Two regularity criteria of solutions to the liquid crystal flows

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Abstract

In this paper, we derive two regularity criteria of solutions to the nematic liquid crystal flows. More precisely, we prove the local smooth solution (u, d) is regular if and only if one of the following two conditions is satisfied: (i) $\nabla_h u_h \in L^{\frac{2p}{2p-3}}(0, T; L^p(\mathbb{R}^3))$, $\partial_3 d \in L^{\frac{2q}{q-3}}(0, T; L^q(\mathbb{R}^3))$, $\frac{3}{2} < p \leq \infty$, $3 < q \leq \infty$; and (ii) $\nabla_h u_h \in L^q(0, T; L^p(\mathbb{R}^3))$, $\frac{3}{p} + \frac{2}{q} \leq 1$, $3 < p \leq 4$.

Key words: liquid crystal flows, regularity criterion, partial components.

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1 Introduction

In this paper, the following Cauchy problem of nematic liquid crystal flows in $\mathbb{R}^3$ will be investigated

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = -\lambda \nabla \cdot (\nabla d \odot \nabla d), \\ \partial_t d + u \cdot \nabla d = \gamma(\Delta d - f(d)), \\ \nabla \cdot u = 0, \\ u(x, 0) = u_0(x), d(x, 0) = d_0(x), \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity field, $d = (d_1, d_2, d_3)$ is the macroscopic average of molecular orientation field and p represents the scalar pressure. The notation $\nabla d \odot \nabla d$ is a 3×3 matrix of which the $(i, j)th$ entry can be denoted by $\sum_{k=1}^{3} \partial_i d_k \partial_j d_k (1 \leq i, j \leq 3)$, and $f(d) = \frac{1}{\eta^2}(|d|^2 - 1)d$. μ, λ, γ, η are positive constants,whose specific value is not crucial, for simplicity, we take $\mu = \lambda = \gamma = \eta = 1$.

The above system (1.1) was first introduced by Lin [5] as a simplification to the Ericksen-Leslie equations which still retains most of the essential features of the hydrodynamic equations for nematic liquid crystal. The global existences of a weak solution and a local strong solution to (1.1) were already established by Lin [6]. However, the regularity of a solution is still a difficult

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problem. It is obvious that the system (1.1) reduces to the well-known Navier-Stokes equations when the orientation field $d \equiv 1$. Therefore, in view of the results of regularity criterion for the 3D Navier-Stokes equations, some Prodi-Serrin type regularity criteria of the nematic liquid crystal flows have been obtained based on the velocity field u, the gradient of the velocity field ∇u and their components, see [2, 3, 4, 7, 8, 9, 12, 13, 14, 15] and the references therein.

In [12], Wei, Li et al. established a regularity criterion for system (1.1). That is, if

$$\int_0^T \|u_3\|_{L^p}^q + \|\nabla d\|_{L^p}^q \mathrm{d}t < \infty, \ \ with \ \frac{3}{p} + \frac{2}{q} \leq \frac{3}{4} + \frac{1}{2p}, \ \frac{10}{3} \leq p. \tag{1.2}$$

then the solution (u, d) can be extended smoothly beyond T. Later in [16], Zhao, Wang et al. improved the above regularity criterion (1.2) to the following

$$\int_0^T \|u_h\|_{L^p}^q \mathrm{d}t < \infty, \ \ with \ \frac{3}{p} + \frac{2}{q} \leq \frac{1}{2}, \ 6 \leq p \leq \infty, \tag{1.3}$$

where $u_h = (u_1, u_2)$ is the horizontal component of velocity. And in [13], Yuan and Li generalized the regularity criterion (1.3) to

$$\int_0^T ||\nabla u_h||_{L^p}^q \mathrm{d}t < \infty, \ \ with \ \frac{3}{p} + \frac{2}{q} \leq \frac{3}{2}, \ 2 \leq p \leq \infty. \tag{1.4}$$

Motivated by the previous studies we mentioned above, we intend to reduce the condition on velocity components u_h in (1.4). Our main results are stated as follows:

Theorem 1.1. *Assume the initial data $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, and $d_0 \in H^2(\mathbb{R}^3)$. Let (u, d) be a smooth solution to the system (1.1) on $[0, T)$ for some $0 < T < \infty$. If $u_h = (u_1, u_2)$ and d satisfy*

$$\int_0^T (\|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}} + \|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}}) dt < \infty, \ \ with \ \frac{3}{2} < p \leq \infty, \ 3 < q \leq \infty, \tag{1.5}$$

here $\nabla_h = (\partial_1, \partial_2)$, then (u, d) can be extended beyond T.

Theorem 1.2. *Assume the initial data $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, and $d_0 \in H^2(\mathbb{R}^3)$. Let (u, d) be a smooth solution to the system (1.1) on $[0, T)$ for some $0 < T < \infty$. If horizontal velocity component u_h satisfies*

$$\int_0^T \|\nabla_h u_h\|_{L^p}^q dt < \infty, \ \ with \ \frac{3}{p} + \frac{2}{q} \leq 1, \ 3 < p \leq 4. \tag{1.6}$$

then the solution (u, d) is regular.

Remark 1.1. *Compared to (1.4), we reduce the derivative in vertical direction on the velocity components u_h in Theorem1.1, but the derivative in vertical direction on the orientation field d is needed. In Theorem1.2, we remove the condition that the derivative in vertical direction on d, which is an improvement.*

Remark 1.2. *In [10, Lemma 2.1], Penel and Pokorný considered the following inequality about divergence-free sufficiently smooth vector field u in $\mathbb{R}^3$ that is*

$$\|\nabla_h u_i\|_{L^p} \leq C(\|\partial_3 u_3\|_{L^p} + \|\omega_3\|_{L^p}), \ 1 < p < \infty, \ i = 1, 2,$$

and therefore the conditions (1.5) and (1.6) can be substituted by

$$\int_0^T (\|\partial_3 u_3\|_{L^p}^{\frac{2p}{2p-3}} + \|\omega_3\|_{L^p}^{\frac{2p}{2p-3}} + \|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}}) dt < \infty, \ \ with \ \frac{3}{2} < p \leq \infty, \ 3 < q \leq \infty. \tag{1.7}$$

and

$$\int_0^T (\|\partial_3 u_3\|_{L^p}^q + \|\omega_3\|_{L^p}^q)dt < \infty, \text{ with } \frac{3}{p} + \frac{2}{q} \le 1, 3 < p \le 4. \tag{1.8}$$

Compared to previous works, (1.7) and (1.8) are also improved and new results.

Throughout this paper, the letter C denotes a generic constant which may vary from line to line, and the directional derivatives of a function φ are denoted by $\partial_i \varphi = \frac{\partial \varphi}{\partial x_i}$ $(i = 1, 2, 3)$. In addition, we use $\|\cdot\|_{L^p}$ to represent the norm of a Lebesgue space L^p.

2 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. To this end we first give the following lemma which will be used in the proof of our theorems.

Lemma 2.1. *([1]) Let $f \in H^1(\mathbb{R}^3)$, then exists a constant C such that*

$$\|f\|_{L^p} \le C \|\partial_1 f\|_{L^{p_1}}^{\frac{1}{3}} \|\partial_2 f\|_{L^{p_2}}^{\frac{1}{3}} \|\partial_3 f\|_{L^{p_3}}^{\frac{1}{3}},$$

where $1 \le p_1, p_2, p_3 < \infty, 1 + \frac{3}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3}$.

It is known that (see, for example [6]) the system (1.1) has a local smooth solution for any initial datum $u_0 \in H^1(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, and $d_0 \in H^2(\mathbb{R}^3)$. Hence, in what follows we shall focus on a priori estimates for a local smooth solution. For the standard L^2 estimate of u and ∇d, by a direct estimate as we did in [13, p.2-3] one has

$$\begin{aligned}
(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + 2\int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \|d|\nabla d|\|_{L^2}^2 + \frac{1}{2}\|\nabla |d|^2\|_{L^2}^2)\mathrm{d}t \\
\le C(\|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2).
\end{aligned} \tag{2.1}$$

By the same process to the system (1.1) as we did in [13, Eq (2.7)], we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 \\
=& \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \mathrm{d}x + \int_{\mathbb{R}^3} \nabla \cdot (\nabla d \odot \nabla d) \cdot \Delta u \mathrm{d}x \\
&- \int_{\mathbb{R}^3} \Delta(u \cdot \nabla d) \cdot \Delta d \mathrm{d}x - \int_{\mathbb{R}^3} \Delta(|d|^2 d - d) \cdot \Delta d \mathrm{d}x \\
:=& I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.2}$$

In the following the above terms I_1, I_2, I_3, I_4 will be estimated one by one. For I_1, making use of the divergence free condition $\nabla \cdot u = 0$ and integrating by parts, we get

$$
\begin{aligned}
I_1 =& \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u \mathrm{d}x = \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} u_i \partial_i u_j \partial_k \partial_k u_j \mathrm{d}x \\
=& - \int_{\mathbb{R}^3} (\sum_{i,j,k=1}^{3} \partial_k u_i \partial_i u_j \partial_k u_j + \sum_{i,j,k=1}^{3} \frac{1}{2} u_i \partial_i (\partial_k u_j)^2) \mathrm{d}x \\
=& - \int_{\mathbb{R}^3} (\sum_{i,k=1}^{2} \sum_{j=1}^{3} \partial_k u_i \partial_i u_j \partial_k u_j + \sum_{i=1}^{2} \sum_{j=1}^{3} \partial_3 u_i \partial_i u_j \partial_3 u_j \\
&+ \sum_{k=1}^{2} \sum_{j=1}^{3} \partial_k u_3 \partial_3 u_j \partial_k u_j + \sum_{j=1}^{3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j) \mathrm{d}x \\
=& - \int_{\mathbb{R}^3} (\sum_{i,k=1}^{2} \sum_{j=1}^{3} \partial_k u_i \partial_i u_j \partial_k u_j + \sum_{i,j=1}^{2} \partial_3 u_i \partial_i u_j \partial_3 u_j + \sum_{i=1}^{2} \partial_3 u_i \partial_i u_3 \partial_3 u_3 \\
&+ \sum_{j,k=1}^{2} \partial_k u_3 \partial_3 u_j \partial_k u_j + \sum_{k=1}^{2} \partial_k u_3 \partial_3 u_3 \partial_k u_3 + \sum_{j=1}^{3} \partial_3 u_3 \partial_3 u_j \partial_3 u_j) \mathrm{d}x
\end{aligned}
$$

Noting that $\partial_3 u_3 = -(\partial_1 u_1 + \partial_2 u_2)$, I_1 can be bounded as follows by Hölder inequality and interpolation inequality

$$
\begin{aligned}
I_1 \quad \leq& \quad C \int_{\mathbb{R}^3} |\nabla_h u_h| |\nabla u|^2 \mathrm{d}x \\
\leq& \quad C \|\nabla_h u_h\|_{L^p} \|\nabla u\|^2_{L^{\frac{2p}{p-1}}} \\
\leq& \quad C \|\nabla_h u_h\|_{L^p} \|\nabla u\|_{L^2}^{\frac{2p-3}{p}} \|\Delta u\|_{L^2}^{\frac{3}{p}} \\
\leq& \quad C \|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}} \|\nabla u\|_{L^2}^2 + \frac{1}{4} \|\Delta u\|_{L^2}^2,
\end{aligned}
\tag{2.3}
$$

where $p > \frac{3}{2}$ implies $2 \leq \frac{2p}{p-1} < 6$. Summing up I_2 and I_3 , it follows by applying the incompressibility condition and integrating by parts

$$
\begin{aligned}
I_2 + I_3 =& \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} [(\partial_i \partial_j d_k \partial_j d_k + \partial_i d_k \partial_j \partial_j d_k) \Delta u_i - (\Delta u_i \partial_i d_k \Delta d_k \\
&+ 2 \partial_j u_i \partial_i \partial_j d_k \Delta d_k + u_i \partial_i \Delta d_k \Delta d_k)] \mathrm{d}x \\
=& \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} [\frac{1}{2} \partial_i (\partial_j d_k)^2 \Delta u_i - 2 \partial_j u_i \partial_i \partial_j d_k \Delta d_k - \frac{1}{2} u_i \partial_i (\Delta d_k)^2] \mathrm{d}x \\
=& - 2 \int_{\mathbb{R}^3} \sum_{i,j}^{3} \partial_j u_i \partial_i \partial_j d \Delta d \mathrm{d}x \\
=& - 2 \int_{\mathbb{R}^3} \sum_{i,j=1}^{2} \partial_j u_i \partial_i \partial_j d \Delta d \mathrm{d}x - 2 \int_{\mathbb{R}^3} \sum_{i=1}^{2} \partial_3 u_i \partial_i \partial_3 d \Delta d \mathrm{d}x \\
&- 2 \int_{\mathbb{R}^3} \sum_{j=1}^{2} \partial_j u_3 \partial_3 \partial_j d \Delta d \mathrm{d}x - 2 \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3 \partial_3 d \Delta d \mathrm{d}x \\
:=& I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}
$$

Similar to the estimate of (2.3), we have

$$
\begin{aligned}
I_{21} &\leq C\int_{\mathbb{R}^3}|\nabla_h u_h||\nabla\nabla d||\Delta d|\mathrm{d}x \\
&\leq C\|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}}\|\Delta d\|_{L^2}^2+\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2.
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
I_{24} &\leq C\int_{\mathbb{R}^3}|\nabla_h u_h||\Delta d||\Delta d|\mathrm{d}x \\
&\leq C\|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}}\|\Delta d\|_{L^2}^2+\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2.
\end{aligned}
\tag{2.5}
$$

For the term I_{22}, we deduce from the Hölder and Gagliardo-Nirenberg inequalities that

$$
\begin{aligned}
I_{22} =&2\int_{\mathbb{R}^3}\sum_{i=1}^{2}(\partial_3\partial_i u_i\partial_3 d\Delta d+\partial_3 u_i\partial_3 d\partial_i\Delta d)\mathrm{d}x \\
\leq& C\int_{\mathbb{R}^3}|\Delta u||\partial_3 d||\Delta d|+|\nabla u||\partial_3 d||\nabla\Delta d|\mathrm{d}x \\
\leq& C\|\partial_3 d\|_{L^q}\|\Delta d\|_{L^{\frac{2q}{q-2}}}\|\Delta u\|_{L^2}+C\|\partial_3 d\|_{L^q}\|\nabla u\|_{L^{\frac{2q}{q-2}}}\|\nabla\Delta d\|_{L^2} \\
\leq& C\|\partial_3 d\|_{L^q}\|\Delta d\|_{L^2}^{\frac{q-3}{q}}\|\nabla\Delta d\|_{L^2}^{\frac{3}{q}}\|\Delta u\|_{L^2}+C\|\partial_3 d\|_{L^q}\|\nabla u\|_{L^2}^{\frac{q-3}{q}}\|\Delta u\|_{L^2}^{\frac{3}{p}}\|\nabla\Delta d\|_{L^2} \\
\leq& C\|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}}\|\Delta d\|_{L^2}^2+C\|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}}\|\nabla u\|_{L^2}^2+\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2+\frac{1}{8}\|\Delta u\|_{L^2}^2,
\end{aligned}
\tag{2.6}
$$

where $q>3$. Using an argument as we did in the above process, we infer that

$$
\begin{aligned}
I_{23} =&2\int_{\mathbb{R}^3}\sum_{j=1}^{2}(\partial_j\partial_j u_3\partial_3 d\Delta d+\partial_j u_3\partial_3 d\partial_j\Delta d)\mathrm{d}x \\
\leq& C\int_{\mathbb{R}^3}(|\Delta u||\partial_3 d||\Delta d|+|\nabla u||\partial_3 d||\nabla\Delta d|)\mathrm{d}x \\
\leq& C\|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}}\|\Delta d\|_{L^2}^2+C\|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}}\|\nabla u\|_{L^2}^2+\frac{1}{16}\|\nabla\Delta d\|_{L^2}^2+\frac{1}{8}\|\Delta u\|_{L^2}^2.
\end{aligned}
\tag{2.7}
$$

Combining the above inequalities (2.4)-(2.7), we arrive at

$$
\begin{aligned}
I_2+I_3\leq& C(\|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}}+\|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}})(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2) \\
&+\frac{1}{4}\|\Delta u\|_{L^2}^2+\frac{1}{4}\|\nabla\Delta d\|_{L^2}^2.
\end{aligned}
\tag{2.8}
$$

For the last term I_4, one has

$$
\begin{aligned}
I_4\leq&\int_{\mathbb{R}^3}|\Delta d|^2+\Delta(|d|^2 d)\cdot\Delta d\mathrm{d}x \\
\leq&\|\Delta d\|_{L^2}^2+C(\|\Delta|d|^2\|_{L^2}\|d\|_{L^6}\|\Delta d\|_{L^3}+\|\Delta d\|_{L^3}\|d\|_{L^6}^2\|\Delta d\|_{L^3}) \\
\leq&\|\Delta d\|_{L^2}^2+C\|\Delta d\|_{L^3}\|d\|_{L^6}^2\|\Delta d\|_{L^3} \\
\leq&\|\Delta d\|_{L^2}^2+C\|\Delta d\|_{L^2}\|\nabla\Delta d\|_{L^2} \\
\leq& C\|\Delta d\|_{L^2}^2+\frac{1}{4}\|\nabla\Delta d\|_{L^2}^2.
\end{aligned}
\tag{2.9}
$$

Inserting the estimates (2.3), (2.8) and (2.9) into (2.2) it yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2)+\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2\\ \leq& C(1+\|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}}+\|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}})(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2).\end{aligned}$$

By the Gronwall inequality it implies that

$$\begin{aligned}&\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\int_0^T(\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2)\mathrm{d}t\\ \leq& C(\|\nabla u_0\|_{L^2}^2+\|\Delta d_0\|_{L^2}^2)e^{\int_0^T C(1+\|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}}+\|\partial_3 d\|_{L^q}^{\frac{2q}{q-3}})\mathrm{d}t}.\end{aligned}$$

Thus the proof of Theorem 1.1 is completed. □

3 Proof of Theorem 1.2

In this section, we prove the Theorem 1.2. The method is that $\|\nabla u\|_{L^2}^2$ and $\|\Delta d\|_{L^2}^2$ will be split into $\|\partial_3 u\|_{L^2}^2, \|\nabla_h u\|_{L^2}^2$ and $\|\nabla\partial_3 d\|_{L^2}^2, \|\nabla\nabla_h d\|_{L^2}^2$, which will be estimated separately. Firstly, applying ∂_3 to $(1.1)_1$ and Δ to $(1.1)_2$, taking inner products with $\partial_3 u$ and $\partial_3\partial_3 d$ respectively, adding them up and integrating by parts, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_3 u\|_{L^2}^2+\|\partial_3\nabla d\|_{L^2}^2)+\|\partial_3\nabla u\|_{L^2}^2+\|\partial_3\Delta d\|_{L^2}^2\\
=&-\int_{\mathbb{R}^3}\partial_3(u\cdot\nabla)u\cdot\partial_3 u\mathrm{d}x-\int_{\mathbb{R}^3}\partial_3\nabla\cdot(\nabla d\odot\nabla d)\cdot\partial_3 u\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\partial_3\partial_3 d\mathrm{d}x-\int_{\mathbb{R}^3}\Delta(|d|^2 d-d)\cdot\partial_3\partial_3 d\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}\sum_{i,j=1}^3\partial_3 u_j\partial_j u_i\partial_3 u_i\mathrm{d}x-\int_{\mathbb{R}^3}\sum_{i,j=1}^3\partial_3 u_i(\partial_i\partial_3 d\partial_j\partial_j d+\partial_i d\partial_3\partial_j\partial_j d)\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\sum_{i,j=1}^3\partial_3\Delta d(\partial_3 u_i\partial_i d+u_i\partial_i\partial_3 d)\mathrm{d}x-\int_{\mathbb{R}^3}\Delta(|d|^2 d-d)\cdot\partial_3\partial_3 d\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}\sum_{i,j=1}^3\partial_3 u_j\partial_j u_i\partial_3 u_i\mathrm{d}x-\int_{\mathbb{R}^3}\sum_{i,j=1}^3\partial_3 u_i\partial_i\partial_3 d\partial_j\partial_j d\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\sum_{i,j=1}^3\partial_3\Delta d u_i\partial_i\partial_3 d\mathrm{d}x-\int_{\mathbb{R}^3}\Delta(|d|^2 d-d)\cdot\partial_3\partial_3 d\mathrm{d}x\\
:=&J_1+J_2+J_3+J_4.\end{aligned}\tag{3.1}$$

To bound J_1, decomposing J_1 into several terms, and making use of Hölder inequality and Gagliardo-Nirenberg inequality, we have

$$\begin{aligned}J_1=&\int_{\mathbb{R}^3}\sum_{i=1}^3\sum_{j=1}^2(\partial_3 u_j\partial_j u_i\partial_3 u_i+\partial_3 u_3\partial_3 u_i\partial_3 u_i)\mathrm{d}x\\
=&\int_{\mathbb{R}^3}[\sum_{i,j=1}^2(\partial_3 u_j\partial_j u_i\partial_3 u_i+\partial_3 u_j\partial_j u_3\partial_3 u_3)+\sum_{i=1}^3\partial_3 u_3\partial_3 u_i\partial_3 u_i]\mathrm{d}x,\end{aligned}$$

$$
\begin{aligned}
J_1 \leq & C\int_{\mathbb{R}^3} |\nabla_h u_h||\partial_3 u||\partial_3 u|\mathrm{d}x + \int_{\mathbb{R}^3} |\nabla_h u_h||\nabla u||\partial_3 u|\mathrm{d}x \\
\leq & C\|\nabla_h u_h\|_{L^p}\|\partial_3 u\|^2_{L^{\frac{2p}{p-1}}} + C\|\nabla_h u_h\|_{L^p}\|\nabla u\|_{L^2}\|\partial_3 u\|_{L^{\frac{2p}{p-2}}} \\
\leq & C\|\nabla_h u_h\|_{L^p}\|\partial_3 u\|^{\frac{2p-3}{p}}_{L^2}\|\partial_3\nabla u\|^{\frac{3}{p}}_{L^2} + C\|\nabla_h u_h\|_{L^p}\|\nabla u\|_{L^2}\|\partial_3 u\|^{\frac{p-3}{p}}_{L^2}\|\partial_3\nabla u\|^{\frac{3}{p}}_{L^2} \\
\leq & C\|\nabla_h u_h\|^{\frac{2p}{2p-3}}_{L^p}\|\partial_3 u\|^2_{L^2} + C\|\nabla_h u_h\|^{\frac{2p}{p-3}}_{L^p}\|\partial_3 u\|^2_{L^2} + C\|\nabla u\|^2_{L^2} + \frac{1}{4}\|\partial_3\nabla u\|^2_{L^2},
\end{aligned} \tag{3.2}
$$

where $p > 3$ makes $2 \leq \frac{2p}{p-2} < 6$ valid. Utilizing integration by parts, J_2 can be rewritten as follows

$$
\begin{aligned}
J_2 = & \int_{\mathbb{R}^3}(-\sum_{i=1}^{2}\sum_{j=1}^{3}\partial_3 u_i\partial_i\partial_3 d\partial_j\partial_j d - \sum_{j=1}^{3}\partial_3 u_3\partial_3\partial_3 d\partial_j\partial_j d)\mathrm{d}x \\
= & \int_{\mathbb{R}^3}[\sum_{i=1}^{2}\sum_{j=1}^{3}(u_i\partial_i\partial_3\partial_3 d\partial_j\partial_j d + u_i\partial_i\partial_3 d\partial_3\partial_j\partial_j d) - \sum_{j=1}^{3}\partial_3 u_3\partial_3\partial_3 d\partial_j\partial_j d]\mathrm{d}x \\
= & \int_{\mathbb{R}^3}[(\sum_{i=1}^{2}\sum_{j=1}^{2}u_i\partial_i\partial_3\partial_3 d\partial_j\partial_j d + \sum_{i=1}^{2}\sum_{j=1}^{3}u_i\partial_i\partial_3 d\partial_3\partial_3\partial_3 d) - \sum_{j=1}^{2}\partial_3 u_3\partial_3\partial_3 d\partial_j\partial_j d]\mathrm{d}x \\
:= & J_{21} + J_{22}.
\end{aligned} \tag{3.3}
$$

For J_{21}, applying Lemma 2.1 with $p_1 = p_2, p_3 = 2$ and Young's inequality, we have

$$
\begin{aligned}
J_{21} \leq & C\|u_h\|_{L^p}\|\nabla\nabla_h d\|_{L^{\frac{2p}{p-2}}}\|\partial_3\Delta d\|_{L^2} + C\|u_h\|_{L^p}\|\partial_3\nabla d\|_{L^{\frac{2p}{p-2}}}\|\partial_3\Delta d\|_{L^2} \\
\leq & C\|\nabla_h u_h\|^{\frac{2}{3}}_{L^{\frac{4p}{6+p}}}\|\partial_3 u_h\|^{\frac{1}{3}}_{L^2}(\|\nabla\nabla_h d\|^{\frac{p-3}{p}}_{L^2}\|\Delta\nabla_h d\|^{\frac{3}{p}}_{L^2} + \|\partial_3\nabla d\|^{\frac{p-3}{p}}_{L^2}\|\partial_3\Delta d\|^{\frac{3}{p}}_{L^2})\|\partial_3\Delta d\|_{L^2} \\
\leq & \frac{1}{8}(\|\partial_3\Delta d\|^2_{L^2} + \|\Delta\nabla_h d\|^2_{L^2}) + C(\|\nabla u\|^2_{L^2} + \|\nabla_h u_h\|^{\frac{4p}{2p-9}}_{L^{\frac{4p}{p+6}}})(\|\nabla\nabla_h d\|^2_{L^2} + \|\partial_3\nabla d\|^2_{L^2}).
\end{aligned} \tag{3.4}
$$

By Hölder and Gagliardo-Nirenberg inequalities, J_{22} can be estimated as

$$
\begin{aligned}
J_{22} \leq & C\|\nabla_h u_h\|_{L^p}\|\Delta d\|_{L^2}\|\partial_3\nabla d\|_{L^{\frac{2p}{p-2}}} \\
\leq & C\|\nabla_h u_h\|_{L^p}\|\Delta d\|_{L^2}\|\partial_3\nabla d\|^{\frac{p-3}{p}}_{L^2}\|\partial_3\Delta d\|^{\frac{3}{p}}_{L^2} \\
\leq & \frac{1}{16}\|\partial_3\Delta d\|^2_{L^2} + C\|\Delta d\|^2_{L^2} + C\|\nabla_h u_h\|^{\frac{2p}{p-3}}_{L^p}\|\partial_3\nabla d\|^2_{L^2}.
\end{aligned} \tag{3.5}
$$

Similarly,

$$
\begin{aligned}
J_3 = & -\int_{\mathbb{R}^3}\sum_{i=1}^{2}\sum_{j=1}^{3}u_i\partial_i\partial_3 d\partial_3\partial_j\partial_j d + \sum_{j=1}^{3}u_3\partial_3\partial_3 d\partial_3\partial_j\partial_j d\mathrm{d}x \\
\leq & C\|u_h\|_{L^p}\|\partial_3\nabla d\|_{L^{\frac{2p}{p-2}}}\|\partial_3\Delta d\|_{L^2} + C\|u_3\|_{L^{3q}}\|\partial_3\nabla d\|_{L^{\frac{6q}{3q-2}}}\|\partial_3\Delta d\|_{L^2} \\
\leq & C\|\nabla_h u_h\|^{\frac{2}{3}}_{L^{\frac{4p}{6+p}}}\|\partial_3 u_h\|^{\frac{1}{3}}_{L^2}\|\partial_3\nabla d\|^{\frac{p-3}{p}}_{L^2}\|\partial_3\Delta d\|^{\frac{3}{p}}_{L^2}\|\partial_3\Delta d\|_{L^2} \\
& + C\|\nabla u\|^{\frac{2}{3}}_{L^2}\|\partial_3 u_3\|^{\frac{1}{3}}_{L^q}\|\partial_3\nabla d\|^{1-\frac{1}{q}}_{L^2}\|\partial_3\Delta d\|^{1+\frac{1}{q}}_{L^2} \\
\leq & \frac{1}{8}\|\partial_3\Delta d\|^2_{L^2} + C\|\nabla_h u_h\|^{\frac{4p}{3(p-3)}}_{L^{\frac{4p}{p+6}}}\|\partial_3 u_h\|^{\frac{2p}{3(p-3)}}_{L^2}\|\partial_3\nabla d\|^2_{L^2} \\
& + C\|\nabla u\|^{\frac{4q}{3(q-1)}}_{L^2}\|\nabla_h u_h\|^{\frac{2q}{3(q-1)}}_{L^q}\|\partial_3\nabla d\|^2_{L^2} \\
\leq & \frac{1}{8}\|\partial_3\Delta d\|^2_{L^2} + C(\|\nabla u\|^2_{L^2} + \|\nabla_h u_h\|^{\frac{4p}{2p-9}}_{L^{\frac{4p}{p+6}}} + \|\nabla_h u_h\|^{\frac{2q}{q-3}}_{L^q})\|\partial_3\nabla d\|^2_{L^2}.
\end{aligned} \tag{3.6}
$$

where $q>3$ guarantees $1<\frac{4q}{3(q-1)}<2$. And here we have used Lemma 2.1 with $p_1=p_2=2,p=3q$, which is

$$\|u_3\|_{L^{3q}}\le C\|\nabla u\|_{L^2}^{\frac{2}{3}}\|\partial_3 u_3\|_{L^q}^{\frac{1}{3}}.$$

For J_4, we have

$$\begin{aligned}
J_4=&\int_{\mathbb{R}^3}\partial_3(|d|^2d-d)\cdot\partial_3\Delta d\mathrm{d}x\\
\le& C\int_{\mathbb{R}^3}(|\partial_3 d||d|^2|\partial_3\Delta d|+|\partial_3 d||\partial_3\Delta d|)\mathrm{d}x\\
\le& C\|\partial_3 d\|_{L^6}\|d\|_{L^6}^2\|\partial_3\Delta d\|_{L^2}+\|\partial_3 d\|_{L^2}\|\partial_3\Delta d\|_{L^2}\\
\le& C\|\partial_3\nabla d\|_{L^2}^2+\frac{1}{16}\|\partial_3\Delta d\|_{L^2}+C\|\nabla d\|_{L^2}^2.
\end{aligned}\tag{3.7}$$

Combining (3.1)-(3.7) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_3 u\|_{L^2}^2+\|\partial_3\nabla d\|_{L^2}^2)+\frac{5}{4}\|\partial_3\nabla u\|_{L^2}^2+\frac{5}{4}\|\partial_3\Delta d\|_{L^2}^2\\
\le&\frac{1}{8}\|\Delta\nabla_h d\|_{L^2}^2+C(1+\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\|\nabla_h u_h\|_{L^{\frac{4p}{p+6}}}^{\frac{4p}{2p-9}}+\|\nabla_h u_h\|_{L^q}^{\frac{2q}{q-3}})\\
&\times(1+\|\partial_3\nabla d\|_{L^2}^2+\|\nabla\nabla_h d\|_{L^2}^2+\|\partial_3 u\|_{L^2}^2).
\end{aligned}\tag{3.8}$$

Next, applying ∇_h to $(1.1)_1$, Δ to $(1.1)_2$, and taking inner products with $\nabla_h u$ and $\Delta_h d$ respectively, then adding them up and integrating by parts, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla_h u\|_{L^2}^2+\|\nabla\nabla_h d\|_{L^2}^2)+\|\nabla\nabla_h u\|_{L^2}^2+\|\Delta\nabla_h d\|_{L^2}^2\\
=&-\int_{\mathbb{R}^3}\nabla_h(u\cdot\nabla)u\cdot\nabla_h u\mathrm{d}x-\int_{\mathbb{R}^3}\nabla_h\nabla\cdot(\nabla d\odot\nabla d)\cdot\nabla_h u\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\Delta_h d\mathrm{d}x-\int_{\mathbb{R}^3}\Delta(|d|^2d-d)\cdot\Delta_h d\mathrm{d}x\\
=&-\int_{\mathbb{R}^3}\sum_{i,j,k=1}^{2}(\partial_k u_i\partial_i u_j\partial_k u_j+\partial_k u_3\partial_3 u_j\partial_k u_j+\partial_k u_i\partial_i u_3\partial_k u_3+\partial_k u_3\partial_3 u_3\partial_k u_3)\mathrm{d}x\\
&+\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\sum_{k=1}^{2}u_i(\partial_i\partial_k\partial_k d\partial_j\partial_j d+\partial_i d\partial_k\partial_k\partial_j\partial_j d+2\partial_i\partial_k d\partial_k\partial_j\partial_j d)\mathrm{d}x\\
&-\int_{\mathbb{R}^3}\sum_{i,j=1}^{3}\sum_{k=1}^{2}u_i\partial_i d\partial_k\partial_k\partial_j\partial_j d\mathrm{d}x+\int_{\mathbb{R}^3}\nabla(|d|^2d-d)\cdot\Delta_h\nabla d\mathrm{d}x\\
:=&K_1+K_2+K_3+K_4.
\end{aligned}\tag{3.9}$$

For K_1, employing Hölder and Gagliardo-Nirenberg inequalities yields

$$\begin{aligned}
K_1\le& C\|\nabla_h u_h\|_{L^p}\|\nabla_h u\|_{L^{\frac{2p}{p-1}}}^2+C\|\nabla_h u_h\|_{L^p}\|\partial_3 u\|_{L^{\frac{2p}{p-2}}}\|\nabla_h u\|_{L^2}\\
\le& C\|\nabla_h u_h\|_{L^p}\|\nabla_h u\|_{L^2}^{\frac{2p-3}{p}}\|\nabla\nabla_h u\|_{L^2}^{\frac{3}{p}}+C\|\nabla_h u_h\|_{L^p}\|\partial_3 u\|_{L^2}^{\frac{p-3}{p}}\|\partial_3\nabla u\|_{L^2}^{\frac{3}{p}}\|\nabla_h u\|_{L^2}\\
\le& C\|\nabla_h u_h\|_{L^p}^{\frac{2p}{2p-3}}\|\nabla_h u\|_{L^2}^2+C\|\nabla_h u_h\|_{L^p}^{\frac{2p}{p-3}}\|\partial_3 u\|_{L^2}^2+C\|\nabla_h u\|_{L^2}^2\\
&+\frac{1}{8}(\|\nabla\nabla_h u\|_{L^2}^2+\|\partial_3\nabla u\|_{L^2}^2),
\end{aligned}\tag{3.10}$$

Because K_2, K_3 contain some terms that can be canceled, combining them and decomposing into several terms, we have

$$
\begin{aligned}
K_2 + K_3 =& - \int_{\mathbb{R}^3} \sum_{i,j=1}^{3} \sum_{k=1}^{2} u_i(\partial_i\partial_k\partial_k d \partial_j\partial_j d + 2\partial_i\partial_k d\partial_k\partial_j\partial_j d)\mathrm{d}x \\
=& - \int_{\mathbb{R}^3} [\sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{2}(u_i\partial_i\partial_k\partial_k d\partial_j\partial_j d + u_i\partial_i\partial_k\partial_k d\partial_3\partial_3 d) + \sum_{i,j=1}^{3}\sum_{k=1}^{2}(2u_i\partial_i\partial_k d\partial_k\partial_j\partial_j d)]\mathrm{d}x. \\
=& - \int_{\mathbb{R}^3} [\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(u_i\partial_i\partial_k\partial_k d\partial_j\partial_j d + u_3\partial_3\partial_k\partial_k d\partial_j\partial_j d + u_i\partial_i\partial_k\partial_k d\partial_3\partial_3 d \\
& + u_3\partial_3\partial_k\partial_k d\partial_3\partial_3 d) + \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} 2(u_i\partial_i\partial_k d\partial_k\partial_j\partial_j d + u_3\partial_3\partial_k d\partial_k\partial_j\partial_j d)]\mathrm{d}x.
\end{aligned}
$$

Similar to J_2, J_3, we can deduce from the above equation that

$$
\begin{aligned}
K_2 + K_3 \le& C\|u_h\|_{L^p}(\|\nabla\nabla_h d\|_{L^{\frac{2p}{p-2}}} + \|\nabla\partial_3 d\|_{L^{\frac{2p}{p-2}}})\|\Delta\nabla_h d\|_{L^2} \\
&+ C\|u_3\|_{L^{3q}}(\|\nabla\nabla_h d\|_{L^{\frac{6q}{3q-2}}} + \|\nabla\partial_3 d\|_{L^{\frac{6q}{3q-2}}})\|\Delta\nabla_h d\|_{L^2} \\
\le& \frac{1}{8}(\|\Delta\nabla_h d\|_{L^2}^2 + \|\Delta\partial_3 d\|_{L^2}^2) + C(\|\nabla u\|_{L^2}^2 + \|\nabla_h u_h\|_{L^{\frac{4p}{p+6}}}^{\frac{4p}{2p-9}} + \|\nabla_h u_h\|_{L^q}^{\frac{2q}{q-3}}) \\
&\times (\|\nabla\nabla_h d\|_{L^2}^2 + \|\partial_3\nabla d\|_{L^2}^2).
\end{aligned}
\tag{3.11}
$$

For last term K_4, we have

$$
\begin{aligned}
K_4 \le& C\int_{\mathbb{R}^3}(|\nabla_h d||d|^2|\Delta\nabla_h d| + |\nabla_h d||\Delta\nabla_h d|)\mathrm{d}x \\
\le& C\|\nabla d\|_{L^6}\|d\|_{L^6}^2\|\Delta\nabla_h d\|_{L^2} + \|\nabla d\|_{L^2}\|\Delta\nabla_h d\|_{L^2} \\
\le& C\|\Delta d\|_{L^2}^2 + \frac{1}{8}\|\Delta\nabla_h d\|_{L^2} + C\|\nabla d\|_{L^2}^2.
\end{aligned}
\tag{3.12}
$$

Collecting (3.8)-(3.12), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|\partial_3 u\|_{L^2}^2 + \|\nabla_h u\|_{L^2}^2 + \|\partial_3\nabla d\|_{L^2}^2 + \|\nabla\nabla_h d\|_{L^2}^2) + \|\partial_3\nabla u\|_{L^2}^2 + \|\nabla\nabla_h u\|_{L^2}^2 + \|\partial_3\Delta d\|_{L^2}^2 \\
&\quad + \|\Delta\nabla_h d\|_{L^2}^2 \le C(1 + \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \|\nabla_h u_h\|_{L^\alpha}^{\frac{8\alpha}{7\alpha-12}} + \|\nabla_h u_h\|_{L^q}^{\frac{2q}{q-3}}) \times (1 + \|\partial_3 u\|_{L^2}^2 \\
&\quad + \|\nabla_h u\|_{L^2}^2 + \|\partial_3\nabla d\|_{L^2}^2 + \|\nabla\nabla_h d\|_{L^2}^2),
\end{aligned}
\tag{3.13}
$$

where $\frac{12}{7} \le \alpha = \frac{4p}{p+6} \le 4$. Combining the condition 1.6, the above inequality indicates that by Gronwall inequality,

$$
\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \int_0^T (\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2)\mathrm{d}t < C.
$$

Thus we complete the proof of Theorem 1.2. □

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