1) and (4.23),

$$\begin{aligned}\||\nabla\psi|^2\|_{L^{r(x)}(\widetilde{\Omega}_1)}&\leqslant\max\left\{\left(\int_{\Omega_1}|\nabla\psi|^{p(x)}dx\right)^{\frac{1}{r^-}},\left(\int_{\Omega_1}|\nabla\psi|^{p(x)}dx\right)^{\frac{1}{r^+}}\right\}\\&=\max\{\rho_p(\nabla\psi)^{\frac{1}{r^-}},\rho_p(\nabla\psi)^{\frac{1}{r^+}}\}.\end{aligned}\tag{4.25}$$

From (2.2), it follows that

$$\rho_p(\nabla\psi)^{\frac{1}{r^{\mp}}}\leqslant\max\{(\|\nabla\psi\|_{p(x)}^{p^-})^{\frac{1}{r^{\mp}}},(\|\nabla\psi\|_{p(x)}^{p^+})^{\frac{1}{r^{\mp}}}\}.$$

Returning to (4.25), we have

$$\||\nabla\psi|^2\|_{L^{r(x)}(\widetilde{\Omega}_1)}\leqslant\max\{\|\nabla\psi\|_{p(x)}^{\frac{p^-}{r^-}},\|\nabla\psi\|_{p(x)}^{\frac{p^+}{r^-}},\|\nabla\psi\|_{p(x)}^{\frac{p^-}{r^+}},\|\nabla\psi\|_{p(x)}^{\frac{p^+}{r^+}}\}:=\xi_{\nabla\psi}.\tag{4.26}$$

Analogously, from (2.1), (2.2) and (4.24) we have

$$\begin{aligned}&\|(|\nabla u|+|\nabla v|)^{p(x)-2}\|_{L^{s(x)}(\widetilde{\Omega}_1)}\\&\qquad\leqslant\max\{\||\nabla u|+|\nabla v|\|_{p(x)}^{\frac{p^-}{s^-}},\||\nabla u|+|\nabla v|\|_{p(x)}^{\frac{p^+}{s^-}},\||\nabla u|+|\nabla v|\|_{p(x)}^{\frac{p^-}{s^+}},\||\nabla u|+|\nabla v|\|_{p(x)}^{\frac{p^+}{s^+}}\}\\&:=\xi_{|\nabla u|+|\nabla v|}.\end{aligned}\tag{4.27}$$

Returning to inequality (4.21) and using (4.22), (4.26) and (4.27), we conclude that

$$\begin{aligned}&\int_{\Omega_1}||\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v||\nabla\psi|dx\\&\leqslant\sqrt[4]{n}\sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{\nabla\psi}\xi_{|\nabla u|+|\nabla v|}\right]^{\frac{1}{2}}\left(\int_{\Omega_1}||\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v||\nabla w|dx\right)^{\frac{1}{2}}.\end{aligned}\tag{4.28}$$

Proceeding in an analogous way, we can conclude that

$$\begin{aligned}&\int_{\Omega}||u|^{p(x)-2}u-|v|^{p(x)-2}v||\psi|dx\\&\leqslant\sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{\psi}\xi_{|u|+|v|}\right]^{\frac{1}{2}}\left(\int_{\Omega}||u|^{p(x)-2}u-|v|^{p(x)-2}v||w|dx\right)^{\frac{1}{2}}.\end{aligned}\tag{4.29}$$

Consider the notations

$$\mathcal{I}_{(f,g)} := \int_{\Omega} ||f|^{p(x)-2}f - |g|^{p(x)-2}g||f-g| \ dx$$

and

$$\mathcal{I}_{1(f,g)} := \int_{\Omega_1} ||f|^{p(x)-2}f - |g|^{p(x)-2}g||f-g| \ dx,$$

Returning to (4.20), using (4.28), (4.29) and the Hölder inequality, it follows that

$$\begin{aligned} |(w_t,\psi)_{H_0}| \leqslant M_0\sqrt[4]{n}\sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{\nabla\psi}\xi_{|\nabla u|+|\nabla v|}\right]^{\frac{1}{2}}\big(\mathcal{I}_{1(\nabla u,\nabla v)}\big)^{\frac{1}{2}} \\ + \sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{\psi}\xi_{|u|+|v|}\right]^{\frac{1}{2}}\big(\mathcal{I}_{(u,v)}\big)^{\frac{1}{2}} + M_0\eta\|\nabla w\|_{H_0}\|\nabla\psi\|_{H_0} \\ + L_B\|w\|_{H_0}\|\psi\|_{H_0}. \end{aligned} \tag{4.30}$$

Since $u, v, \psi \in V_0$, $p(x) \geqslant p^- > 2$ and $|\Omega| < \infty$, we can conclude that $u, v, \psi \in W_0^{1,2}(\Omega)$. From Poincaré inequality, there exists $\alpha > 0$ such that

$$\|w\|_{H_0} \leqslant \alpha\|\nabla w\|_{H_0} \quad \text{and} \quad \|\psi\|_{H_0} \leqslant \alpha\|\nabla\psi\|_{H_0}.$$

It follows from (4.30) acting in $t = 1+\theta$, where $\theta \in [0,1]$, that

$$\begin{aligned} \sup_{\|\psi\|_{V_0}\leqslant 1} |(w_t(1+\theta),\psi)_{H_0}| \\ \leqslant M_0\sqrt[4]{n}\sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{|\nabla u(1+\theta)|+|\nabla v(1+\theta)|}\right]^{\frac{1}{2}}\sup_{\|\psi\|_{V_0}\leqslant 1}(\xi_{\nabla\psi})^{\frac{1}{2}}\big(\mathcal{I}_{1(\nabla u(1+\theta),\nabla v(1+\theta))}\big)^{\frac{1}{2}} \\ + \sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{|u(1+\theta)|+|v(1+\theta)|}\right]^{\frac{1}{2}}\sup_{\|\psi\|_{V_0}\leqslant 1}(\xi_{\psi})^{\frac{1}{2}}\big(\mathcal{I}_{(u(1+\theta),v(1+\theta))}\big)^{\frac{1}{2}} \\ + M_0\eta\|\nabla w(1+\theta)\|_{H_0}\sup_{\|\psi\|_{V_0}\leqslant 1}\|\nabla\psi\|_{H_0} + L_B\alpha\|\nabla w(1+\theta)\|_{H_0}\alpha\sup_{\|\psi\|_{V_0}\leqslant 1}\|\nabla\psi\|_{H_0}. \end{aligned} \tag{4.31}$$

If $\|\psi\|_{V_0} \leqslant 1$, then $\xi_{\nabla\psi} \leqslant 1$ and $\xi_{\psi} \leqslant 1$. Since $L^{p(x)} \hookrightarrow L^2$, there exists a constant $\alpha_1 > 0$, such that

$$\|\nabla\psi\|_{H_0} \leqslant \alpha_1\|\nabla\psi\|_{p(x)} \leqslant \alpha_1\|\psi\|_{V_0}.$$

We conclude, from (4.31), that

$$
\begin{aligned}
&\sup_{\|\psi\|_{V_0}\leqslant 1} |(w_t(1+\theta),\psi)_{H_0}| \\
&\qquad \leqslant M_0\sqrt[4]{n}\sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{|\nabla u(1+\theta)|+|\nabla v(1+\theta)|}\right]^{\frac{1}{2}}\left(\mathcal{I}_{1(\nabla u(1+\theta),\nabla v(1+\theta))}\right)^{\frac{1}{2}} \\
&\qquad\qquad + \sqrt{p^+-1}\left[\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{|u(1+\theta)|+|v(1+\theta)|}\right]^{\frac{1}{2}}\left(\mathcal{I}_{(u(1+\theta),v(1+\theta))}\right)^{\frac{1}{2}} \\
&\qquad\qquad\qquad + M_0\eta\alpha_1\|\nabla w(1+\theta)\|_{H_0} + L_B\alpha^2\alpha_1\|\nabla w(1+\theta)\|_{H_0}.
\end{aligned}
$$

Knowing that $(a+b)^2 \leqslant 2(a^2+b^2)$, that is, $(a+b+c)^2 \leqslant 4(a^2+b^2)+2c^2$, we have

$$
\begin{aligned}
&\left(\sup_{\|\psi\|_{V_0}\leqslant 1} |(w_t(1+\theta),\psi)_{H_0}|\right)^2 \\
&\qquad \leqslant 4M_0^2\sqrt{n}(p^+-1)\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{|\nabla u(1+\theta)|+|\nabla v(1+\theta)|}\mathcal{I}_{1(\nabla u(1+\theta),\nabla v(1+\theta))} \\
&\qquad\qquad + 4(p^+-1)\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\xi_{|u(1+\theta)|+|v(1+\theta)|}\mathcal{I}_{(u(1+\theta),v(1+\theta))} \\
&\qquad\qquad\qquad + 2(M_0\eta\alpha_1+L_B\alpha^2\alpha_1)^2\|\nabla w(1+\theta)\|_{H_0}^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|(L(1)\chi_1 - L(1)\chi_2)_t\|^2_{L^2(0,1;V_0')} \\
&\qquad = \int_0^1 \|w_t(1+\theta)\|^2_{V_0'}\, d\theta \\
&\qquad = \int_0^1 \left(\sup_{\|\psi\|_{V_0}\leqslant 1} |(w_t(1+\theta),\psi)_{H_0}|\right)^2 d\theta \\
&\qquad \leqslant 4M_0^2\sqrt{n}(p^+-1)\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\int_0^1 \xi_{|\nabla u(1+\theta)|+|\nabla v(1+\theta)|}\mathcal{I}_{1(\nabla u(1+\theta),\nabla v(1+\theta))}\, d\theta \\
&\qquad\qquad + 4(p^+-1)\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\int_0^1 \xi_{|u(1+\theta)|+|v(1+\theta)|}\mathcal{I}_{(u(1+\theta),v(1+\theta))}\, d\theta \\
&\qquad\qquad\qquad + 2(M_0\eta\alpha_1+L_B\alpha^2\alpha_1)^2\int_0^1 \|\nabla w(1+\theta)\|^2_{H_0}\, d\theta.
\end{aligned}
\tag{4.32}
$$

In turn, it follows from the demonstration of Lemma 3.1, considering $t_0 = \frac{1}{2}$ and $k = \frac{1}{2}$, the existence of $r_2 > 0$ such that $\|u(t)\|_{V_0} \leqslant r_2$, for all $t \geqslant t_1 = t_0 + k = 1$, then

$$\begin{aligned}\| \, |\nabla u(1+\theta)| + |\nabla v(1+\theta)| \, \|_{p(x)} &\leqslant \sup_{\theta\in[0,1]} (\|\nabla u(1+\theta)\|_{p(x)} + \|\nabla v(1+\theta)\|_{p(x)}) \\ &\leqslant \sup_{t\in[1,2]} (\|u(t)\|_{V_0} + \|v(t)\|_{V_0}) \\ &\leqslant 2r_2, \quad \forall \theta \in [0,1].\end{aligned}$$

So,

$$\xi_{|\nabla u(1+\theta)|+|\nabla v(1+\theta)|} \leqslant \max\{(2r_2)^{\frac{p^-}{s^-}}, (2r_2)^{\frac{p^+}{s^-}}, (2r_2)^{\frac{p^-}{s^+}}, (2r_2)^{\frac{p^+}{s^+}}\} := \kappa_1.$$

Likewise, we can conclude that $\xi_{|u(1+\theta)|+|v(1+\theta)|} \leqslant \kappa_1$. Hence, in (4.32), we have

$$\begin{aligned}&\|(L(1)\chi_1 - L(1)\chi_2)_t\|^2_{L^2(0,1;V_0')} \\ &\leqslant \gamma \left(\int_0^1 \mathcal{I}_{1(\nabla u(1+\theta),\nabla v(1+\theta))} \, d\theta + \int_0^1 \mathcal{I}_{(u(1+\theta),v(1+\theta))} \, d\theta + \int_0^1 \|\nabla w(1+\theta)\|^2_{H_0} \, d\theta \right),\end{aligned} \tag{4.33}$$

where $\gamma = \max\left\{4M_0^2\sqrt{n}(p^+-1)\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\kappa_1, 4(p^+-1)\left(\frac{1}{r^-}+\frac{1}{s^-}\right)\kappa_1, 2(M_0\eta\alpha_1 + L_B\alpha^2\alpha_1)^2\right\}$.

Our next goal is to show the existence of a constant, $\beta_1 > 0$, such that

$$\int_0^1 \mathcal{I}_{1(\nabla u(1+\theta),\nabla v(1+\theta))} \, d\theta \leqslant \beta_1 \int_0^1 \|w(\theta)\|^2_{H_0} \, d\theta. \tag{4.34}$$

Making the product $(\cdot, w)_{H_0}$ in equation (4.19) we have

$$(w_t, w)_{H_0} + (A_0^{H_0}u - A_0^{H_0}v, w)_{H_0} = (Bu - Bv, w)_{H_0},$$

where

$$\begin{aligned}(A_0^{H_0}u - A_0^{H_0}v, w)_{H_0} &= \langle A_0u - A_0v, w\rangle_{V_0',V_0} \\ &= \int_{\Omega_1} d_0(x)\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla w \, dx + \int_{\Omega_1} d_0(x)\eta(\nabla u - \nabla v)\nabla w \, dx \\ &\quad + \int_{\Omega}(|u|^{p(x)-2}u - |v|^{p(x)-2}v)w \, dx \\ &\geqslant m_0\mathcal{I}_{1(\nabla u,\nabla v)} + m_0\eta\int_{\Omega_1}|\nabla w|^2 \, dx + \mathcal{I}_{(u,v)}.\end{aligned}$$

It follows from Lemma 2.1 in [27], with $\delta = 0$, that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|^2_{H_0} + m_0\int_{\Omega_1} \frac{1}{2^{p(x)-1}}|\nabla w|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\, dx + m_0\eta\int_{\Omega_1} |\nabla w|^2\, dx \\
+ \int_{\Omega} \frac{1}{2^{p(x)-1}}|w|^2(|u| + |v|)^{p(x)-2}\, dx \\
\leqslant \frac{1}{2}\frac{d}{dt}\|w\|^2_{H_0} + m_0\mathcal{I}_{1(\nabla u,\nabla v)} + m_0\eta\int_{\Omega_1} |\nabla w|^2\, dx + \mathcal{I}_{(u,v)} \\
\leqslant (w_t, w)_{H_0} + (A_0^{H_0}u - A_0^{H_0}v, w)_{H_0} \\
= (Bu - Bv, w)_{H_0} \\
\leqslant \|Bu - Bv\|_{H_0}\|w\|_{H_0} \\
\leqslant L_B\|w\|^2_{H_0}. \qquad (4.35)
\end{aligned}$$

Since $p(x) \leqslant p^+$ we can write

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{H_0} + m_0\frac{1}{2^{p^+-1}}\int_{\Omega_1} |\nabla w|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\, dx \leqslant L_B\|w\|^2_{H_0}, \qquad (4.36)$$

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{H_0} + m_0\eta\int_{\Omega_1} |\nabla w|^2\, dx \leqslant L_B\|w\|^2_{H_0} \qquad (4.37)$$

and

$$\frac{1}{2}\frac{d}{dt}\|w\|^2_{H_0} + \frac{1}{2^{p^+-1}}\int_{\Omega} |w|^2(|u| + |v|)^{p(x)-2}\, dx \leqslant L_B\|w\|^2_{H_0}. \qquad (4.38)$$

Neglecting the second term of the sum in (4.36) and integrating to θ varying in interval $[s, t]$, where $0 \leqslant s < t$, it follows that

$$\|w(t)\|^2_{H_0} \leqslant \|w(s)\|^2_{H_0} + 2L_B\int_s^t \|w(\theta)\|^2_{H_0}\, d\theta.$$

From the Grönwall Bellman Lemma, we have

$$\|w(t)\|^2_{H_0} \leqslant \|w(s)\|^2_{H_0}e^{2L_B(t-s)}, \quad \text{for } 0 \leqslant s < t. \qquad (4.39)$$

Returning to (4.36) and integrating in interval $[\tau, 2]$, with $\tau \in [0, 1]$, we obtain

$$\frac{1}{2}\|w(2)\|^2_{H_0} - \frac{1}{2}\|w(\tau)\|^2_{H_0} + \frac{m_0}{2^{p^+-1}}\int_\tau^2\int_{\Omega_1} |\nabla w|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\, dx\, d\theta \leqslant L_B\int_\tau^2 \|w(\theta)\|^2_{H_0}\, d\theta.$$

Then,

$$\begin{aligned}
\frac{m_0}{2^{p^+-2}}\int_\tau^2\int_{\Omega_1} |\nabla w|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\, dx\, d\theta \\
\leqslant \frac{m_0}{2^{p^+-2}}\int_\tau^2\int_{\Omega_1} |\nabla w|^2(|\nabla u| + |\nabla v|)^{p(x)-2}\, dx\, d\theta + \|w(2)\|^2_{H_0} \\
\leqslant \|w(\tau)\|^2_{H_0} + 2L_B\int_\tau^2 \|w(\theta)\|^2_{H_0}\, d\theta. \qquad (4.40)
\end{aligned}$$

We will now estimate the left side of inequality (4.40). Since $0 \leqslant \tau \leqslant \theta \leqslant 2$ then for (4.39), with $s = \tau$ and $t = \theta$, we have

$$\|w(\theta)\|_{H_0}^2 \leqslant \|w(\tau)\|_{H_0}^2 e^{2L_B(\theta-\tau)}.$$

Integrating this last inequality to θ varying in $[\tau, 2]$, we obtain

$$2L_B \int_\tau^2 \|w(\theta)\|_{H_0}^2 \, d\theta \leqslant \|w(\tau)\|_{H_0}^2 e^{2L_B(2-\tau)} - \|w(\tau)\|_{H_0}^2. \tag{4.41}$$

Substituting in (4.40) we conclude that

$$\frac{m_0}{2^{p^+-2}} \int_\tau^2 \int_{\Omega_1} |\nabla w|^2 (|\nabla u| + |\nabla v|)^{p(x)-2} \, dx \, d\theta \leqslant \|w(\tau)\|_{H_0}^2 e^{4L_B}, \quad \forall \tau \in [0,1]. \tag{4.42}$$

So, from (4.42) and from Lemma 2.1 in [27], we have

$$\begin{aligned}
\int_0^1 \mathcal{I}_{1(\nabla u(1+\theta), \nabla v(1+\theta))} d\theta &= \int_1^2 \int_{\Omega_1} ||\nabla u(t)|^{p(x)-2} \nabla u(t) - |\nabla v(t)|^{p(x)-2} \nabla v(t)||\nabla w(t)| \; dxdt \\
&\leqslant \int_\tau^2 \int_{\Omega_1} ||\nabla u(t)|^{p(x)-2} \nabla u(t) - |\nabla v(t)|^{p(x)-2} \nabla v(t)||\nabla w(t)| \; dxdt \\
&\leqslant \sqrt{n}(p^+ - 1) \int_\tau^2 \int_{\Omega_1} |\nabla w|^2 (|\nabla u| + |\nabla v|)^{p(x)-2} \; dxdt \\
&\leqslant \sqrt{n}(p^+ - 1) \left(\frac{2^{p^+-2}}{m_0} \|w(\tau)\|_{H_0}^2 e^{4L_B} \right), \quad \forall \tau \in [0,1].
\end{aligned}$$

Integrating this last inequality for τ varying in $[0, 1]$ we have

$$\int_0^1 \mathcal{I}_{1(\nabla u(1+\theta), \nabla v(1+\theta))} \; d\theta \leqslant \frac{2^{p^+-2}}{m_0} \sqrt{n}(p^+ - 1) e^{4L_B} \int_0^1 \|w(\tau)\|_{H_0}^2 \, d\tau.$$

Therefore (4.34) occurs for $\beta_1 = 2^{p^+-2} m_0^{-1} \sqrt{n}(p^+ - 1) e^{4L_B}$.

Proceeding analogously, we conclude the existence of $\beta_2 > 0$, such that

$$\int_0^1 \mathcal{I}_{(u(1+\theta), v(1+\theta))} \; d\theta \leqslant \beta_2 \int_0^1 \|w(\theta)\|_{H_0}^2 \, d\theta. \tag{4.43}$$

In fact, using the same argumentation made in (4.36), for (4.38), it follows from (4.41) that

$$\begin{aligned}
\frac{1}{2^{p^+-2}} \int_\tau^2 \int_\Omega |w|^2 (|u| + |v|)^{p(x)-2} \, dx \, d\theta \leqslant & \frac{1}{2^{p^+-2}} \int_\tau^2 \int_\Omega |w|^2 (|u| + |v|)^{p(x)-2} \, dx \, d\theta + \|w(2)\|_{H_0}^2 \\
\leqslant & \|w(\tau)\|_{H_0}^2 + 2L_B \int_\tau^2 \|w(\theta)\|_{H_0}^2 \, d\theta \\
\leqslant & \|w(\tau)\|_{H_0}^2 e^{4L_B}, \quad \forall \tau \in [0,1]. \qquad (4.44)
\end{aligned}$$

Then, from Lemma 2.1 in [27], with $\delta = 0$, and (4.44), we conclude that

$$\begin{aligned}
\int_0^1 \mathcal{I}_{(u(1+\theta),v(1+\theta))}\ d\theta &= \int_1^2 \int_\Omega ||u(t)|^{p(x)-2}u(t) - |v(t)|^{p(x)-2}v(t)||w(t)|\ dxdt \\
&\leqslant \int_\tau^2 \int_\Omega ||u(t)|^{p(x)-2}u(t) - |v(t)|^{p(x)-2}v(t)||w(t)|\ dxdt \\
&\leqslant (p^+ - 1) \int_\tau^2 \int_\Omega |w|^2(|u| + |v|)^{p(x)-2}\ dxdt \\
&\leqslant (p^+ - 1) \left(2^{p^+-2}\|w(\tau)\|_{H_0}^2 e^{4L_B}\right), \quad \forall \tau \in [0,1].
\end{aligned}$$

Integrating this last inequality for τ varying in $[0,1]$ we have

$$\int_0^1 \mathcal{I}_{(u(1+\theta),v(1+\theta))}\ d\theta \leqslant 2^{p^+-2}(p^+ - 1)e^{4L_B} \int_0^1 \|w(\tau)\|_{H_0}^2\, d\tau,$$

obtaining (4.43), with $\beta_2 = 2^{p^+-2}(p^+ - 1)e^{4L_B}$.

Besides, we have the existence of $\beta_3 > 0$ such that

$$\|\nabla w(1+\cdot)\|_{L^2(0,1;H_0)}^2 \leqslant \beta_3 \int_0^1 \|w(\theta)\|_{H_0}^2\ d\theta. \tag{4.45}$$

In fact, integrating both sides of (4.37) under the interval $[\tau, 2]$, for $\tau \in [0,1]$, it follows from (4.41), that

$$\begin{aligned}
2m_0\eta \int_\tau^2 \int_{\Omega_1} |\nabla w(\theta,x)|^2\, dx\, d\theta &\leqslant 2m_0\eta \int_\tau^2 \int_{\Omega_1} |\nabla w(\theta,x)|^2\, dx\, d\theta + \|w(2)\|_{H_0}^2 \\
&\leqslant \|w(\tau)\|_{H_0}^2 + 2L_B \int_\tau^2 \|w(\theta)\|_{H_0}^2\, d\theta \\
&\leqslant \|w(\tau)\|_{H_0}^2 e^{4L_B}, \quad \forall \tau \in [0,1].
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_0^1 \int_\Omega |\nabla w(1+\theta,x)|^2\, dx\, d\theta &\leqslant \int_\tau^2 \int_{\Omega_1} |\nabla w(t,x)|^2\, dx\, dt \\
&\leqslant \frac{e^{4L_B}}{2m_0\eta}\|w(\tau)\|_{H_0}^2, \quad \forall \tau \in [0,1].
\end{aligned}$$

Integrating this last inequality for τ varying in $[0,1]$, we obtain (4.45) with
$\beta_3 = e^{4L_B}(2m_0\eta)^{-1}$.

Finally, it follows from (4.33), (4.34), (4.43) and (4.45) that

$$\begin{aligned}
\|L(1)\chi_1 - L(1)\chi_2\|_{Y_0} &= \|\nabla w(1+\cdot)\|_{L^2(0,1;H_0)} + \|(L(1)\chi_1 - L(1)\chi_2)_t\|_{L^2(0,1;V_0')} \\
&\leqslant \|\nabla w(1+\cdot)\|_{L^2(0,1;H_0)} \\
&\quad + \gamma^{\frac{1}{2}} \left(\int_0^1 \mathcal{I}_{1(\nabla u(1+\theta), \nabla v(1+\theta))}\, d\theta + \|\nabla w(1+\cdot)\|^2_{L^2(0,1;H_0)} + \int_0^1 \mathcal{I}_{(u(1+\theta), v(1+\theta))}\, d\theta \right)^{\frac{1}{2}} \\
&\leqslant \|\nabla w(1+\cdot)\|_{L^2(0,1;H_0)} + \gamma^{\frac{1}{2}} \left(\int_0^1 \mathcal{I}_{1(\nabla u(1+\theta), \nabla v(1+\theta))}\, d\theta \right)^{\frac{1}{2}} + \gamma^{\frac{1}{2}} \|\nabla w(1+\cdot)\|_{L^2(0,1;H_0)} \\
&\quad + \gamma^{\frac{1}{2}} \left(\int_0^1 \mathcal{I}_{(u(1+\theta), v(1+\theta))}\, d\theta \right)^{\frac{1}{2}} \\
&\leqslant (1+\gamma^{\frac{1}{2}}) \left(\beta_3 \int_0^1 \|w(\theta)\|^2_{H_0}\, d\theta \right)^{\frac{1}{2}} + \gamma^{\frac{1}{2}} \left(\beta_1 \int_0^1 \|w(\theta)\|^2_{H_0}\, d\theta \right)^{\frac{1}{2}} \\
&\quad + \gamma^{\frac{1}{2}} \left(\beta_2 \int_0^1 \|w(\theta)\|^2_{H_0}\, d\theta \right)^{\frac{1}{2}} \\
&= \omega_1 \|\chi_1 - \chi_2\|_{L^2(0,1;H_0)},
\end{aligned}$$

where $\omega_1 = [(1+\gamma^{\frac{1}{2}})\beta_3^{\frac{1}{2}} + \gamma^{\frac{1}{2}}\beta_1^{\frac{1}{2}} + \gamma^{\frac{1}{2}}\beta_2^{\frac{1}{2}}]$. Thus, concluding the demonstration. □

We emphasize that in this last Lemma the constant $\omega_1^\lambda > 0$ cannot be taken uniformly with respect to the parameter $\lambda \in (0,1]$, given that the constant $M_\lambda > 0$ determined by the limitation of diffusion d_λ, composes the constant ω_1^λ.

Consider the applications

$$\begin{aligned}
\mathbf{e} : &\mathfrak{X} \to H_0 \\
&\chi \mapsto \chi(1)
\end{aligned} \tag{4.46}$$

and

$$\begin{aligned}
\mathbf{e}_\lambda : &\mathfrak{X}_\lambda \to H \\
&\chi \mapsto \chi(1),
\end{aligned} \tag{4.47}$$

for all $\lambda \in (0,1]$.

Lemma 4.4. *Applications* $\mathbf{e}$ *and* $\mathbf{e}_\lambda$ *defined, respectively, in* (4.46) *and* (4.47)*, are Lipschitz continuous in* $\mathcal{B}_0$ *and* $\mathcal{B}_0^\lambda$ *respectively.*

Proof. We will demonstrate that $\mathbf{e}$ is Lipschitz continuous. The demonstration for $\mathbf{e}_\lambda$ is made the same way.

Let $\chi_1, \chi_2 \in \mathcal{B}_0$, then there exits only u and v solutions of (2.9) with $u(0), v(0) \in B_0$ such that $u|_{[0,1]} = \chi_1$ and $v|_{[0,1]} = \chi_2$, we denote by w the difference $u - v$. Proceeding as in the demonstration of Lemma 4.3, we can conclude (4.39). In particular,

$$\|w(1)\|^2_{H_0} \leqslant \|w(\theta)\|^2_{H_0} e^{2L_B(1-\theta)} \leqslant \|w(\theta)\|^2_{H_0} e^{2L_B}, \quad \forall \theta \in [0,1].$$

Integrating this last inequality, for θ varying in $[0,1]$, we have

$$\|w(1)\|_{H_0} \leqslant e^{L_B} \left(\int_0^1 \|w(\theta)\|_{H_0}^2 \, d\theta \right)^{\frac{1}{2}}.$$

Then,

$$\|\mathbf{e}(\chi_1) - \mathbf{e}(\chi_2)\|_{H_0} = \|w(1)\|_{H_0} \leqslant e^{L_B} \|w\|_{L^2(0,1;H_0)} = e^{L_B} \|\chi_1 - \chi_2\|_{L^2(0,1;H_0)}.$$

□

Proposition 4.1. *There exists constant* $c_3 = c_3(u_0) > 0$ *such that*

$$\|L(s)\chi_1 - L(t)\chi_2\|_{L^2(0,1;H_0)} \leqslant c_3(|s-t|^{\frac{1}{2}} + \|\chi_1 - \chi_2\|_{L^2(0,1;H_0)}),$$

for all $t, s \in [0,1]$ *and for all* $\chi_1, \chi_2 \in \mathcal{B}_0$. *For each* $\lambda \in (0,1]$, *there exists* $\tilde{c}_3 = \tilde{c}_3(u_0^\lambda) > 0$ *such that*

$$\|L_\lambda(s)\chi_1 - L_\lambda(t)\chi_2\|_{L^2(0,1;H)} \leqslant \tilde{c}_3(|s-t|^{\frac{1}{2}} + \|\chi_1 - \chi_2\|_{L^2(0,1;H)}),$$

for all $t, s \in [0,1]$ *and for all* $\chi_1, \chi_2 \in \mathcal{B}_0^\lambda$.

Proof. First of all, we will demonstrate that there exists $c > 0$ so that

$$\|T_0(s)u_0 - T_0(t)v_0\|_{H_0} \leqslant c(|s-t|^{\frac{1}{2}} + \|u_0 - v_0\|_{H_0}),$$

for all $s, t \in [0,1]$ and $u_0, v_0 \in B_0$. Indeed, for all $s, t \in [0,1]$ and $u_0, v_0 \in B_0$

$$\|T_0(s)u_0 - T_0(t)v_0\|_{H_0} \leqslant \|T_0(s)u_0 - T_0(t)u_0\|_{H_0} + \|T_0(t)u_0 - T_0(t)v_0\|_{H_0}. \tag{4.48}$$

By the Fundamental Theorem of Calculus, Hölder inequality, Fubini Theorem and Theorem 3.6 from [8], once $u_0 \in B_0 \subset V_0 = D(\varphi)$, we have

$$\begin{aligned}
\|T_0(s)u_0 - T_0(t)u_0\|_{H_0}^2 &= \int_\Omega |T_0(s)u_0(x) - T_0(t)u_0(x)|^2 \, dx \\
&= \int_\Omega \left| \int_t^s (T_0(\theta)u_0(x))_\theta \, d\theta \right|^2 \, dx \\
&\leqslant |s-t| \int_\Omega \int_t^s |(T_0(\theta)u_0(x))_\theta|^2 \, d\theta \, dx \\
&= |s-t| \int_t^s \|(T_0(\theta)u_0)_\theta\|_{H_0}^2 \, d\theta \\
&\leqslant |s-t| \|(T_0(\cdot)u_0)_t\|_{L^2(0,1;H_0)}^2 \\
&\leqslant |s-t| c_1^2,
\end{aligned}$$

where $c_1 = c_1(u_0)$. Besides, it follows from (4.39) that,

$$\|T_0(t)u_0 - T_0(t)v_0\|_{H_0} \leqslant c_2 \|u_0 - v_0\|_{H_0}.$$

So, returning to (4.48), we have that

$$\|T_0(s)u_0 - T_0(t)v_0\|_{H_0} \leqslant c_1|s-t|^{\frac{1}{2}} + c_2\|u_0 - v_0\|_{H_0} \tag{4.49}$$

for all $t, s \in [0,1]$ and $u_0, v_0 \in B_0$.

From Lemma 4.1, $T_0(\theta)u_0, T_0(\theta)v_0 \in B_0$, for all $\theta \in [0,1]$. From (4.49), we have

$$\begin{aligned}\|L(s)\chi_1 - L(t)\chi_2\|^2_{L^2(0,1;H_0)} &= \int_0^1 \|T_0(s)T_0(\theta)u_0 - T_0(t)T_0(\theta)v_0\|^2_{H_0}\, d\theta \\ &\leqslant \int_0^1 (c_1+c_2)^2(|s-t|^{\frac{1}{2}} + \|T_0(\theta)u_0 - T_0(\theta)v_0\|_{H_0})^2\, d\theta \\ &\leqslant 4(c_1+c_2)^2(|s-t| + \|\chi_1-\chi_2\|^2_{L^2(0,1;H_0)}).\end{aligned}$$

Therefore, for $c_3 = 2(c_1+c_2)$, we conclude the result for $\{L(t)\}_{t\geqslant 0}$. The estimative for case $\lambda \in (0,1]$, follows analogously. $\square$

Now, we state the main result of the work.

Theorem 4.1. *The dynamic system associated with* (2.9)*, has a global attractor $\mathcal{A}_0$. Furthermore, there exists a subset B of H_0, positively invariant, with $\mathcal{A}_0 \subset B$ so that the dynamic system $(T_0(t), B)$ admits an exponential attractor $\mathcal{E}_0$.*

For each $\lambda \in (0,1]$, the dynamic system associated with (2.8)*, has a global attractor $\mathcal{A}_\lambda$. Moreover, there exists a subset B_λ of H, positively invariant, with $\mathcal{A}_\lambda \subset B_\lambda$ so that the dynamic system $(T_\lambda(t), B_\lambda)$ admits an exponential attractor $\mathcal{E}_\lambda$.*

Proof. Let $u_0 \in H_0$ and $T > 0$, we have seen that (2.9) admits a unique $u \in C([0,T]; H_0)$ weak solution in $[0,T]$. Due to Lemma 3.4, we can apply Theorem 3.6 from [8], concluding, with this, that u is the only strong solution of (2.9). From the demonstration of Lemma 3.1, fixing values of $t_0 = k = 1$, we have

$$\|u(t)\|_{V_0} \leqslant r, \quad \forall t \geqslant 2.$$

That is, $\|u(t)\|_{p(x)} \leqslant r$ and $\|\nabla u(t)\|_{p(x)} \leqslant r$, for all $t \geqslant 2$. Therefore,

$$u(t) \in B_1 \subset \bigcup_{s\in[0,2]} T_0(s)B_1 = B_0, \quad \forall t \geqslant 2. \tag{4.50}$$

Lemma 4.1 and (4.50), guarantee Hypothesis (H2) of [1]. From Proposition 4.1, Hypothesis (H4), (H9) and (H10), from [1], are satisfied. From Lemmas 4.3 and 4.4, we have, respectively, that Hypothesis (H6) and (H8) from [1] are verified.

Moreover, from Hypothesis (H2) and (H4), it follows from Lemma 4.2, that $\mathcal{B}^0_\ell$ satisfies the hypothesis from Lemma 1.1 in [1]. Then, from the demonstration of Theorem 2.1, in [1], $(L(t), \mathfrak{X})$ admits a global attractor $\mathcal{A}_0$.

It follows from Theorem 2.5, in [1], that the dynamic system $(L(t), \mathcal{B}_0)$ admits an exponential attractor. Finally, Theorem 2.6, in [1], guarantee that $(T_0(t), \mathrm{e}(\mathcal{B}_0))$ has an exponential attractor in $\mathcal{E}_0$.

Proceeding in an analogous way, we conclude the existence of an exponential attractor for the dynamic system $(T_\lambda(t), \mathrm{e}_\lambda(\mathcal{B}^\lambda_0))$, where $\lambda \in (0,1]$. $\square$

Remark 4.1. *In the method of the $\ell-$trajectories, in* [1]*, we can consider the following alternative hypothesis for (H1) and (H5), respectively,*

- $(H1)'$ *For all $u_0 \in X$ and any $T > 0$, exists, not necessarily a unique solution u for* (1.6) *in $[0,T]$, with $u(0) = u_0$ and $u\colon [0,T] \to (X, \sigma(X,X'))$ continuous.*
- $(H5)'$ *$\mathcal{B}^\ell_0$ is compact in $X_\ell = L^2(0,\ell;X)$.*

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