Received: Added at production | Revised: Added at production | Accepted: Added at production
DOI: xxx/xxxx

ARTICLE TYPE

CMMSE: Estimates of singular numbers (s-numbers) and eigenvalues of a mixed elliptic-hyperbolic type operator with parabolic degeneration†

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Summary

This paper is concerned with a mixed type differential operator

$$Lu = k(y)\,u_{xx} - u_{yy} + b(y)\,u_x + q(y)\,u,$$

which is initially defined with $C^{\infty}_{0,\pi}\left(\overline{\Omega}\right)$, where $\overline{\Omega} = \{(x, y) : -\pi \le x \le \pi, \ -\infty < y < \infty\}$, $C^{\infty}_{0,\pi}$ is a set of infinitely differentiable functions with compact support with respect to the variable y and satisfying the conditions:

$$u_x^{(i)}(-\pi, \ y) = u_x^{(i)}(\pi, \ y) \qquad i = 0, \ 1.$$

Regarding the coefficient $k(y)$, with supposition that $k(y)$ satisfies the condition:
a) $|k(y)| \ge 0$ is a piecewise continuous and bounded function in $\mathbb{R} = (-\infty, \infty)$.
The coefficients $b(y)$ and $q(y)$ are continuous functions in $\mathbb{R}$ and can be unbounded at infinity.
The operator L admits closure in the space $L_2(\Omega)$ and the closure is also denoted by L.
Taking into consideration certain constraints on the coefficients $b(y)$ $q(y)$, apart from the above-mentioned conditions, the existence of a bounded inverse operator is proved in this paper; a condition guaranteeing compactness of the resolvent kernel is found; and we also obtained two-sided estimates for singular numbers (s-numbers). Here we note that the estimate of singular numbers (s-numbers) shows the rate of approximation of the resolvent of the operator L by linear finite-dimensional operators. It is given an example of how the obtained estimates for the s-numbers enable to identify the estimates for the eigenvalues of the operator L. We note that the above results are apparently obtained for the first time for a mixed-type operator in the case of an unbounded domain with rapidly oscillating and greatly growing coefficients at infinity.

KEYWORDS:
singular numbers (s-numbers), eigenvalues, compactness, resolvent, separability, mixed type operator.
AMS: 34B24, 34L05, 35M10, 47A10

1 | INTRODUCTION. STATEMENT OF RESULTS

Consider the differential operator

$$(L+\mu I)u = k(y)\,u_{xx} - u_{yy} + b(y)\,u_x + q(y)\,u + \mu u, \tag{1}$$

which is initially defined with $C^{\infty}_{0,\pi}\left(\overline{\Omega}\right)$, where $\overline{\Omega} = \{(x,y) : -\pi \le x \le \pi,\ -\infty < y < \infty\}$, $C^{\infty}_{0,\pi}$ is a set of infinitely differentiable functions with compact support with respect to the variable y and satisfying the conditions:

$$u_x^{(i)}(-\pi,\ y) = u_x^{(i)}(\pi,\ y) \qquad i = 0,\ 1.$$

Regarding the coefficients $b(y)$ and $q(y)$, they are presumed as continuous functions.

In the sequel, it is assumed that the coefficients $k(y)$, $b(y)$, $q(y)$ satisfy the conditions:

a) $|k(y)| \ge 0$ is a piecewise continuous, bounded function in $\mathbb{R} = (-\infty, \infty)$;

i) $|b(y)| \ge \delta_0 > 0,\ q(y) \ge \delta > 0$ are continuous functions in $\mathbb{R}$.

It is worth noting that functions $b(y)$ and $q(y)$ can be unbounded at infinity.

It is easily ascertainable that, depending on the signs of taken functions $k(y)$ in $\mathbb{R}$, this operator L pertains to different types. In this regard, it is to be recalled that the mixed type elliptic-hyperbolic operators with parabolic degeneration are differential operators, that either belongs to the elliptic type as one part of the considered domain or pertains to the hyperbolic type in the other part of the domain. These parts are separated by a line of transition on which the operator degenerates into the parabolic type [1-2] [1,2].

In the case of a bounded domain, depending on the boundary conditions and the geometry of the domain, the spectral properties of operators of mixed type were studied in [3-11] [3,4,5,6,7,8,9,10,11] and the papers cited there.

However, in applications one often has to deal with such cases when a mixed type operator is given in an unbounded domain with rapidly oscillating and greatly growing coefficients at infinity.

For example, an operator of the form

$$Lu = \sin e^{10|y|}u_{xx} - u_{yy} + e^{100\,|y|}u_x + e^{100\,|y|}u, u \in D(L)$$

where $-\infty < y < \infty,\ \ -\infty < x < \infty$.

Here you can see that the function $k(y) = \sin e^{10\,|y|}$ oscillates rapidly at infinity when $|y| \to \infty$ and the operator often changes its type. Hence it follows that at the points where the operator changes its type, the condition of uniform ellipticity and hyperbolicity is violated. Thus, functions from the domain of the operator do not preserve their smoothness. Consequently, in this case, various difficulties arise associated with the behavior of functions from the domain of the operator, and these difficulties, in turn, affect the spectral characteristics of the operator of mixed type. It should also be noted here that the estimates of the eigenvalues are influenced by the growth and oscillation of the coefficients $b(y)$ and $q(y)$ of the operator (1).

In this paper, we are interested in the following questions for the mixed type operator (1) with rapidly oscillating and greatly growing coefficients:

- the existence of the resolvent;
- the existence of the estimate

$$\left\|k(y)\,u_{xx} - u_{yy}\right\|_2 + \left\|b(y)\,u_x\right\|_2 + \|q(y)\,u\|_2 \le c\left(\|Lu\|_2 + \|u\|_2\right) \tag{2}$$

for all $u \in D(L)$, where $D(L)$ is the domain of the operator L, $\|\cdot\|_2$ is the norm in $L_2(\Omega)$, $c > 0$ is a constant;

- singular numbers (s-numbers) estimates;
- eigenvalues estimates.

It is not difficult to verify that under condition *a*) and *i*) the operator $L + \mu\, I$ admits closure and the closure is also denoted by $L + \mu\, I$, $\mu\ \ge 0$.

Following the works [12-13] [12,13], we introduce the following definition.

Definition 1.1. *We say that the operator L of mixed type is separable if estimate* (2) *holds for all* $u \in D(L)$.

Here are the formulations of the main results.

[†]This work was supported by grants AP08856339 and AP08855802 of the Ministry of Education and Science of the Republic of Kazakhstan.

Theorem 1.1. Let the conditions *a*) and *i*) be fulfilled. Then the operator $L+\mu I$ is continuously invertible in the space $L_2(\Omega)$ for $\mu \geq 0$ and the equality

$$u(x,y)=(L+\mu I)^{-1} f=\sum_{n=-\infty}^{\infty}\left(l_n+\mu I\right)^{-1} f_n(y)\cdot e^{inx}, \tag{3}$$

holds, where $f(x,y)\in L_2(\Omega)$, $f(x,y)=\sum_{n=-\infty}^{\infty} f_n(y)\cdot e^{inx}$, $f_n(y)=< f(x,y),\ e^{inx} >$, $i^2=-1$, $<\cdot\,,\ \cdot>$ is the scalar product in $L_2(\Omega)$,

$$\left(l_n+\mu\ I\right)u=-u''(y)+\left(-n^2k(y)+inb(y)+q(y)+\mu\right)u(y),\quad u\in D(l_n)$$

$D(l_n)$ is the domains of the operators l_n, $n=0,\pm1,\pm2,\ldots$.

Suppose that the coefficients $b(y)$, $q(y)$, in addition to conditions *a*)-*i*), satisfy the conditions

ii) $\mu_0=\sup\limits_{|y-t|\leq 1}\frac{b(y)}{b(t)}<\infty;\quad \mu_1=\sup\limits_{|y-t|\leq 1}\frac{q(y)}{q(t)}<\infty;$

iii) $q(y)\leq C_0\cdot b^2(y)$, for $y\in\mathbb{R}$, $C_0>0$ is a constant.

Theorem 1.2. Let the conditions *a*) and *i*) − *iii*) be fulfilled. Then the operator $L+\mu I$ is separable for $\mu\geq 0$.

Theorem 1.3. Let the conditions *a*) and *i*) − *iii*) be fulfilled. Then the resolvent of the operator L is compact if and only if

$$\lim_{|y|\to\infty} q(y)=\infty.$$

Definition 1.2.[14] *Let A be a linear completely continuous operator and let* $|A|=\sqrt{A^*\cdot A}$, *where* A^* *is the adjoint operator to A. The eigenvalues of the operator* $|A|$ *are called s-numbers of the operator A.*

The nonzero s-numbers of the operator $(L+\mu I)^{-1}$ be numbered according to decreasing magnitude and observing their multiplicities, so that[14]

$$s_k(L+\mu\ I)^{-1}=\lambda_k\left[\left((L+\mu\ I)^{-1}\right)^*(L+\mu\ I)^{-1}\right]^{\frac{1}{2}},\quad k=1,\,2,\ldots$$

We introduce the counting function $N(\lambda)=\sum_{s_k>\lambda}1$ of those s_k greater than $\lambda>0$.

Theorem 1.4. Let the conditions of Theorem 1.3 be satisfied. Then the estimate

$$c^{-1}\sum_{n=-\infty}^{\infty}\lambda^{-\frac{1}{2}}\,mes\left(y\in\mathbb{R}:Q_n(y)\leq c^{-1}\lambda^{-1}\right)\leq N(\lambda)\leq c\sum_{n=-\infty}^{\infty}\lambda^{-1}mes\left(y\in\mathbb{R}:K_n(y)\leq c\lambda^{-1}\right)$$

holds for $N(\lambda)$, where $Q_n(y)=\left|(k(y)+\varepsilon)n^2+in\,b(y)+c(y)\right|$, $K_n(y)=|n\cdot\ b(y)|+q(y)u$, $\varepsilon>0$ is such a number that the inequality $k(y)+\varepsilon>\varepsilon_0>0$ holds.

Example 1. Consider the operator

$$(L+\mu\ I)u=\sin e^{10|y|}u_{xx}-u_{yy}+e^{100|y|}u_x+e^{100|y|}u+\mu u,$$

$$u\in D(L),\mu\geq 0.$$

It is easy to verify that all conditions of Theorems 1.1 and 1.2 are satisfied. Therefore, the operator $L+\mu I$ is continuously invertible in $L_2(\Omega)$ and separable, i.e. the estimate

$$\left\|\sin e^{10|y|}\,u_{xx}-u_{yy}\right\|_2+\left\|e^{100|y|}\,u_x\right\|_2+\left\|e^{100|y|}\,u\right\|\leq c\left(\|Lu\|_2+\|u\|_2\right),$$

holds, where $c>0$ is a constant, $\|\cdot\|_2$ is the norm in $L_2(\Omega)$.

Example 2. Now, we show how we can use Theorem 1.4 to find estimates for the eigenvalues. As an example, for simplicity of computation, consider the operator

$$(L+\mu\ I)u=\sin e^{10|y|}u_{xx}-u_{yy}+(|y|+1)\,u_x+(|y|+1)\,u+\mu u,$$

$$u\in D(L),\mu\geq 0.$$

Theorem 1.1 implies that if s is a singular point of an operator, then s is a singular number of one of the operators $\left(l_n+\mu I\right)^{-1}$ $(n=0,\pm1,\pm2,\ldots)$, and vice versa. Further, we denote by $s_{k,n}$ the singular numbers of the operator $\left(l_n+\mu I\right)^{-1}$ $(n=0,\pm1,\pm2,\ldots)$ when $\mu\geq 0$. Therefore, taking into account the last statement, according to Theorem 1.1 and Lemma 4.5, we find

$$\frac{c^{-1}}{(K(y)+\varepsilon)^{2/3}(|n|+1)^{4/3}k^{2/3}}\leq s_{k,n}\leq\frac{c}{(|n|+1)^{1/2}k^{1/2}},k=1,\ 2,\ \ldots,\quad n=0,\ \pm1,\ \pm2,\ldots \tag{4}$$

where $c > 0$ is constant (independent of n, k).

From the result of Theorem 1.1, that is, it follows from representation (3) that the operator $(L + \mu I)^{-1}$ has an infinite number of eigenvalues; the last proposition follows from the fact that the operator $\left(l_0 + \mu I\right)^{-1}$ is self-adjoint and compact operator if $n = 0$. The compactness of the operator $\left(l_0 + \mu I\right)^{-1}$ follows from Theorem 1.3. Therefore, using the estimate (4) and Weyl's inequality [14], as well as inequality $e^k \cdot k! \ \geq k^k,\ k = 1, 2, 3, ...,$ we obtain that

$$\left|\lambda_{n,k}\right|^k \leq \prod_{j=1}^{k} \left|\lambda_{j,n}\right| \leq \prod_{j=1}^{k} s_{j,n} \leq \frac{c^k (k!)^{-\frac{1}{2}}}{(|n|+1)^{\frac{1}{2}k}} \leq \frac{c^k \cdot e^{\frac{1}{2}k}}{(|n|+1)^{\frac{1}{2}k}\, k^{\frac{1}{2}k}}$$

Hence

$$\left|\lambda_{n,k}\right| \leq \frac{c \cdot e^{\frac{1}{2}}}{(|n|+1)^{\frac{1}{2}}\, k^{\frac{1}{2}}},\ k = 1, 2, 3...,\ n = 0, \pm 1, \pm 2, ..., \tag{5}$$

where $\lambda_{n,k}$ are eigenvalues of the operator $(L + \mu I)^{-1}$.

An operator of mixed type has been studied in the paper [15] [15] for the case when the coefficient satisfies the condition: $y \cdot k(y) > 0$ for $y \neq 0$ and $k(0) = 0$.

Questions on the existence and compactness of the resolvent of a mixed-type operator has been studied in [16] [16], when the coefficient $k(y)$ satisfies the condition: $k(y)$ is a piecewise continuous and bounded function in $\mathbb{R} = (-\infty, \infty)$ and is not identically zero in any interval.

In contrast to these works, in this paper it is shown that operator (1) is separable for a large class of rapidly oscillating coefficients $k(y)$ (for example, $k(y) = \sin e^{100|y|}$). In addition, in this paper, a two-sided estimate for the distribution function of singular numbers (s-numbers) is obtained for the resolvent of the operator $(L + \mu I)$. The found estimate shows that the growth of the coefficients $b(y)$, $q(y)$ of the operator (1) affects the estimates of the singular and eigenvalues. An example is given.

2 | PROOF OF THEOREM 1.1

Lemma 2.1. Let conditions a) and i) be satisfied and $\mu \geq 0$. Then the estimate

$$\|(L + \mu I)\, u\|_2 \geq c\, \|u\|_2,$$

holds for all $u \in D(L)$, where $\|\cdot\|_2$ is the norm in the space $L_2(\Omega)$, $c = c\left(\delta_0, \delta\right) > 0$.

Proof. Taking the conditions a) and i) into account and using the functionals $< (L + \mu I)\, u,\ u >$ and $< (L + \mu I)\, u,\ u_x >$ we obtain the proof of Lemma 2.1, where $< \cdot\ ,\ \cdot >$ is the scalar product in $L_2(\Omega)$. ☐

Direct computations show that the study of operator (1) can be reduced, using the Fourier method, to the study of the following second-order differential operator with a sign-variable parameter

$$\left(l_n + \mu I\right) u = -u''(y) + \left(-k(y)\, n^2 + in\ b(y) + q(y) + \mu\right)\ u, \quad u \in D(l_n), n = 0, \pm 1, \pm 2, ...$$

If $n = 0$ then the above operator is the well-known Sturm-Liouville operator.

When $|n| \to \infty$ in the coefficient $\left(-k(y)\, n^2 + in\ b(y) + q(y) + \mu\right)$ and when $k(y) \equiv 1$, the term $-k(y)\, n^2 \to -\infty$. Consequently, the differential operator is not semi-bounded. In this case, a completely different situation arises compared to the Sturm-Liouville operator.

Let $\Delta_j = (j-1,\ j+1)$, $j \in Z$. Then $\bigcup_{\{j\}} \Delta_j = \mathbb{R}$. Take a set of non-negative functions $\left\{\varphi_j\right\}_{j=-\infty}^{\infty}$ $(j \in Z)$ from $C_0^\infty(\mathbb{R})$ such that supp $\varphi_j \subseteq \Delta_j$, $\sum_{j=-\infty}^{\infty} \varphi_j^2(y) \equiv 1$.

Let us extend $b(y)$, $q(y)$ from Δ_j to the whole space $\mathbb{R}$ so that their extensions $b_j(y)$, $q_j(y)$ be bounded and periodic functions of the same period.

We denote by $l_{n,j,\alpha} + \mu I$ the closure of the operator

$$\left(l_{n,j,\alpha} + \mu\ I\right)\ u = -u''(y) + \left(-k(y)\, n^2 + in\ \left(b_j(y) + \alpha\right) + q_j(y) + \mu\right)\ u$$

defined on $C_0^\infty(\mathbb{R})$, where the sign of the real number α coincides with the sign of $b(y)$, i.e. $\alpha \cdot b(y) > 0$ for $y \in \mathbb{R}$. The number α was introduced in order to obtain estimates for the norm of the operator $\frac{d}{dy}\left(l_{n,j,\alpha} + \mu I\right)$. At the end of the paper, we will get rid of this number.

Lemma 2.2. Let the conditions $a)$ and $i)$ be satisfied and $\mu \geq 0$. Then the operator $l_{n,j,\alpha} + \mu I$ has a continuous inverse operator $\left(l_{n,j,\alpha} + \mu I\right)^{-1}$ for $\mu \geq 0$ defined on the whole $L_2(\mathbb{R})$.

Lemma 2.2 is proved using the computations used in the proof of Lemma 2.2 from [17][17].

We denote by $l_{n,\alpha} + \mu I$ the closure of the differential operator

$$\left(l_{n,\alpha} + \mu I\right) u = -u''(y) + \left(-k(y) n^2 + in \ (b(y) + \alpha) + q(y) + \mu\right) u$$

in the space $L_2(\mathbb{R})$, originally defined on $C_0^\infty(\mathbb{R})$.

We introduce the following bounded operator in $L_2(\mathbb{R})$:

$$K_{\mu,\alpha} f = \sum_{\{j\}} \varphi_j \left(l_{n,j,\alpha} + \mu I\right)^{-1} \varphi_j f.$$

The following lemma is proved by repeating the computations and arguments used in [17][17].

Lemma 2.3. Let the conditions $a)$ and $i)$ be satisfied. Then there exists a number $\mu_0 > 0$ such that the operator $l_{n,\alpha} + \mu I$ for $\mu \geq \mu_0$ is boundedly invertible, and the resolvent of the operator $l_{n,\alpha}$ satisfies the equality

$$\left(l_{n,\alpha} + \mu I\right)^{-1} f = K_{\mu,\alpha} \left(I - B_{\mu,\alpha}\right)^{-1} f,$$

where $B_{\mu,\alpha} f = \sum_{\{j\}} \varphi_j'' \left(l_{n,j,\alpha} + \mu I\right)^{-1} \varphi_j f + 2 \sum_{\{j\}} \varphi_j' \frac{d}{dy} \left(l_{n,j,\alpha} + \mu I\right)^{-1} \varphi_j f, \quad f \in L_2(\mathbb{R}).$

Lemma 2.4. Let the conditions $a)$ and $i)$ be satisfied and $\mu_0 > 0$. Then the operator $l_n + \mu I$ is boundedly invertible and the equality

$$\left(l_n + \mu I\right)^{-1} f = \left(l_{n,\alpha} + \mu I\right)^{-1} \left(I - A_{\mu,\alpha}\right)^{-1} f, \quad f \in L_2(\mathbb{R}),$$

holds, where $A_{\mu,\alpha} = in\alpha \left(l_{n,\alpha} + \mu I\right)^{-1}$, and $\left\|A_{\mu,\alpha}\right\|_{2\to 2} < 1$.

This lemma is proved by the same method as Lemma 3.4 in [17][17].

Proof of Theorem 1.1. Lemma 2.4 implies that

$$u_k(x, y) = \sum_{n=-k}^{k} \left(l_n + \mu I\right)^{-1} f_n(y) \cdot e^{inx} \tag{6}$$

is the solution to the problem

$$\begin{gathered}(L + \mu I) \ u_k(x, y) = f_k(x, y), \\ u_{k,x}^{(i)}(-\pi, y) = u_{k,x}^{(i)}(\pi, y), \quad i = 0, \ 1,\end{gathered}$$

where $f_k(x, y) \overset{L_2}{\to} f(x, y)$, $f_k(x, y) = \sum_{n=-\kappa}^{k} f_n(y) \cdot e^{inx}, \quad i^2 = -1$.

Lemma 2.1 implies that

$$\left\|u_k(x, y) - u_m(x, y)\right\|_2 \leq \frac{1}{c} \left\|f_k(x, y) - f_m(x, y)\right\|_2 \to 0, \text{ as } k, m \to \infty.$$

Hence, due to the completeness of space $L_2(\mathbb{R})$, it follows that

$$u_k(x, y) \overset{L_2}{\to} u(x, y) \text{ as } k \to \infty. \tag{7}$$

Using equality (6) and (7), we have that

$$u(x, y) = (L + \mu I)^{-1} f(x, y) = \sum_{n=-\infty}^{\infty} \left(l_n + \mu I\right)^{-1} f_n(y) \cdot e^{inx} \tag{8}$$

is a strong solution to the problem

$$(L + \lambda I) \ u = f, u_x^{(i)}(-\pi, y) = u_x^{(i)}(\pi, y), \ i = 0, \ 1 \tag{9}$$

for any $f(x, y) \in L_2(\Omega)$.

Definition 2.1. *A function* $u(x, y) \in L_2(\Omega)$ *is called a strong solution to the problem* (9) *if there is a sequence* $\left\{u_k(x, y)\right\}_{k=1}^{\infty} \subset C_{0,\pi}^{\infty}(\Omega)$ *such that*

$$\left\|u_k - u\right\|_2 \to 0, \quad \left\|(L + \mu I) u_k - f\right\|_2 \to 0 \text{ as } k \to \infty.$$

Using the last definition, it is easy to verify that formula (8) is an inverse operator to the closed operator $L + \mu I$. Hence, by virtue of Lemma 2.1 and the equality (9), we obtain that Theorem 1.1 holds for all $\mu \geq 0$. Theorem 1.1 is completely proved. □

3 | PROOF OF THEOREM 1.2.

The existence of the resolvent of the operator $l_{n,j,\alpha}$ is proved in Lemma 2.2. Let us show several properties of the resolvent of the operator $l_{n,j,\alpha}$ in the following lemma.

Lemma 3.1. Let the conditions $a)$ and $i) - iii)$ be satisfied. Then the following inequalities

$$\left\| \left(l_{n,j,\alpha} + \mu I\right)^{-1} \right\|_{2\to 2} \le \frac{1}{|n| \cdot \left| b\left(\tilde{y}_j\right) \right|}, \quad n \neq 0; \tag{10}$$

$$\left\| \left(l_{n,j,\alpha} + \mu I\right)^{-1} \right\|_{2\to 2} \le \frac{2 \cdot c}{q\left(\bar{y}_j\right) + \mu}, \quad c > 0; \tag{11}$$

$$\left\| \left(l_{n,j,\alpha} + \mu I\right)^{-1} \right\|_{2\to 2} \le \frac{c}{(\delta + \mu)^{\frac{1}{2}}}, \quad c = c(\delta) > 0; \tag{12}$$

$$\left\| \frac{d}{dy} \left(l_{n,j,\alpha} + \mu I\right)^{-1} \right\|_{2\to 2} \le \frac{c}{(\delta + \mu)^{\frac{1}{4}}}, \quad c > 0, \tag{13}$$

hold, where $\|\cdot\|_{2\to 2}$ is the norm of the operator $l_{n,j,\alpha} + \mu I$ from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$, $\left| b\left(\tilde{y}_j\right) \right| = \min\limits_{y\in\bar{\Delta}_j} |b(y)|$, $q\left(\bar{y}_j\right) = \min\limits_{y\in\bar{\Delta}_j} q(y)$.

Proof. Let $u \in C_0^\infty(\mathbb{R})$. Then, we have

$$< \left(l_{n,j,\alpha} + \mu I\right) u, u >= \int\limits_{\mathbb{R}} \left(\left|u'\right|^2 + \left(-k(y) n^2 + q_j(y) + \mu\right) |u|^2 \right) dy + \int\limits_{\mathbb{R}} in\left(b_j(y) + \alpha\right) |u|^2 \, dy \tag{14}$$

Hence, taking the conditions $a)$ and $i)$ into account and using the property of complex numbers, and also by virtue of the Cauchy-Bunyakovsky inequality, we obtain

$$\left\| \left(l_{n,j,\alpha} + \mu I\right) u \right\|_2^2 \ge |n|^2 \left(\left| b_j\left(\tilde{y}_j\right) \right| + |\alpha| \right)^2 \|u\|_2^2 \tag{15}$$

Considering that $b\left(\tilde{y}_j\right) = \min\limits_{y\in\bar{\Delta}_j} \left| b_j(y) \right| = \min\limits_{y\in\bar{\Delta}_j} |b(y)|$ on the segment Δ_j, from (15) we find

$$\left\| \left(l_{n,j,\alpha} + \mu I\right) u \right\|_2^2 \ge |n|^2 \cdot \left| b\left(\tilde{y}_j\right) \right|^2 \cdot \|u\|_2^2,$$

where $b\left(\tilde{y}_j\right) = \min\limits_{y\in\bar{\Delta}_j} |b(y)|$.

By virtue of the continuity of the norm, the last inequality holds for all $u \in D\left(l_{n,j,\alpha}\right)$. Hence, we finally have

$$\left\| \left(l_{n,j,\alpha} + \mu I\right)^{-1} \right\|_{2\to 2} \le \frac{1}{|n| \cdot \left| b\left(\tilde{y}_j\right) \right|}, \quad n \neq 0.$$

The inequality (10) is proved.

The inequalities (12) and (13) are proved using the computations used in the proof of Lemma 6 from [18][18] and Lemma 2.3 from [17][17].

From equality (14), by virtue of the Cauchy inequality with "$\varepsilon > 0$", we obtain

$$\frac{1}{2\left(q\left(\bar{y}_j\right) + \mu\right)} \left\| \left(l_{n,j,\alpha} + \mu I\right) u \right\|_2^2 + \frac{q\left(\bar{y}_j\right) + \mu}{2} \|u\|_2^2 \ge$$
$$\ge \int\limits_{\mathbb{R}} \left[\left|u'\right|^2 + \left(q_j(y) + \mu I\right) |u|^2 \right] dy - n^2 \int\limits_{\mathbb{R}} |k(y)| \, |u|^2 \, dy,$$

where $\varepsilon = q\left(\bar{y}_j\right) + \mu$.
Hence

$$\frac{1}{2\left(q\left(\bar{y}_j\right) + \mu\right)} \left\| \left(l_{n,j,\alpha} + \mu I\right) u \right\|_2^2 \ge \int\limits_{\mathbb{R}} \left|u'\right|^2 dy + \frac{q\left(\bar{y}_j\right) + \mu}{2} \int\limits_{\mathbb{R}} |u|^2 \, dy - n^2 \int\limits_{\mathbb{R}} |k(y)| \cdot |u|^2 \, dy, \tag{16}$$

where $q\left(\bar{y}_j\right) = \min\limits_{y\in\bar{\Delta}_j} q(y)$.

Multiplying both sides of the inequality (15) by the number $\frac{c}{2(q(\bar{y}_j)+\mu)}$ and taking the condition $ii)$ into account, we find

$$\frac{c}{2\left(q\left(\bar{y}_j\right)+\mu\right)}\left\|\left(l_{n,j,\alpha}+\mu\, I\right)\, u\right\|_2^2 \geq \frac{c\cdot n^2\left(b\left(\tilde{y}_j\right)+|\alpha|\right)^2}{2\,\mu_1\left(q\left(\bar{y}_j\right)+\mu\right)}\cdot \|u\|_2^2, \tag{17}$$

where $c>0$ is a constant.

Combining (16) and (17) we come to the inequality

$$\frac{c}{q\left(\bar{y}_j\right)+\mu}\left\|\left(l_{n,j,\alpha}+\mu\, I\right)\, u\right\|_2^2 \geq \|u'\|_2^2+\frac{q\left(\bar{y}_j\right)+\mu}{2}\cdot\|u\|_2^2+$$
$$+n^2\int\limits_{\mathbb{R}}\left[\frac{c\cdot\left(\left|b\left(\tilde{y}_j\right)\right|+|\alpha|\right)^2}{2\mu_1\cdot\left(q\left(\bar{y}_j\right)+\mu\right)}-|k\left(y\right)|\right]|u|^2\, dy.$$

From the last inequality, taking the conditions $a)$, $iii)$ into account and choosing α and $c>0$, so that $\frac{c\cdot\left(\left|b(\tilde{y}_j)\right|+|\alpha|\right)^2}{2\mu_1\cdot(q(\bar{y}_j)+\mu)}-|k\left(y\right)|\geq 0$, we obtain

$$2\cdot c\cdot\left\|\left(l_{n,j,\alpha}+\mu\, I\right)u\right\|_2^2 \geq \left(q\left(\bar{y}_j\right)+\mu\right)^2\cdot\|u\|_2^2. \tag{18}$$

From (18), by virtue of the definition of the norm of the operator $l_{n,j,\alpha}+\mu\, I$, we find

$$\left\|\left(l_{n,j,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}\leq\frac{2\cdot c}{q\left(\bar{y}_j\right)+\mu}.$$

Lemma 3.1 is proved. □

Lemma 3.2. Let the conditions $a)$ and $i)-iii)$ be satisfied and let $\mu>0$ such that $\left\|B_{\mu,\alpha}\right\|_{2\to 2}<1$. Then the estimate

$$\left\|p\left(y\right)\,|n|^{\alpha}\left(l_{n,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq c\left(\lambda\right)\sup_{\{j\}}\left\|p\left(y\right)\,|n|^{\alpha}\cdot\varphi_j\left(l_{n,j,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2,$$

holds, where $\alpha=0,1$, $p\left(y\right)$is a continuous function in $\mathbb{R}$.

Lemma 3.2 is proved by the same method as Lemma 3.7 in [17][17].

Lemma 3.3. Let the conditions $a)$ and $i)-iii)$ be satisfied and let $\mu>0$ such that $\left\|B_{\mu,\alpha}\right\|_{2\to 2}<1$. Then the estimates

a) $\left\|q\left(y\right)\left(l_{n,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq c_1<\infty$;

b) $\left\|in\, b\left(y\right)\left(l_{n,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq c_2<\infty$;

c) $\left\|\frac{d}{dy}\left(l_{n,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq c_3<\infty$, $c_1>0$, $c_2>0$, $c_3>0$ are constants

hold.

Proof. By virtue of Lemma 3.2, we find

$$\left\|q\left(y\right)\left(l_{n,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq c\left(\mu\right)\sup_{\{j\}}\left\|q\left(y\right)\,\varphi_j\left(l_{n,j,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2,$$

From this and Lemma 3.1, taking the condition $ii)$ into account, we find that

$$\left\|q\left(y\right)\left(l_{n,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq c\left(\mu\right)\sup_{\{j\}}\left\|q\left(y\right)\,\varphi_j\left(l_{n,j,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq$$
$$\leq c\left(\mu\right)\sup_{|y-t|\leq 1}\frac{q\left(y\right)}{q\left(t\right)}\leq c_1<\infty.$$

The inequality $a)$ is proved.

Repeating the above computations and arguments, and also using Lemmas 3.1 and 3.2, we obtain

$$\left\|in\, b\left(y\right)\left(l_{n,\alpha}+\mu\, I\right)^{-1}\right\|_{2\to 2}^2\leq c\left(\mu\right)\sup_{|y-t|\leq 1}\frac{b\left(y\right)}{b\left(t\right)}\leq c_2<\infty.$$

The inequality $b)$ is proved.

In the same way, repeating the computations and arguments that were used in the proof of the inequalities $a)$ and $b)$, we obtain the proof of the item $c)$. □

Now, using Lemma 3.3 for the resolvent of the operator l_n, we have the following lemma.

Lemma 3.4. Let the conditions *a*) and *i*) $-$ *iii*) be satisfied. Then the following estimates

a) $\left\|q(y)\left(l_n+\mu I\right)^{-1}\right\|_{2\to 2}^2 \le c_4 < \infty$;

b) $\left\|in\,b(y)\left(l_n+\mu I\right)^{-1}\right\|_{2\to 2}^2 \le c_5 < \infty$;

c) $\left\|\frac{d}{dy}\left(l_n+\mu I\right)^{-1}\right\|_{2\to 2}^2 \le c_6 < \infty$

hold, where $\|\cdot\|_{2\to 2}$ is the norm of an operator from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$, $c_4>0,\ c_5>0,\ c_6>0$ are constants.

Proof. Lemma 2.4 implies

$$\left\|q(y)\left(l_n+\mu I\right)^{-1} f\right\|_2^2 = \left\|q(y)\left(l_{n,\alpha}+\mu I\right)^{-1}\left(I-A_{\mu,\alpha}\right) f\right\|_2^2.$$

Since the operators $q(y)\left(l_{n,\alpha}+\mu I\right)^{-1}$ and $\left(I-A_{\mu,\alpha}\right)^{-1}$ are bounded, from the last equality we obtain that

$$\left\|q(y)\left(l_n+\mu I\right)^{-1} f\right\|_2^2 \le c_4\cdot\|f\|_2^2$$

or

$$\|q(y)\,u\|_2^2 \le c_4\cdot\left\|\left(l_n+\mu I\right)u\right\|_2^2, \tag{19}$$

where $\left(l_n+\mu I\right)u = f$, $u=\left(l_n+\mu I\right)^{-1} f$, $c_4>0$ is a constant. Here we note that the boundedness of the operators $q(y)\left(l_{n,\alpha}+\mu I\right)^{-1}$ and $\left(I-A_{\mu,\alpha}\right)^{-1}$ follows from Lemmas 2.4 and 3.3. From (19), according to the definition of the operator norm, we have

$$\left\|q(y)\left(l_n+\mu I\right)^{-1}\right\|_{2\to 2} = c_4 < \infty.$$

The item *a*) of Lemma 3.4 is proved.

The items *b*) and *c*) are proved by the same method as the item *a*) of Lemma 3.4, that is, the following inequalities

$$\|in\,b(y)\,u\|_2^2 \le c_5\cdot\left\|\left(l_n+\mu I\right)u\right\|_2^2, \tag{20}$$

$$\left\|\frac{d}{dy}u\right\|_2^2 \le c_6\cdot\left\|\left(l_n+\mu I\right)u\right\|_2^2, \tag{21}$$

hold, where $c_5>0,\ c_6>0$ are constants. □

Lemma 3.5. Let the conditions *a*) and *i*) $-$ *iii*) be satisfied. Then the inequality

$$\left\|u'\right\|_2 + \|q(y)\,u\|_2 + \|in\,b(y)\,u\|_2 \le c\cdot\left(\left\|l_n u\right\| + \|u\|_2\right),$$

holds, where $c>0$ is a constant.

The proof of Lemma 3.5 follows from the inequalities (19)-(21).

Proof of Theorem 1.2. The representation (8) implies that

$$b(y)\,u_x = b(y)\frac{\partial}{\partial x}(L+\mu I)^{-1} f = b(y)\frac{\partial}{\partial x}\sum_{n=-\infty}^{\infty}\left(l_n+\mu I\right)^{-1} f_n(y)\cdot e^{inx} =$$

$$= b(y)\sum_{n=-\infty}^{\infty} in\left(l_n+\mu I\right)^{-1} f_n(y)\cdot e^{inx} = \sum_{n=-\infty}^{\infty} b(y)\ in\left(l_n+\mu I\right)^{-1} f_n(y)\cdot e^{inx}.$$

Hence, since the system $\left\{e^{inx}\right\}_{n=-\infty}^{\infty}$ is orthonormal, we obtain

$$\left\|b(y)\,u_x\right\|_2^2 \le \sup_{\{j\}}\left\|in\,b(y)\left(l_n+\mu I\right)^{-1}\right\|_{2\to 2}^2\cdot\|f(x,y)\|_2^2.$$

From the last inequality and using the item *b*) of Lemma 3.4, we find that

$$\left\|b(y)\,u_x(x,y)\right\|_2^2 \le c_5\cdot\|(L+\mu I)u\|_2, \tag{22}$$

where $c_5>0$ is the constant from Lemma 3.4, $(L+\mu I)u = f(x,y)$ from the equality (9).

Repeating the above computations and arguments and taking Lemma 3.4 into account, we have

$$\|q(y)\,u(x,y)\|_2^2 \le c_4\cdot\|(L+\mu I)u\|_2, \tag{23}$$

$$\left\| \frac{\partial u(x,y)}{\partial y} \right\|_2^2 \le c_6 \cdot \|(L+\mu I)u\|_2 , \tag{24}$$

where $c_4 > 0$, $c_6 > 0$ from Lemma 3.4.

Now, using inequalities (22) - (23), we have

$$\left\| k(y)\, u_{xx} - u_{yy} \right\|_2 = \left\| (L+\mu\, I) - b(y)\, u_x - q(y)\, u - \mu\, u \right\|_2 \le c_7 \cdot \|(L+\mu\, I)\, u\|_2^2 , \tag{25}$$

where $c_7 > 0$ is a constant.

Hence, taking the inequalities (22)-(24) into account, we obtain that

$$\left\| k(y)\, u_{xx} - u_{yy} \right\|_2 + \|b(y)\ u_x\|_2 + \|q(y)\ u\|_2 + \left\| u_y \right\| \le c \cdot \left(\|Lu\|_2 + \|u\|_2 \right),$$

where $c > 0$ is independent of $u(x,y)$. Theorem 1.2 is proved. □

4 | PROOFS OF THEOREMS 1.3-1.4. COMPACTNESS AND ESTIMATION OF SINGULAR NUMBERS (S-NUMBERS) OF THE RESOLVENT OF THE OPERATOR $L + \mu I$

Proof of Theorem 1.3. In order to prove Theorem 1.3, we first give the following lemma.

Lemma 4.1. Let the conditions $a)$ and $i) - iii)$ be satisfied. Then the resolvent of the operator l_n $(n = 0, \pm 1, \pm 2, ...)$ is compact if and only if

$$\lim_{|y|\to\infty} q(y) = \infty.$$

Lemma 4.1 is proved in exactly the same way as Theorems 1.2 and 1.3 from [17][17].

Now, let us prove Theorem 1.3. Since the operator $\left(l_n + \mu\, I\right)^{-1}$, $\mu \ge 0$ is completely continuous for each n $(n = 0, \pm 1, \pm 2, ...)$, by virtue of Lemma 4.1, then it can be shown from Theorem 1.1 and from representation (3) with the help of well-known methods with a ε-net that the operator $(L + \mu\, I)^{-1}$ is completely continuous if and only if

$$\lim_{|n|\to\infty} \left\| \left(l_n + \mu\, I\right)^{-1} \right\|_{2\to 2} = 0. \tag{26}$$

It is easy to see that equality (26) follows from Lemmas 3.4-3.5. Theorem 1.3 is proved. □

To prove Theorem 1.4, we need the following lemmas.

We introduce the following sets, which are closely related to the domain of the operator l_n:

$$M = \left\{ u \in L_2(\mathbb{R}) \,:\, \|l_n u\|_2^2 + \|u\|_2^2 \le 1 \right\},$$

$$\tilde{M}_{c_0} = \left\{ u \in L_2(\mathbb{R}) \,:\, \|u'(y)\|_2^2 + \|inb(y)\, u\|_2^2 + \|q(y)\, u\|_2^2 \le c_0 \right\},$$

$$\tilde{\tilde{M}}_{c_0^{-1}} = \left\{ u \in L_2(\mathbb{R}) \,:\, \|u''(y)\|_2^2 + \left\|(k(y)+\varepsilon)\, n^2 u\right\|_2^2 + \|inb(y)\, u\|_2^2 + \|q(y)\, u\|_2^2 \le c_0^{-1} \right\},$$

where c_0 is a constant number independent of $u(y)$ and n, $k(y) + \varepsilon > \varepsilon_0 > 0$.

Lemma 4.2. Let the conditions $a)$ and $i) - iii)$ be satisfied. Then the inclusions

$$\tilde{\tilde{M}}_{c_0^{-1}} \subseteq M \subseteq \tilde{M}_{c_0}$$

hold.

Proof. Taking Lemma 3.5 into account and using the computations used in the proof of Lemma 2.4 [19][19], we prove the inclusion $M \subseteq \tilde{M}_{c_0}$. The inclusion $\tilde{\tilde{M}}_{c_0^{-1}} \subseteq M$ is proved by the same method as Lemma 2.4 in [19][19]. Lemma 4.2 is proved. □

Definition 4.1.[14] *The magnitude*

$$d_k = \inf_{\{y_k\}} \sup_{u\in M} \inf_{\vartheta \in y_k} \|u - \vartheta\|_2 ,$$

is called Kolmogorov k-widths (diameter) of the set M in the space $L_2(\mathbb{R})$, where y_k is the set of all subspaces in $L_2(\mathbb{R})$, the dimension of which does not exceed k.

The following lemmas hold.

Lemma 4.3. Let the conditions $a)$ and $i) - iii)$ be satisfied. Then the estimate

$$c^{-1}\tilde{\tilde{d}}_k \leq s_{k+1} \leq c\, \tilde{d}_k, \quad k = 1, 2, ...,$$

holds, where $c > 0$ is a constant, S_k is the s-number of the operator $\left(l_n + \mu I\right)^{-1}$, $\mu \geq 0$, d_k, $\tilde{d}_k$, $\tilde{\tilde{d}}_k$ are the Kolmogorov k-widths of the corresponding sets M, $\tilde{M}$, $\tilde{\tilde{M}}$.

Lemma 4.4. Let the conditions $a)$ and $i) - iii)$ be satisfied. Then the estimate

$$\tilde{\tilde{N}}(c\lambda) \leq N(\lambda) \leq \tilde{N}\left(c^{-1}\lambda\right),$$

holds, where the counting function $N(\lambda) = \sum_{s_{k+1} > \lambda} 1$ of those s_{k+1} of the operator $\left(l_n + \mu I\right)^{-1}$ greater than $\lambda > 0$, the counting function $\tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1$ of those $\tilde{d}_k$ greater than $\lambda > 0$, the counting function $\tilde{\tilde{N}}(\lambda) = \sum_{\tilde{\tilde{d}}_k > \lambda} 1$ of those $\tilde{\tilde{d}}_k$ greater than $\lambda > 0$.

Lemmas 4.3-4.4 are proved in exactly the same way as Lemmas 2.5-2.6 in [19] [19].

Lemma 4.5. Let the conditions $a)$ and $i) - iii)$ be satisfied. Then the estimate

$$c^{-1}\lambda^{-\frac{1}{2}}\, mes\left(y \in \mathbb{R} : Q_n(y) \leq c^{-1}\lambda^{-1}\right) \leq N(\lambda) \leq c\lambda^{-1} mes\left(y \in \mathbb{R} : K_n(y) \leq c\lambda^{-1}\right)$$

holds, where $c > 0$ is a constant, the functions $Q_n(y)$ and $K_n(y)$ are from Theorem 1.4.

Proof. We denote by $L_2^2\left(\mathbb{R}, Q_n(y)\right)$, $L_2'\left(\mathbb{R}, K_n(y)\right)$ the spaces obtained by the completion of $C_0^\infty(\mathbb{R})$ with respect to the norms

$$\|u\|_{L_2^2(\mathbb{R}, Q_n(y))} = \left(\int_{-\infty}^{\infty} \left|u''_{yy}\right|^2 + Q_n^2(y)\,|u|^2\,dy\right)^{\frac{1}{2}},$$

$$\|u\|_{L_2'(\mathbb{R}, K_n(y))} = \left(\int_{-\infty}^{\infty} \left|u'(y)\right|^2 + K_n^2(y)\,|u|^2\,dy\right)^{\frac{1}{2}},$$

where the functions $Q_n(y)$ and $K_n(y)$ from Theorem 1.4.

It is easy to see that $\tilde{M} \subset L_2'\left(\mathbb{R}, K_n(y)\right)$, $\tilde{\tilde{M}} \subset L_2^2\left(\mathbb{R}, Q_n(y)\right)$. Now, repeating the computations and arguments from Lemma 2.7 [19] [19], we obtain the proof of Lemma 4.5. □

Proof of Theorem 1.4. Theorem 1.1 and the representation (3) imply that

$$u(x, y) = (L + \mu I)^{-1} f = \sum_{n=-\infty}^{\infty} \left(l_n + \mu I\right)^{-1} f_n(y) \cdot e^{inx}.$$

The last equality implies that if the s is a singular point of the operator $(L + \mu I)^{-1}$, then s is a singular number of one of the operators $\left(l_n + \mu I\right)^{-1}$ $(n = 0, \pm 1, \pm 2, ...)$ and vice versa, if s is a singular number of one of the operators $\left(l_n + \mu I\right)^{-1}$, then s is a singular point of the operator $(L + \mu I)^{-1}$. From this and from Lemma 4.5 the proof of Theorem 1.4 follows. □

ACKNOWLEDGMENTS

This work was supported by Ministry of Education and Science of the Republic of Kazakhstan [grants numbers (IRN): AP08856339 and AP08855802]

Author contributions

Mussakan Muratbekov: conceptualization, formulation of the problem, methodology, investigation, writing original draft.
Akbota Abylayeva: supervision, methodology, investigation, writing and editing original manuscript, funding acquisition, project administration.
Madi Muratbekov: investigation, some computations, writing and editing original manuscript, project administration, bibliography.

Conflict of interest

The authors declare no potential conflict of interests.

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