

# Scattering by a biisotropic obstacle and the properties of the Beltrami spherical vector waves

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## Abstract

In this paper, we take a fresh look at the determination of the transition matrix for a homogeneous, biisotropic particle employing the Null-field approach. The condition for passivity of the material is discussed. In particular, we focus on some previously unproved properties of the Beltrami spherical vector waves, such as completeness, orthogonality, and linear independence, which are all instrumental in the analysis of the scattering problem with the Null-field approach. Some numerical examples illustrate the approach.

## KEY WORDS

Scattering of electromagnetic waves, Biisotropic material, Completeness, Null-field approach

*This paper is a dedication to professor Ioannis Stratis, recently retired, for his life-long scientific achievements.*

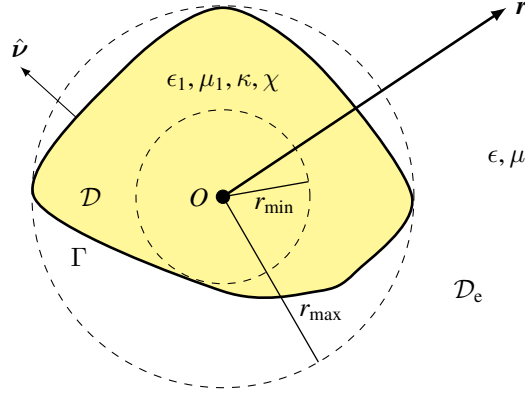
## 1 | INTRODUCTION

Scattering of electromagnetic waves by a biisotropic particle has off and on been in focus in electromagnetic research. Chiral materials — media that show handedness — belong to this category of materials, *i.e.*, reciprocal, biisotropic material, and a majority of the existing literature treats this case. Chiral materials possess optical activity, *i.e.*, the plane of polarization rotates as the wave propagates through the medium. Chiral materials occur both as natural and artificial materials. For a survey of the literature, see [1, 2, 3, 4, 5, 21, 22, 23, 24, 25] and references therein.

The question whether linear, non-reciprocal, biisotropic materials exist or not has been debated intensively over recent decades, see [13, 14, 26, 27, 28, 34, 35, 37], but the question has never been fully resolved, and the question of existence of non-reciprocal, biisotropic media (Tellegen media) is unsettled still. The analysis in this paper is general and applies to both non-reciprocal as well as reciprocal biisotropic materials.

The scattering problem by a chiral particle is traditionally solved by a layer potential approach and the use of boundary integral operators to show the uniqueness and existence of the solution to the problem. Another, less frequently used method, is the generalization of the Null-field approach to solve this scattering problem. This method was originally devised by Peter C Waterman in the late 60's and early 70's in a series of well-cited papers [39, 40, 41, 42]. In this paper, we address some open questions, and we take a fresh look at the scattering problem with the Null-field approach, and at the same time address the completeness properties of the expansion functions used. Completeness of the spherical vector waves, used in isotropic problems, has been treated and proved in the literature [6, 29], but the corresponding expansion functions in the biisotropic case seem not to have been addressed before. The linear independence of the appropriate spherical vector waves in biisotropic problems seems also to be an unaddressed topic. These topics are treated in detail in this paper.

The outline of the paper is as follows: In Section 2, we set up the problem and define the geometry and the mathematical prerequisites. The Null-field approach is reviewed in Section 3, which also contains the explicit expressions of the construction of the transition matrix. This section also contains the completeness of the Beltrami spherical vector waves and a proof of their linear independence. In Section 4, we present some numerical computations illustrating the theory presented in this paper. The



**FIGURE 1** The geometry and material parameters of the biisotropic particle and its maximum inscribed and minimum circumscribed spheres with radii  $r_{\min}$  and  $r_{\max}$ , respectively.

paper is concluded with a discussion in Section 5. The vector spherical harmonics and the spherical vector waves are defined in Appendix A, the definitions of closure and completeness in Hilbert spaces are reviewed in Appendix B, and Appendix C contains the details of the proof of equal expansion coefficients of the electric and the magnetic surface fields.

## 2 | SCATTERING PROBLEM — DESCRIPTION

A typical geometry is depicted in Figure 1. The particle occupies the bounded, open domain  $\mathcal{D}$  with connected exterior domain  $\mathcal{D}_e = \mathbb{R}^3 \setminus \overline{\mathcal{D}}$ , and the particle is excited by an incident field with sources in  $\mathcal{D}_e$ . The origin  $O$  is assumed located in  $\mathcal{D}$ . We denote the bounding surface of  $\mathcal{D}$  by  $\Gamma$ , which we assume consists of a finite number of disjoint, closed, bounded surfaces belonging to the class  $C^2$ . We denote by  $\hat{\nu}$  the outward pointing unit normal of the surface.

The goal is to solve the time-harmonic Maxwell equations (we use the time convention  $e^{-i\omega t}$ , where the angular frequency is denoted  $\omega$ ) [19]<sup>†</sup>

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}) = i\omega \mathbf{B}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) = -i\omega \mathbf{D}(\mathbf{r}) \end{cases} \quad \mathbf{r} \in \mathbb{R}^3 \quad (1)$$

The complex-valued electric and magnetic field intensities are denoted  $\mathbf{E}(\mathbf{r})$  and  $\mathbf{H}(\mathbf{r})$ , respectively, and the corresponding electric and magnetic flux densities are denoted  $\mathbf{D}(\mathbf{r})$  and  $\mathbf{B}(\mathbf{r})$ , respectively.

The fields are subject to the usual boundary conditions — continuity of the tangential electric and magnetic fields on  $\Gamma$

$$\begin{cases} \hat{\nu} \times \mathbf{E}(\mathbf{r})|_+ = \hat{\nu} \times \mathbf{E}(\mathbf{r})|_- \\ \hat{\nu} \times \mathbf{H}(\mathbf{r})|_+ = \hat{\nu} \times \mathbf{H}(\mathbf{r})|_- \end{cases} \quad \mathbf{r} \in \Gamma \quad (2)$$

where the subscript  $+$ ( $-$ ) denotes the limit value from the outside(inside) of  $\Gamma$ .

We are looking for classical solutions to the biisotropic scattering problem in  $\mathbb{R}^3$ , *i.e.*, both inside the particle  $\mathcal{D}$  and outside the particle  $\mathcal{D}_e$ . More precisely, we look for solutions in the space [9]

$$F(\mathcal{D}) = \{\mathbf{f} \in (C^1(\mathcal{D}))^3 \cap (C(\overline{\mathcal{D}}))^3\}$$

and similarly for the exterior region  $\mathcal{D}_e$ . Other spaces we use in this paper are [17] (the Hölder space  $C^{0,\alpha}(\Gamma)$  is defined in *e.g.*, [33, Sec. 3.2])

$$\begin{cases} T(\Gamma) = \{\mathbf{f} \in (C(\Gamma))^3 : \mathbf{f} \cdot \hat{\nu} = 0 \text{ on } \Gamma\} \\ T^{0,\alpha}(\Gamma) = \{\mathbf{f} \in (C^{0,\alpha}(\Gamma))^3 : \mathbf{f} \cdot \hat{\nu} = 0 \text{ on } \Gamma\} \\ T_d^{0,\alpha}(\Gamma) = \{\mathbf{f} \in T^{0,\alpha}(\Gamma) : \text{Div} \mathbf{f} \in C^{0,\alpha}(\Gamma)\} \end{cases}$$

<sup>†</sup> Vector-valued quantities are denoted in *italic boldface*, and vectors of unit length have a “hat” or caret (^) over a symbol.

for  $0 < \alpha < 1$ , and where  $\text{Div}$  denotes the surface divergence [9]. We also use the space of square integrable, tangential functions

$$L_1^2(\Gamma) = \{\mathbf{f} \in (L^2(\Gamma))^3 : \mathbf{f} \cdot \hat{\nu} = 0 \text{ on } \Gamma\}$$

In the lossless, isotropic exterior region  $\mathcal{D}_e$ , characterized by the (real-valued) relative permittivity  $\epsilon$  and permeability  $\mu$ , we represent the electric field as a sum of two parts — incident and scattered fields

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_i(\mathbf{r}) + \mathbf{E}_s(\mathbf{r}), \quad \mathbf{r} \in \mathcal{D}_e$$

where the sources of the incident field  $\mathbf{E}_i$  are located in  $\mathcal{D}_e$ , while the sources of the scattered field  $\mathbf{E}_s$  are located in  $\mathcal{D}$ . The magnetic field  $\mathbf{H}(\mathbf{r})$  has a similar decomposition in two parts. The scattered fields satisfy one of the Silver-Müller radiation conditions, see [19, 32, 36]

$$\begin{cases} \hat{\mathbf{r}} \times \mathbf{E}_s(\mathbf{r}) - \eta_0 \eta \mathbf{H}_s(\mathbf{r}) = o((kr)^{-1}) \\ \eta_0 \eta (\hat{\mathbf{r}} \times \mathbf{H}_s(\mathbf{r})) + \mathbf{E}_s(\mathbf{r}) = o((kr)^{-1}) \end{cases} \quad \text{uniformly in the direction } \hat{\mathbf{r}} \text{ as } r \rightarrow \infty$$

The wave impedance of vacuum is denoted  $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ , and the relative wave impedance of the exterior material is denoted  $\eta = \sqrt{\mu/\epsilon}$ . The permittivity and permeability of vacuum are denoted  $\epsilon_0$  and  $\mu_0$ , respectively. The wavenumber of the exterior region is  $k = k_0 \sqrt{\epsilon\mu}$ , where  $k_0 = \omega \sqrt{\epsilon_0 \mu_0}$  is the wavenumber in vacuum.

The dimensionless material parameters of the homogeneous, biisotropic scatterer  $\mathcal{D}$  are  $\epsilon_1$ ,  $\mu_1$ ,  $\kappa$ , and  $\chi$ . The constitutive relations of a biisotropic media is [19, Chap. 1]<sup>‡</sup>

$$\begin{cases} \mathbf{D}(\mathbf{r}) = \epsilon_0 \{\epsilon_1 \mathbf{E}(\mathbf{r}) + (\kappa + i\chi) \eta_0 \mathbf{H}(\mathbf{r})\} \\ \mathbf{B}(\mathbf{r}) = \sqrt{\epsilon_0 \mu_0} \{(\kappa - i\chi) \mathbf{E}(\mathbf{r}) + \mu_1 \eta_0 \mathbf{H}(\mathbf{r})\} \end{cases} \quad (3)$$

The material parameters  $\epsilon_1$ ,  $\mu_1$ ,  $\kappa$ , and  $\chi$  are in general complex-valued, and the parameters  $\kappa$  and  $\chi$ , are usually referred to as the reciprocity and the chirality parameters, respectively. As usual, the parameters  $\epsilon_1$  and  $\mu_1$  are referred to as the permittivity and the permeability of the medium, respectively. If the reciprocity parameter  $\kappa = 0$  and  $\chi \neq 0$ , the medium is called chiral.

The Maxwell equations in the biisotropic media read

$$\begin{cases} \nabla \times \mathbf{E}(\mathbf{r}) = ik_0 \{(\kappa - i\chi) \mathbf{E}(\mathbf{r}) + \mu_1 \eta_0 \mathbf{H}(\mathbf{r})\} \\ \eta_0 \nabla \times \mathbf{H}(\mathbf{r}) = -ik_0 \{\epsilon_1 \mathbf{E}(\mathbf{r}) + (\kappa + i\chi) \eta_0 \mathbf{H}(\mathbf{r})\} \end{cases} \quad \mathbf{r} \in \mathcal{D} \quad (4)$$

The Maxwell equation and the constitute relations are used to obtain an equation in the electric field. The result is

$$\nabla \times (\nabla \times \mathbf{E}(\mathbf{r})) + \alpha \nabla \times \mathbf{E}(\mathbf{r}) - \beta^2 \mathbf{E}(\mathbf{r}) = \mathbf{0} \quad (5)$$

where  $\alpha = -2k_0\chi$  and  $\beta^2 = k_0^2 (\epsilon_1 \mu_1 - \kappa^2 - \chi^2)$ . Here, we observe the different role the chirality parameter  $\chi$  has in contrast to the reciprocity parameter  $\kappa$ , which simply modify the  $\beta$  coefficient.

<sup>‡</sup> An alternative description of the material is the Fedorov constitutive relations

$$\begin{cases} \mathbf{D}(\mathbf{r}) = \epsilon_F \{\mathbf{E}(\mathbf{r}) + \alpha \nabla \times \mathbf{E}(\mathbf{r})\} \\ \mathbf{B}(\mathbf{r}) = \mu_F \{\mathbf{H}(\mathbf{r}) + \beta \nabla \times \mathbf{H}(\mathbf{r})\} \end{cases}$$

This model is equivalent to our material parameters by

$$\begin{cases} \epsilon_1 = \frac{\epsilon_F/\epsilon_0}{1 - \omega^2 \epsilon_F \mu_F \alpha \beta} \\ \mu_1 = \frac{\mu_F/\mu_0}{1 - \omega^2 \epsilon_F \mu_F \alpha \beta} \end{cases} \quad \begin{cases} \kappa = i \frac{\omega \epsilon_F \mu_F (\alpha - \beta)/2}{\sqrt{\epsilon_0 \mu_0} (1 - \omega^2 \epsilon_F \mu_F \alpha \beta)} \\ \chi = \frac{\omega \epsilon_F \mu_F (\alpha + \beta)/2}{\sqrt{\epsilon_0 \mu_0} (1 - \omega^2 \epsilon_F \mu_F \alpha \beta)} \end{cases}$$

with inverse

$$\begin{cases} \epsilon_F = \epsilon_0 \frac{\epsilon_1 \mu_1 - \kappa^2 - \chi^2}{\mu_1} \\ \mu_F = \mu_0 \frac{\epsilon_1 \mu_1 - \kappa^2 - \chi^2}{\epsilon_1} \end{cases} \quad \begin{cases} \alpha = -\frac{i(\kappa + i\chi)}{\omega \sqrt{\epsilon_0 \mu_0} (\epsilon_1 \mu_1 - \kappa^2 - \chi^2)} \\ \beta = \frac{i(\kappa - i\chi)}{\omega \sqrt{\epsilon_0 \mu_0} (\epsilon_1 \mu_1 - \kappa^2 - \chi^2)} \end{cases}$$

The biisotropic material of the scatterer in this paper is assumed to be passive, *i.e.*, it produces no energy, but only consumes energy. A biisotropic material is passive if the matrix [19]

$$\begin{pmatrix} \text{Im } \epsilon_1 & \text{Im } \kappa + i \text{Im } \chi \\ \text{Im } \kappa - i \text{Im } \chi & \text{Im } \mu_1 \end{pmatrix}$$

is non-negative definitive, *i.e.*, both eigenvalues of the matrix are non-negative

$$\lambda_{1,2} = \frac{\text{Im } \epsilon_1 + \text{Im } \mu_1 \pm \sqrt{(\text{Im } \epsilon_1 - \text{Im } \mu_1)^2 + 4(\text{Im } \kappa)^2 + 4(\text{Im } \chi)^2}}{2} \geq 0 \quad (6)$$

The medium is lossless if both eigenvalues are zero, *i.e.*,  $\epsilon_1$ ,  $\mu_1$ ,  $\kappa$ , and  $\chi$  are all real-valued [19].

Of special importance in this paper is uniqueness of the interior Maxwell boundary value problem stated as

**Definition 1.** Find two vector fields  $\mathbf{E}, \mathbf{H} \in F(\mathcal{D})$  satisfying the Maxwell equations (4) in  $\mathcal{D}$  and the boundary condition  $\hat{\nu} \times \mathbf{E} = \mathbf{f}$  on  $\Gamma$ . Here,  $\mathbf{f} \in T_d^{0,\alpha}(\Gamma)$ ,  $0 < \alpha < 1$ , is a given tangential surface field.

If the biisotropic medium is lossless, this problem has a unique solution except for a discrete set of angular frequencies,  $\omega \in \sigma(\mathcal{D})$ , see *e.g.*, [33, Sec. 4.2], with the only accumulation point at infinity. For a strictly passive material, *i.e.*, both eigenvalues  $\lambda_{1,2} \geq \lambda_0 > 0$  at all points,  $\mathbf{r} \in \mathcal{D}$ , we have always uniqueness.

**Lemma 1.** *The interior Maxwell boundary-value problem in Definition 1 is uniquely solvable provided the material is strictly passive.*

**Proof.** We prove uniqueness by proving that the only solution to the interior Maxwell boundary-value problem when  $\mathbf{f} = \mathbf{0}$  is the trivial solution  $\mathbf{E} = \eta_0 \mathbf{H} = \mathbf{0}$ .

Use the Gauss' theorem in the region  $\mathcal{D}$  on the following surface integral:

$$0 = \text{Re} \iint_{\Gamma} (\hat{\nu} \times \mathbf{E}|_{-}) \cdot \eta_0 \mathbf{H}|_{-}^* \, dS = \text{Re} \iint_{\Gamma} (\mathbf{E}|_{-} \times \eta_0 \mathbf{H}|_{-}^*) \cdot \hat{\nu} \, dS = \text{Re} \iiint_{\mathcal{D}} \nabla \times (\mathbf{E} \times \eta_0 \mathbf{H}^*) \, dv$$

The Maxwell equation (4) implies

$$\begin{aligned} 0 &= k_0 \text{Im} \iiint_{\mathcal{D}} (\{(\kappa - i\chi)\mathbf{E} + \mu_1 \eta_0 \mathbf{H}\}^* \cdot \eta_0 \mathbf{H} + \{\epsilon_1 \mathbf{E} + (\kappa + i\chi)\eta_0 \mathbf{H}\}^* \cdot \mathbf{E}) \, dv \\ &= -k_0 \iiint_{\mathcal{D}} \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix}^{\dagger} \cdot \begin{pmatrix} \text{Im } \epsilon_1 & \text{Im } \kappa + i \text{Im } \chi \\ \text{Im } \kappa - i \text{Im } \chi & \text{Im } \mu_1 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix} \, dv \leq -\lambda_0 k_0 \iiint_{\mathcal{D}} |\mathbf{E}|^2 + |\eta_0 \mathbf{H}|^2 \, dv \end{aligned}$$

where  $*$  denotes the complex conjugate, and  $\dagger$  denotes the Hermitian conjugate. This inequality implies that  $\mathbf{E} = \eta_0 \mathbf{H} = \mathbf{0}$ ,  $\mathbf{r} \in \mathcal{D}$ , and the lemma is proved.  $\square$

**Lemma 2.** *If the biisotropic medium of the scatterer is passive, *i.e.*, the material parameters satisfy (6) at every point in  $\mathcal{D}$ , then*

$$\text{Re} \iint_{\Gamma} (\mathbf{E}(\mathbf{r})|_{-} \times \mathbf{H}^*(\mathbf{r})|_{-}) \cdot \hat{\nu}(\mathbf{r}) \, dS \leq 0$$

for all electric and magnetic fields satisfying the Maxwell equations.<sup>§</sup>

**Proof.** Proceed as in Lemma 1, and we obtain

$$\text{Re} \iint_{\Gamma} (\mathbf{E}|_{-} \times \eta_0 \mathbf{H}^*|_{-}) \cdot \hat{\nu} \, dS = -k_0 \iiint_{\mathcal{D}} \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix}^{\dagger} \cdot \begin{pmatrix} \text{Im } \epsilon_1 & \text{Im } \kappa + i \text{Im } \chi \\ \text{Im } \kappa - i \text{Im } \chi & \text{Im } \mu_1 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix} \, dv \leq 0$$

since the eigenvalues of the matrix in the integrand,  $\lambda_{1,2}$ , both are non-negative, if the passivity condition (6) holds.  $\square$

<sup>§</sup> Alternatively, this can be taken as the definition of a passive material. The integral inequality states that energy flows inwards into the region  $\mathcal{D}$ .

### 3 | NULL-FIELD APPROACH

The Null-Field approach was originally presented by Peter C Waterman in a series of papers [39, 40, 41, 42], see also [19, Chap. 9] for an introductory text on the approach. The method is semi-analytic, and employs global expansion functions in contrast to several numerical methods that use local expansion functions.

For  $\mathbf{E} \in F(\mathcal{D}_e)$ , the integral representation of the solution to the Maxwell equation in the exterior homogeneous region  $\mathcal{D}_e$  is [9, 15, 19]

$$\begin{aligned} -\frac{\eta_0\eta}{ik}\nabla \times \left\{ \nabla \times \iint_{\Gamma} \mathbf{G}_e(k, \mathbf{r}-\mathbf{r}') \cdot (\hat{\mathbf{v}}(\mathbf{r}') \times \mathbf{H}(\mathbf{r}')) dS' \right\} \\ + \nabla \times \iint_{\Gamma} \mathbf{G}_e(k, \mathbf{r}-\mathbf{r}') \cdot (\hat{\mathbf{v}}(\mathbf{r}') \times \mathbf{E}(\mathbf{r}')) dS' = \begin{cases} \mathbf{E}_s(\mathbf{r}), & \mathbf{r} \in \mathcal{D}_e \\ -\mathbf{E}_i(\mathbf{r}), & \mathbf{r} \in \mathcal{D} \end{cases} \quad (7) \end{aligned}$$

where the Green dyadic in free space is [19]

$$\mathbf{G}_e(k, \mathbf{r}-\mathbf{r}') = \left( \mathbf{I}_3 + \frac{1}{k^2} \nabla \nabla \right) g(k, |\mathbf{r}-\mathbf{r}'|) = \left( \mathbf{I}_3 + \frac{1}{k^2} \nabla \nabla \right) \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi |\mathbf{r}-\mathbf{r}'|}, \quad \mathbf{r} \neq \mathbf{r}' \quad (8)$$

The boundary conditions in (2) show, that the surface fields in these integrals are identical to their limit values from the inside. As a consequence, we do not need to indicate from which side the limit is taken.

The Green dyadic in free space has an expansion in spherical vector waves (see Appendix A for the definition of the radiating spherical vector waves  $\mathbf{u}_{\tau n}(k\mathbf{r})$  and the regular spherical vector waves  $\mathbf{v}_{\tau n}(k\mathbf{r})$ )

$$\mathbf{G}_e(k, \mathbf{r}-\mathbf{r}') = ik \sum_{\tau n} \mathbf{v}_{\tau n}(k\mathbf{r}_{<}) \mathbf{u}_{\tau n}(k\mathbf{r}_{>}), \quad \mathbf{r} \neq \mathbf{r}' \quad (9)$$

where  $\mathbf{r}_{<}(\mathbf{r}_{>})$  is the position vector with the smallest (largest) distance to the origin, and the index  $n$  is a multi-index  $n = \{\sigma, m, l\}$  and  $\tau = 1, 2$ , see Appendix A. The expansion (9) is uniformly convergent in compact domains, provided  $\mathbf{r} \neq \mathbf{r}'$  [16, 31].

#### 3.1 | Scattered and incident fields

We apply the integral representation (7) and (8) to the exterior region  $\mathcal{D}_e$ , and let the position vector  $\mathbf{r}$  be such that  $r = |\mathbf{r}| > r_{\max}$ . A change of summation and integration gives us an expansion of the scattered field outside the smallest circumscribed sphere of the scatterer. Thus, the scattered field has an convergent expansion in radiating spherical vector waves  $\mathbf{u}_{\tau n}(k\mathbf{r})$  outside the smallest circumscribed sphere, *i.e.*,

$$\mathbf{E}_s(\mathbf{r}) = \sum_{\tau n} f_{\tau n} \mathbf{u}_{\tau n}(k\mathbf{r}), \quad r > r_{\max} \quad (10)$$

where the expansion coefficients are

$$f_{\tau n} = ik^2 \iint_{\Gamma} i\eta_0\eta \mathbf{v}_{\tau n}(k\mathbf{r}) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})) + \mathbf{v}_{\bar{\tau}n}(k\mathbf{r}) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{E}(\mathbf{r})) dS \quad (11)$$

The dual index  $\bar{\tau}$  is defined by  $\bar{1} = 2$  and  $\bar{2} = 1$ .

We now apply the integral representation (7) to the interior region  $\mathcal{D}$ . By applying (8), we obtain a convergent expansion of the incident field inside the largest inscribed sphere of the scatterer in terms of regular spherical vector waves  $\mathbf{v}_{\tau n}(k\mathbf{r})$ , *i.e.*,

$$\mathbf{E}_i(\mathbf{r}) = \sum_{\tau n} a_{\tau n} \mathbf{v}_{\tau n}(k\mathbf{r}), \quad r < r_{\min}$$

where the expansion coefficients are

$$a_{\tau n} = -ik^2 \iint_{\Gamma} i\eta_0\eta \mathbf{u}_{\tau n}(k\mathbf{r}) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})) + \mathbf{u}_{\bar{\tau}n}(k\mathbf{r}) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{E}(\mathbf{r})) dS \quad (12)$$

The incident wave is known, *i.e.*, the expansion coefficients  $a_{\tau n}$  are known, and we seek the scattered field, *i.e.*, the expansion coefficients  $f_{\tau n}$ . Formally, this can be accomplished by eliminating the surface fields  $\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{E}(\mathbf{r})$  and  $\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})$  in (11) and (12). In this paper, we eliminate the surface field with the Null-field approach formulated by Peter C Waterman.

The transition matrix  $T_{\tau n, \tau' n'}$  is defined as the linear transformation between the expansion coefficients of the incident and scattered fields, *i.e.*,

$$f_{\tau n} = \sum_{\tau' n'} T_{\tau n, \tau' n'} a_{\tau' n'}$$

### 3.2 | Bohren transformation and Beltrami fields

We now focus on the fields inside the obstacle  $\mathcal{D}$ . To find the appropriate form of the Maxwell equations suited for a biisotropic media, we introduce the Bohren transformation [8]

$$\begin{pmatrix} \mathbf{E} \\ i\eta_0 \mathbf{H} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -Y_- & Y_+ \end{pmatrix} \begin{pmatrix} \mathbf{Q}_L \\ \mathbf{Q}_R \end{pmatrix}$$

where the (relative) material admittances  $Y_{\pm}$  of the biisotropic material are defined as

$$Y_{\pm} = \frac{(\epsilon_1 \mu_1 - \kappa^2)^{1/2} \mp i\kappa}{\mu_1}$$

The inverse of Bohren transformation is

$$\begin{pmatrix} \mathbf{Q}_L \\ \mathbf{Q}_R \end{pmatrix} = \frac{1}{Y_+ + Y_-} \begin{pmatrix} Y_+ & -1 \\ Y_- & 1 \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ i\eta_0 \mathbf{H} \end{pmatrix} \quad (13)$$

The new fields  $\mathbf{Q}_{L,R}$  are called Beltrami fields and satisfy [20, 33]

$$\begin{pmatrix} \nabla \times \mathbf{Q}_L(\mathbf{r}) \\ \nabla \times \mathbf{Q}_R(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} -k_- & 0 \\ 0 & k_+ \end{pmatrix} \begin{pmatrix} \mathbf{Q}_L(\mathbf{r}) \\ \mathbf{Q}_R(\mathbf{r}) \end{pmatrix} \quad (14)$$

where  $k_{\pm}$  are the wavenumbers of the right- and left-circularly polarized waves, respectively<sup>¶</sup>

$$k_{\pm} = k_0 \left\{ (\epsilon_1 \mu_1 - \kappa^2)^{1/2} \pm \chi \right\}$$

and, consequently, the new fields  $\mathbf{Q}_L(\mathbf{r})$  and  $\mathbf{Q}_R(\mathbf{r})$  diagonalize the Maxwell equation.

The spherical vector waves  $\mathbf{v}_{\tau n}(k\mathbf{r})$  and  $\mathbf{u}_{\tau n}(k\mathbf{r})$  for  $\tau = 1, 2$  (see Appendix A for their definitions) satisfy

$$\begin{cases} \nabla \times \mathbf{v}_{\tau n}(k\mathbf{r}) = k\mathbf{v}_{\tau n}(k\mathbf{r}) \\ \nabla \times \mathbf{u}_{\tau n}(k\mathbf{r}) = k\mathbf{u}_{\tau n}(k\mathbf{r}) \end{cases}$$

These spherical vector waves are not suited as expansion functions for a biisotropic material, since the biisotropic material is characterized by two wavenumbers  $k_{\pm}$ . Instead, we form the regular and radiating Beltrami spherical vector waves as linear combinations of  $\mathbf{v}_{\tau n}(k_{\pm}\mathbf{r})$  and  $\mathbf{u}_{\tau n}(k_{\pm}\mathbf{r})$ , *i.e.*,

$$\begin{cases} \mathbf{v}_{Ln}(\mathbf{r}) = \mathbf{v}_{1n}(k_- \mathbf{r}) - \mathbf{v}_{2n}(k_- \mathbf{r}) \\ \mathbf{v}_{Rn}(\mathbf{r}) = \mathbf{v}_{1n}(k_+ \mathbf{r}) + \mathbf{v}_{2n}(k_+ \mathbf{r}) \end{cases} \quad \begin{cases} \mathbf{u}_{Ln}(\mathbf{r}) = \mathbf{u}_{1n}(k_- \mathbf{r}) - \mathbf{u}_{2n}(k_- \mathbf{r}) \\ \mathbf{u}_{Rn}(\mathbf{r}) = \mathbf{u}_{1n}(k_+ \mathbf{r}) + \mathbf{u}_{2n}(k_+ \mathbf{r}) \end{cases} \quad (15)$$

The Beltrami spherical vector waves satisfy

$$\begin{cases} \nabla \times \mathbf{v}_{Ln}(\mathbf{r}) = -k_- \mathbf{v}_{Ln}(\mathbf{r}) \\ \nabla \times \mathbf{v}_{Rn}(\mathbf{r}) = k_+ \mathbf{v}_{Rn}(\mathbf{r}) \end{cases} \quad \begin{cases} \nabla \times \mathbf{u}_{Ln}(\mathbf{r}) = -k_- \mathbf{u}_{Ln}(\mathbf{r}) \\ \nabla \times \mathbf{u}_{Rn}(\mathbf{r}) = k_+ \mathbf{u}_{Rn}(\mathbf{r}) \end{cases} \quad (16)$$

<sup>¶</sup> The definitions of left- and right-handed waves differ between authors. In this paper, we follow [19].

The Beltrami spherical vector waves  $\{\mathbf{v}_{Ln}(\mathbf{r}), \mathbf{v}_{Rn}(\mathbf{r})\}_n$  is a complete and linearly independent set of vector waves. These statements are proved in Sections 3.3–3.4. Moreover, they satisfy the following orthogonality relation:

**Lemma 3.** *If the Beltrami fields  $\mathbf{Q}_{L,R}(\mathbf{r})$  and  $\mathbf{Q}'_{L,R}(\mathbf{r})$  satisfy (14) in the region  $\mathcal{D}$ , then the following orthogonality properties hold:*

$$\begin{cases} \iint_{\Gamma} \mathbf{Q}_L(\mathbf{r}) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{Q}'_L(\mathbf{r})) \, dS = 0 \\ \iint_{\Gamma} \mathbf{Q}_R(\mathbf{r}) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{Q}'_R(\mathbf{r})) \, dS = 0 \end{cases}$$

on the bounding surface  $\Gamma$ .

**Proof.** The lemma is a simple application Gauss' theorem and (14), i.e.,

$$\begin{aligned} \iint_{\Gamma} \mathbf{Q}_L \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{Q}'_L) \, dS &= \iiint_{\mathcal{D}} \nabla \cdot (\mathbf{Q}'_L \times \mathbf{Q}_L) \, dv = \iiint_{\mathcal{D}} \mathbf{Q}_L \cdot (\nabla \times \mathbf{Q}'_L) \, dv - \iiint_{\mathcal{D}} (\nabla \times \mathbf{Q}_L) \cdot \mathbf{Q}'_L \, dv \\ &= -k_- \iiint_{\mathcal{D}} \mathbf{Q}_L \cdot \mathbf{Q}'_L \, dv + k_- \iiint_{\mathcal{D}} \mathbf{Q}_L \cdot \mathbf{Q}'_L \, dv = 0 \end{aligned}$$

The orthogonality relation for the functions  $\mathbf{Q}_R(\mathbf{r})$  and  $\mathbf{Q}'_R(\mathbf{r})$  is proved in a similar manner.  $\square$

As a special case, we have for  $\mathbf{Q}_{L,R}(\mathbf{r}) = \mathbf{v}_{L,Rn}(\mathbf{r})$  and  $\mathbf{Q}'_{L,R}(\mathbf{r}) = \mathbf{v}_{L,Rn'}(\mathbf{r})$  the following corollary:

**Corollary 1.** *The regular Beltrami spherical vector waves  $\{\mathbf{v}_{Ln}(\mathbf{r}), \mathbf{v}_{Rn}(\mathbf{r})\}_n$  satisfy the following orthogonality property:*

$$\begin{cases} \iint_{\Gamma} \mathbf{v}_{Ln}(\mathbf{r}) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r})) \, dS = 0 \\ \iint_{\Gamma} \mathbf{v}_{Rn}(\mathbf{r}) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) \, dS = 0 \end{cases} \quad (17)$$

for all closed surfaces  $\Gamma$ , and all indices  $n$  and  $n'$ . The location of the origin is arbitrary.

### 3.2.1 | Green dyadic in biisotropic media

The appropriate Green dyadic in a biisotropic material satisfies

$$\nabla \times (\nabla \times \mathbf{G}_{bi}(\mathbf{r} - \mathbf{r}')) + \alpha \nabla \times \mathbf{G}_{bi}(\mathbf{r} - \mathbf{r}') - \beta^2 \mathbf{G}_{bi}(\mathbf{r} - \mathbf{r}') = \mathbf{I}_3 \delta(\mathbf{r} - \mathbf{r}')$$

where  $\alpha = -2k_0\chi$  and  $\beta^2 = k_0^2 (\epsilon_1\mu_1 - \kappa^2 - \chi^2)$ , and  $\mathbf{I}_3$  denotes the unit dyadic in  $\mathbb{R}^3$ . The solution is given by [20]

$$\mathbf{G}_{bi}(\mathbf{r} - \mathbf{r}') = \mathbf{G}_{bi}^+(\mathbf{r} - \mathbf{r}') + \mathbf{G}_{bi}^-(\mathbf{r} - \mathbf{r}') \quad (18)$$

where the decomposed parts read

$$\mathbf{G}_{bi}^{\pm}(\mathbf{r} - \mathbf{r}') = \frac{(k_{\pm} \mathbf{I}_3 \pm \nabla \times \mathbf{I}_3) \cdot \mathbf{G}_e(k_{\pm}, \mathbf{r} - \mathbf{r}')}{k_+ + k_-} \quad (19)$$

where  $\mathbf{G}_e(k_{\pm}, \mathbf{r} - \mathbf{r}')$  is given in (8).

The Green dyadic  $\mathbf{G}_{bi}^{\pm}$  has a decomposition in spherical and Beltrami spherical vector waves. By the use of (9) and (15), we get

$$\mathbf{G}_{bi}^{\pm}(\mathbf{r} - \mathbf{r}') = \frac{ik_{\pm}^2}{k_+ + k_-} \sum_n \mathbf{v}_{R,Ln}(\mathbf{r}_<) \mathbf{u}_{R,Ln}(\mathbf{r}_>), \quad \mathbf{r} \neq \mathbf{r}' \quad (20)$$

where the  $+$ ( $-$ ) sign corresponds to R(L) indices. This expansion is uniformly convergent in compact domains, provided  $\mathbf{r} \neq \mathbf{r}'$ .

### 3.3 | Completeness proof

The completeness of the regular and outgoing spherical vector waves,  $\{\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{\tau n}(\mathbf{r})\}_{\tau n}$  and  $\{\hat{\nu}(\mathbf{r}) \times \mathbf{u}_{\tau n}(\mathbf{r})\}_{\tau n}$ , as well as the scalar spherical waves on a closed boundary  $\Gamma$  is well documented [6, 18, 29, 30]. In general, the radiating functions are complete for all wavenumbers, while for the regular waves, we have to exclude a set of wavenumbers related to the eigenvalues of a specific interior Maxwell boundary value problem. For Beltrami spherical vector waves, we have the following result:

**Theorem 1.** *The regular Beltrami spherical vector waves  $\{\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L,n}(\mathbf{r}), \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R,n}(\mathbf{r})\}_n$  form a complete system in the space of square integrable tangential functions  $L_t^2(\Gamma)$ , except for a discrete set of eigenfrequencies  $\sigma(\mathcal{D})$  that corresponds to the eigenfrequencies of the cavity problem with perfectly conducting boundary, see Definition 1.*

**Proof.** We prove this theorem by showing that the system is closed, i.e., for  $\mathbf{f} \in L_t^2(\Gamma)$ , we have [11, Th. 11.1.7] (the definitions of complete and closed system are given in Appendix B)

$$\iint_{\Gamma} (\hat{\nu}(\mathbf{r}') \times \mathbf{v}_{L,R,n}(\mathbf{r}')) \cdot \mathbf{f}(\mathbf{r}') \, dS' = 0, \quad \text{for L, R, and for all } n \Rightarrow \mathbf{f} = \mathbf{0}$$

To prove the theorem it suffices to work with surface fields that are dense in  $L_t^2(\Gamma)$ , e.g., continuous, tangential functions  $\mathcal{C}_t(\Gamma) = \{\mathbf{f} \in (C(\Gamma))^3 : \mathbf{f} \cdot \hat{\nu} = 0 \text{ on } \Gamma\}$ , and use the continuity of the scalar product.

Introduce the layer potentials  $\mathbf{M}(k)$  and  $\mathbf{N}(k)$  [17, 33]

$$(\mathbf{M}(k)\mathbf{a})(\mathbf{r}) := \nabla \times \left\{ \iint_{\Gamma} g(k, |\mathbf{r} - \mathbf{r}'|) \mathbf{a}(\mathbf{r}') \, dS' \right\} = \nabla \times \left\{ \iint_{\Gamma} \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') \cdot \mathbf{a}(\mathbf{r}') \, dS' \right\}, \quad \mathbf{r} \in \mathcal{D} \cup \mathcal{D}_e$$

and

$$\begin{aligned} (\mathbf{N}(k)\mathbf{a})(\mathbf{r}) &:= \nabla \times \left\{ \nabla \times \left( \iint_{\Gamma} g(k, |\mathbf{r} - \mathbf{r}'|) \mathbf{a}(\mathbf{r}') \, dS' \right) \right\} \\ &= \nabla \times \left\{ \nabla \times \left( \iint_{\Gamma} \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') \cdot \mathbf{a}(\mathbf{r}') \, dS' \right) \right\} = k^2 \iint_{\Gamma} \mathbf{G}_e(k, \mathbf{r} - \mathbf{r}') \cdot \mathbf{a}(\mathbf{r}') \, dS', \quad \mathbf{r} \in \mathcal{D} \cup \mathcal{D}_e \end{aligned}$$

where the Green dyadic in free space,  $\mathbf{G}_e(k, \mathbf{r} - \mathbf{r}')$ , is given by (8).

We have

$$\nabla \times (\mathbf{M}(k)\mathbf{a})(\mathbf{r}) = (\mathbf{N}(k)\mathbf{a})(\mathbf{r}), \quad \nabla \times (\mathbf{N}(k)\mathbf{a})(\mathbf{r}) = k^2 (\mathbf{M}(k)\mathbf{a})(\mathbf{r}), \quad \mathbf{r} \in \mathcal{D} \cup \mathcal{D}_e$$

We also need two boundary integral operators (principal values) [33]

$$(\tilde{\mathbf{M}}(k)\mathbf{a})(\mathbf{r}) := \hat{\nu}(\mathbf{r}) \times \iint_{\Gamma \setminus \{\mathbf{r}\}} \nabla g(k, |\mathbf{r} - \mathbf{r}'|) \times \mathbf{a}(\mathbf{r}') \, dS', \quad \mathbf{r} \in \Gamma$$

and

$$(\tilde{\mathbf{N}}(k)\mathbf{a})(\mathbf{r}) := \hat{\nu}(\mathbf{r}) \times \left\{ \nabla \iint_{\Gamma} g(k, |\mathbf{r} - \mathbf{r}'|) \text{Div} \mathbf{a}(\mathbf{r}') \, dS' + k^2 \iint_{\Gamma} g(k, |\mathbf{r} - \mathbf{r}'|) \mathbf{a}(\mathbf{r}') \, dS' \right\}, \quad \mathbf{r} \in \Gamma$$

where Div denotes the surface divergence of  $\Gamma$ , and we have the limit values [33]

$$\begin{cases} \hat{\nu}(\mathbf{r}) \times (\mathbf{M}(k)(\mathbf{a}))(\mathbf{r}) = (\tilde{\mathbf{M}}(k)(\mathbf{a}))(\mathbf{r}) \pm \frac{1}{2} \mathbf{a}(\mathbf{r}) \\ \hat{\nu}(\mathbf{r}) \times (\mathbf{N}(k)(\mathbf{a}))(\mathbf{r}) = (\tilde{\mathbf{N}}(k)(\mathbf{a}))(\mathbf{r}) \end{cases} \quad \mathbf{r} \in \Gamma$$

where the upper(lower) sign holds for the limit from the outside(inside) of the boundary  $\Gamma$ . The boundary layer operators map [9]

$$\tilde{\mathbf{M}}(k), \tilde{\mathbf{N}}(k) : T(\Gamma) \rightarrow T^{0,\alpha}(\Gamma)$$

where  $0 < \alpha < 1$ .



We now prove that the system  $\{\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Ln}(\mathbf{r}), \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Rn}(\mathbf{r})\}_n$  is closed, *i.e.*, for  $\mathbf{f} \in \mathcal{C}_t(\Gamma)$ .

$$\iint_{\Gamma} (\hat{\nu}(\mathbf{r}') \times \mathbf{v}_{L,Rn}(\mathbf{r}')) \cdot \mathbf{f}(\mathbf{r}') \, dS' = 0, \quad \text{for } L, R, \text{ and for all } n \quad (21)$$

and we prove that the only solution to this statement is  $\mathbf{f} = \mathbf{0}$ .

Let  $\mathbf{r}$  be located outside the smallest circumscribed sphere, and multiply the integral in (21) containing  $\mathbf{v}_{Ln}(\mathbf{r}')$  by  $k_-^2 \mathbf{v}_{Ln}(\mathbf{r})$ , the integral containing  $\mathbf{v}_{Rn}(\mathbf{r}')$  by  $k_+^2 \mathbf{u}_{Rn}(\mathbf{r})$ , add and sum over  $n$ . We get, see (18) and (20)

$$\mathbf{E}(\mathbf{r}) = \iint_{\Gamma} \mathbf{G}_{bi}(\mathbf{r} - \mathbf{r}') \cdot \mathbf{a}(\mathbf{r}') \, dS' = \mathbf{0}, \quad r > r_{max} \quad (22)$$

where  $\mathbf{a} = \hat{\nu} \times \mathbf{f}$ . The function  $\mathbf{E}(\mathbf{r})$  satisfies (5) outside the smallest circumscribed sphere of  $\Gamma$ , and by analyticity everywhere in  $\mathcal{D}_e$  [12]. The limits from the outside are

$$\mathbf{E}(\mathbf{r})|_+ = \mathbf{0}, \quad \hat{\nu}(\mathbf{r}) \times \mathbf{E}(\mathbf{r})|_+ = \mathbf{0}, \quad \hat{\nu}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})|_+ = 0$$

We express the field  $\mathbf{E}$  in (22) in terms of the layer potentials  $\mathbf{N}(k)$  and  $\mathbf{M}(k)$  in the exterior region  $\mathcal{D}_e$ , *i.e.*,

$$\mathbf{E}_e(\mathbf{r}) = \mathbf{E}(\mathbf{r}) = \frac{1}{k_+ + k_-} \left\{ \frac{1}{k_+} (\mathbf{N}(k_+) \mathbf{a})(\mathbf{r}) + \frac{1}{k_-} (\mathbf{N}(k_-) \mathbf{a})(\mathbf{r}) + (\mathbf{M}(k_+) \mathbf{a})(\mathbf{r}) - (\mathbf{M}(k_-) \mathbf{a})(\mathbf{r}) \right\} = \mathbf{0}, \quad \mathbf{r} \in \mathcal{D}_e$$

The limit value at the boundary (limit from the outside) is

$$\hat{\nu}(\mathbf{r}) \times \mathbf{E}_e(\mathbf{r})|_+ = \frac{1}{k_+ + k_-} \left\{ \frac{1}{k_+} (\tilde{\mathbf{N}}(k_+) \mathbf{a})(\mathbf{r}) + \frac{1}{k_-} (\tilde{\mathbf{N}}(k_-) \mathbf{a})(\mathbf{r}) + (\tilde{\mathbf{M}}(k_+) \mathbf{a})(\mathbf{r}) - (\tilde{\mathbf{M}}(k_-) \mathbf{a})(\mathbf{r}) \right\} = \mathbf{0}, \quad \mathbf{r} \in \Gamma \quad (23)$$

The function  $\mathbf{E}(\mathbf{r})$  in (22) is also well defined inside the obstacle  $\mathcal{D}$ , but its value is not known. Define the field in  $\mathcal{D}$

$$\mathbf{E}_i(\mathbf{r}) = \frac{1}{k_+ + k_-} \left\{ \frac{1}{k_+} (\mathbf{N}(k_+) \mathbf{a})(\mathbf{r}) + \frac{1}{k_-} (\mathbf{N}(k_-) \mathbf{a})(\mathbf{r}) + (\mathbf{M}(k_+) \mathbf{a})(\mathbf{r}) - (\mathbf{M}(k_-) \mathbf{a})(\mathbf{r}) \right\}, \quad \mathbf{r} \in \mathcal{D}$$

with limit value at the boundary (limit from the inside)

$$\hat{\nu}(\mathbf{r}) \times \mathbf{E}_i(\mathbf{r})|_- = \frac{1}{k_+ + k_-} \left\{ \frac{1}{k_+} (\tilde{\mathbf{N}}(k_+) \mathbf{a})(\mathbf{r}) + \frac{1}{k_-} (\tilde{\mathbf{N}}(k_-) \mathbf{a})(\mathbf{r}) + (\tilde{\mathbf{M}}(k_+) \mathbf{a})(\mathbf{r}) - (\tilde{\mathbf{M}}(k_-) \mathbf{a})(\mathbf{r}) \right\}, \quad \mathbf{r} \in \Gamma$$

By the use of (23), we obtain  $\hat{\nu}(\mathbf{r}) \times \mathbf{E}_i|_-(\mathbf{r}) = \mathbf{0}$ . If we exclude the frequencies of the internal cavity problem with PEC boundary,  $\sigma(\mathcal{D})$ , see Definition 1, we have a unique solution of the internal problem, *i.e.*,  $\mathbf{E}_i(\mathbf{r}) = \mathbf{0}$ ,  $\mathbf{r} \in \mathcal{D}$ .

Form the curl of the fields  $\mathbf{E}_{e,i}(\mathbf{r})$ , which are zero in  $\mathbf{r} \in \mathcal{D} \cup \mathcal{D}_e$ .

$$\nabla \times \mathbf{E}_{e,i}(\mathbf{r}) = \frac{1}{k_+ + k_-} \left\{ k_+ (\mathbf{M}(k_+) \mathbf{a})(\mathbf{r}) + k_- (\mathbf{M}(k_-) \mathbf{a})(\mathbf{r}) + (\mathbf{N}(k_+) \mathbf{a})(\mathbf{r}) - (\mathbf{N}(k_-) \mathbf{a})(\mathbf{r}) \right\} = \mathbf{0}, \quad \mathbf{r} \in \mathcal{D} \cup \mathcal{D}_e$$

The tangential limit values of the vector field  $\nabla \times \mathbf{E}_{e,i}(\mathbf{r})$  on the surface leads to the following boundary integral equations:

$$\begin{aligned} \hat{\nu}(\mathbf{r}) \times (\nabla \times \mathbf{E}_{e,i}(\mathbf{r}))|_{\pm} &= \frac{1}{k_+ + k_-} \left\{ k_+ (\tilde{\mathbf{M}}(k_+) \mathbf{a})(\mathbf{r}) + k_- (\tilde{\mathbf{M}}(k_-) \mathbf{a})(\mathbf{r}) \right. \\ &\quad \left. + (\tilde{\mathbf{N}}(k_+) \mathbf{a})(\mathbf{r}) - (\tilde{\mathbf{N}}(k_-) \mathbf{a})(\mathbf{r}) \pm \frac{k_+ + k_-}{2} \mathbf{a}(\mathbf{r}) \right\} = \mathbf{0}, \quad \mathbf{r} \in \Gamma \end{aligned}$$

from the outside(inside), respectively. The difference of these two boundary integral equations is zero, and gives the value of  $\mathbf{a} = \hat{\nu} \times \mathbf{f} = \mathbf{0}$ , and the proof of closure is completed, which also implies completeness [11].  $\square$

### 3.4 | Linear independence

In this section, we prove the linear independence of the set  $\{\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ll}(\mathbf{r}), \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rl}(\mathbf{r})\}_n$ .

**Lemma 4.** *The set of tangential components of the regular Beltrami spherical vector waves  $\{\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ll}(\mathbf{r}), \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rl}(\mathbf{r})\}_n$  is a linear independent set of functions on  $\Gamma$  except for a discrete set of eigenfunctions that corresponds to the eigenfrequencies of the cavity problem with perfectly conducting boundary, see Definition 1.*

**Proof.** To prove this theorem, consider

$$\sum_{\sigma=e,o} \sum_{l=0}^{l_{\max}} \sum_{m=0}^l \{c_{L\sigma ml} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{L\sigma ml}(\mathbf{r}) + c_{R\sigma ml} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{R\sigma ml}(\mathbf{r})\} = \mathbf{0}, \quad \mathbf{r} \in \Gamma$$

We have to prove that  $c_{L\sigma ml} = c_{R\sigma ml} = 0$  for all indices  $\{\sigma, m, l\}$  and all  $l_{\max} \geq 0$ . Form the vector field

$$\mathbf{E}(\mathbf{r}) = \sum_{\sigma=e,o} \sum_{l=0}^{l_{\max}} \sum_{m=0}^l \left\{ c_{L\sigma ml} (\mathbf{v}_{1\sigma ml}(k_- \mathbf{r}) - \mathbf{v}_{2\sigma ml}(k_- \mathbf{r})) + c_{R\sigma ml} (\mathbf{v}_{1\sigma ml}(k_+ \mathbf{r}) + \mathbf{v}_{2\sigma ml}(k_+ \mathbf{r})) \right\}, \quad \mathbf{r} \in \mathcal{D}$$

This function is well defined, and it satisfies the electric field equation (5) in region  $\mathcal{D}$  (no convergence problems since the sum is finite). By construction, its tangential components,  $\hat{\mathbf{v}} \times \mathbf{E}$ , vanish on  $\Gamma$ . Due to uniqueness of the interior problem, we have  $\mathbf{E} = \mathbf{0}$  everywhere in  $\mathcal{D}$ . In particular, the vector field  $\mathbf{E}$  vanishes on a spherical surface  $S_R$  of radius  $R \leq r_{\min}$  centered at the origin  $O$ . Orthogonality of the vector spherical harmonics  $\mathbf{A}_{\tau n}(\hat{\mathbf{r}})$ , see Appendix A, on the sphere  $S_R$  implies

$$\begin{aligned} 0 &= \iint_{S_R} |\mathbf{E}(\mathbf{r})|^2 dS = \sum_{\sigma=e,o} \sum_{l=0}^{l_{\max}} \sum_{m=0}^l \left| c_{R\sigma ml} j_l(k_+ R) + c_{L\sigma ml} j_l(k_- R) \right|^2 \\ &\quad + \sum_{\sigma=e,o} \sum_{l=0}^{l_{\max}} \sum_{m=0}^l \left| c_{R\sigma ml} \frac{(k_+ R j_l(k_+ R))'}{k_+ R} - c_{L\sigma ml} \frac{(k_- R j_l(k_- R))'}{k_- R} \right|^2 + \sum_{\sigma=e,o} \sum_{l=0}^{l_{\max}} \sum_{m=0}^l l(l+1) \left| c_{R\sigma ml} \frac{j_l(k_+ R)}{k_+ R} - c_{L\sigma ml} \frac{j_l(k_- R)}{k_- R} \right|^2 \end{aligned}$$

Each term in the sum is positive and must therefore be identically zero. We get the system of equations

$$\begin{cases} c_{R\sigma ml} j_l(k_+ R) + c_{L\sigma ml} j_l(k_- R) = 0 \\ c_{R\sigma ml} \frac{(k_+ R j_l(k_+ R))'}{k_+ R} - c_{L\sigma ml} \frac{(k_- R j_l(k_- R))'}{k_- R} = 0 \\ c_{R\sigma ml} \frac{j_l(k_+ R)}{k_+ R} - c_{L\sigma ml} \frac{j_l(k_- R)}{k_- R} = 0 \end{cases}$$

or in a reduced form

$$\begin{cases} c_{R\sigma ml} j_l(k_+ R) + c_{L\sigma ml} j_l(k_- R) = 0 \\ c_{R\sigma ml} j_l'(k_+ R) - c_{L\sigma ml} j_l'(k_- R) = 0 \end{cases}$$

which has a unique solution  $c_{L\sigma ml} = c_{R\sigma ml} = 0$  for all indices  $n = \{\sigma, m, l\}$  and all  $l_{\max} \geq 0$ , provided

$$j_l(k_+ R) j_l'(k_- R) + j_l'(k_+ R) j_l(k_- R) \neq 0 \text{ for all } l$$

which is true for a sufficiently small value of  $R$ , and the lemma is proved.  $\square$

### 3.5 | Elimination of the surface fields

The elimination of the surface fields in (11) and (12) can be made by an expansion of the surface fields of the scatterer in a complete set of vector-valued expansion functions on the surface  $\Gamma$ . Here, we employ the set  $\{\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ll}(\mathbf{r}), \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rl}(\mathbf{r})\}_n$ , where the wavenumbers of the material in the obstacle are  $k_{\pm}$ . This is a complete system in  $L^2_t(\Gamma)$ , see Theorem 1, and we use

the following expansion of the surface fields on the inside of the scatterer:

$$\begin{cases} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{E}|_{-}(\mathbf{r}) = \sum_{n'} (\alpha_{Ln'} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r}) + \alpha_{Rn'} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) \\ i\eta_0 \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{H}|_{-}(\mathbf{r}) = \sum_{n'} (-\alpha_{Ln'} Y_- \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r}) + \alpha_{Rn'} Y_+ \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) \end{cases} \quad \mathbf{r} \in \Gamma \quad (24)$$

Note that these expansions hold in  $L^2_t(\Gamma)$  on the surface  $\Gamma$ , and that, in general, there is no guarantee that the expansions converge or represent the field in some other way in the domain inside the scatterer. Also, notice that both the expansions of the tangential electric and magnetic fields have the same expansion coefficients. This is in general not true, but for this particular choice of expansion functions,  $\{\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln}(\mathbf{r}), \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn}(\mathbf{r})\}_n$ , it is possible, see Appendix C.

We proceed by introducing the  $R$  and  $Q$  matrices of the obstacle. The explicit form of the matrices are:

$$\begin{cases} R_{\tau n, Ln'} = ik^2 \iint_{\Gamma} (\eta Y_- \mathbf{v}_{\tau n}(k\mathbf{r}) - \mathbf{v}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r})) \, dS \\ R_{\tau n, Rn'} = -ik^2 \iint_{\Gamma} (\eta Y_+ \mathbf{v}_{\tau n}(k\mathbf{r}) + \mathbf{v}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) \, dS \end{cases} \quad (25)$$

and

$$\begin{cases} Q_{\tau n, Ln'} = ik^2 \iint_{\Gamma} (\eta Y_- \mathbf{u}_{\tau n}(k\mathbf{r}) - \mathbf{u}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r})) \, dS \\ Q_{\tau n, Rn'} = -ik^2 \iint_{\Gamma} (\eta Y_+ \mathbf{u}_{\tau n}(k\mathbf{r}) + \mathbf{u}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) \, dS \end{cases} \quad (26)$$

These matrices contain the geometry and the material parameters of the scatterer. Once the geometry and the material parameters of the particle are defined, these matrices can be computed by numerical integration. They contain no unknown fields. For a spherical particle, the integrals have an exact expression [19].

Insert the surface field expansions (24) in (11) and (12) and use (25) and (26). We readily obtain the following relations:

$$\begin{cases} a_{\tau n} = \sum_{n'} \{ Q_{\tau n, Ln'} \alpha_{Ln'} + Q_{\tau n, Rn'} \alpha_{Rn'} \} \\ f_{\tau n} = - \sum_{n'} \{ R_{\tau n, Ln'} \alpha_{Ln'} + R_{\tau n, Rn'} \alpha_{Rn'} \} \end{cases}$$

Formal elimination of the expansion coefficients  $\alpha_{Ln'}$  and  $\alpha_{Rn'}$  gives the transition matrix  $T_{\tau n \tau' n'}$ . As matrices in the indices  $\{n, n'\}$  and column vectors in the index  $n$  and  $n'$ , we write

$$\begin{cases} a_1 = Q_{1L} \alpha_L + Q_{1R} \alpha_R \\ a_2 = Q_{2L} \alpha_L + Q_{2R} \alpha_R \end{cases} \quad \begin{cases} f_1 = -R_{1L} \alpha_L - R_{1R} \alpha_R \\ f_2 = -R_{2L} \alpha_L - R_{2R} \alpha_R \end{cases} \quad (27)$$

From these expressions it is straightforward to express the expansion coefficients  $\alpha_L$  and  $\alpha_R$  in terms of the known expansion coefficients  $a_{\tau}$  and combinations of the  $Q_{\tau L}$  and  $Q_{\tau R}$  matrices and their inverses. However, here, we prefer to solve for another linear combination of the coefficients  $\alpha_L$  and  $\alpha_R$ , which connects more directly to the analysis of the Null-field approach for an isotropic particle.

Introduce the following combinations of the  $Q_{\tau L}$  and  $Q_{\tau R}$  matrices

$$Q_{\tau 1} = \frac{1}{2} (Q_{\tau R} + Q_{\tau L}), \quad Q_{\tau 2} = \frac{1}{2} (Q_{\tau R} - Q_{\tau L})$$

or

$$Q_{\tau L} = Q_{\tau 1} - Q_{\tau 2}, \quad Q_{\tau R} = Q_{\tau 1} + Q_{\tau 2}$$

where the matrices  $Q_{\tau n, \tau' n'}$  have the form

$$\begin{cases} Q_{\tau n, 1n'} = -\frac{ik^2}{2} \iint_{\Gamma} (\eta Y_+ \mathbf{u}_{\tau n}(k\mathbf{r}) + \mathbf{u}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) \, dS + \frac{ik^2}{2} \iint_{\Gamma} (\eta Y_- \mathbf{u}_{\tau n}(k\mathbf{r}) - \mathbf{u}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r})) \, dS \\ Q_{\tau n, 2n'} = -\frac{ik^2}{2} \iint_{\Gamma} (\eta Y_+ \mathbf{u}_{\tau n}(k\mathbf{r}) + \mathbf{u}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) \, dS - \frac{ik^2}{2} \iint_{\Gamma} (\eta Y_- \mathbf{u}_{\tau n}(k\mathbf{r}) - \mathbf{u}_{\bar{\tau} n}(k\mathbf{r})) \cdot (\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r})) \, dS \end{cases} \quad (28)$$

Similar expressions exist for the regular matrices  $R_{\tau L}$  and  $R_{\tau R}$  — replace  $\mathbf{u}_{\tau n}(k\mathbf{r})$  by  $\mathbf{v}_{\tau n}(k\mathbf{r})$ . The second index, 1 and 2, in these expressions does not play the same role as a  $\tau$  index, but the index becomes identical to the  $\tau$  index if we specialize the material parameters to be isotropic.

With these new  $Q$  matrices, equation (27) reads

$$\begin{cases} a_1 = (Q_{11} - Q_{12}) \alpha_L + (Q_{11} + Q_{12}) \alpha_R = Q_{11} (\alpha_R + \alpha_L) + Q_{12} (\alpha_R - \alpha_L) \\ a_2 = (Q_{21} - Q_{22}) \alpha_L + (Q_{21} + Q_{22}) \alpha_R = Q_{21} (\alpha_R + \alpha_L) + Q_{22} (\alpha_R - \alpha_L) \end{cases}$$

with inverse

$$\begin{pmatrix} \alpha_R + \alpha_L \\ \alpha_R - \alpha_L \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}^{-1} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Insert in the expression for  $f_{\tau n}$ , and we get

$$f_{\tau} = -R_{\tau L} \alpha_L - R_{\tau R} \alpha_R = -(R_{\tau 1} - R_{\tau 2}) \alpha_L - (R_{\tau 1} + R_{\tau 2}) \alpha_R = -R_{\tau 1} (\alpha_R + \alpha_L) - R_{\tau 2} (\alpha_R - \alpha_L)$$

which implies an explicit expression of the transition matrix

$$T_{\tau \tau'} = - \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}^{-1} \Big|_{\tau \tau'}$$

This is the final expression of the transition matrix expressed in terms of the known matrices  $R$  and  $Q$ .

## 4 | NUMERICAL ILLUSTRATIONS

We present a few examples of the results in this paper by illustrating the scattering cross section of a biisotropic oblate spheroid and cylinder. Several calculations, using other methods, have been published in the literature, *e.g.*, [7, 10, 21].

For axially symmetric particles the surface integrals in the  $Q$  and  $R$  matrices simplify to a line integral, and, due to axial symmetry, all matrices are diagonal in the  $m$  index.

The incident field  $\mathbf{E}_i(\mathbf{r})$  in this section is a plane wave, *i.e.*,

$$\mathbf{E}_i(\mathbf{r}) = \mathbf{E}_0 e^{ik\hat{\mathbf{k}}_i \cdot \mathbf{r}}$$

where the direction of the incident wave is denoted  $\hat{\mathbf{k}}_i$ , and specified by the spherical angles  $\alpha$  and  $\beta$ , *i.e.*,  $\hat{\mathbf{k}}_i = \sin \alpha \cos \beta \hat{\mathbf{x}} + \sin \alpha \sin \beta \hat{\mathbf{y}} + \cos \alpha \hat{\mathbf{z}}$ . The complex vector  $\mathbf{E}_0$  characterizes the polarization state of the incident wave. The expansion coefficients  $a_{\tau n}$  have a closed form expression [19].

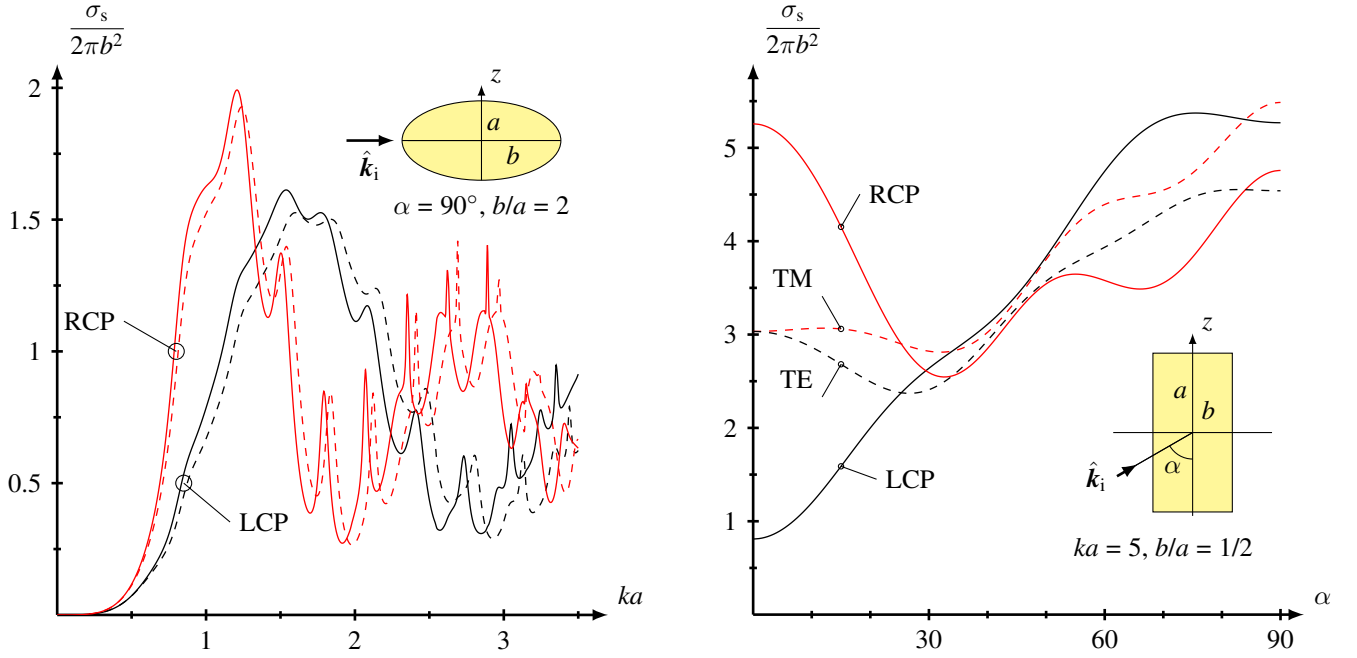
$$a_{\tau n} = 4\pi i^{l+1-\tau} \mathbf{A}_{\tau n}(\hat{\mathbf{k}}_i) \cdot \mathbf{E}_0, \quad \tau = 1, 2$$

The scattering cross section,  $\sigma_s(\hat{\mathbf{k}}_i)$ , is defined as [19]

$$\sigma_s(\hat{\mathbf{r}}, \hat{\mathbf{k}}_i) = \frac{1}{k^2 |\mathbf{E}_0|^2} \sum_{\tau n} |f_{\tau n}|^2$$

where  $f_{\tau n}$  are the expansion coefficients of the scattered field, see (10).

In the left figure in Figure 2, the normalized scattering cross section,  $\sigma_s/2\pi b^2$ , for a lossless, biisotropic oblate spheroid,  $b/a = 2$  ( $a$  along the symmetry axis  $z$ ), as a function of frequency,  $ka$ , for two different reciprocity values is depicted. The material



**FIGURE 2** **Left:** The normalized scattering cross section  $\sigma_s/2\pi b^2$  of a biisotropic oblate spheroid as a function of frequency,  $ka$ , for different incident circular polarization. The solid curves show  $\kappa/\sqrt{\epsilon\mu} = 0.5$  and the dashed curves  $\kappa/\sqrt{\epsilon\mu} = 0$ . **Right:** The normalized scattering cross section  $\sigma_s/2\pi b^2$  of a chiral cylinder as a function of the incident angle  $\alpha$  for different polarizations of the incident plane wave. The material parameters are given in the text.

parameters of the oblate spheroid are  $\epsilon_1/\epsilon = 4$ ,  $\mu_1/\mu = 1$ , and  $\chi/\sqrt{\epsilon\mu} = 0.2$  and the reciprocity parameters are  $\kappa/\sqrt{\epsilon\mu} = 0, 0.5$ . The incident plane wave is circularly polarized and the incident angle is  $\alpha = 90^\circ$  w.r.t. the symmetry axis  $z$ . The main effect of the variation of the reciprocity parameter  $\kappa$  is a frequency shift of the curves.

Scattering by a cylindrical particle is presented in the right figure in Figure 2. In this figure, we illustrate the normalized scattering cross section,  $\sigma_s/2\pi b^2$ , as a function of the incident angle  $\alpha$  for different states of polarization of the incident plane wave. The chiral material of the scatterer is  $\epsilon_1/\epsilon = 4$ ,  $\mu_1/\mu = 1$ ,  $\chi/\sqrt{\epsilon\mu} = 0.1$ , and  $\kappa = 0$ , and the geometry is  $b/a = 1/2$  and  $ka = 5$ . At small incident angles  $\alpha$ , the two circular polarizations, RCP and LCP, differ more than the two linear polarizations, TE and TM, which are identical at  $\alpha = 0^\circ$ .

## 5 | DISCUSSION AND CONCLUSIONS

In this paper, the transition matrix of a biisotropic particle is constructed by the Null-field approach. The particle is assumed to consist of a passive biisotropic material, *i.e.*, we allow both chirality and non-reciprocity of the material. The condition for passivity of the material is discussed. The Bohren transformation is employed, and the Beltrami combinations of the spherical vector waves are analyzed. In particular, we showed the completeness and linear independence of the tangential components of the regular Beltrami spherical vector waves for a general smooth boundary  $\Gamma$ , if we exclude frequencies in the set  $\sigma(\mathcal{D})$  of internal resonances. We also show that the expansion coefficients of the electric and the magnetic surface fields are the same for the special choice of expansion functions used in this paper. A few numerical examples are presented and discussed.

## APPENDIX

### A SPHERICAL VECTOR WAVES

We follow the definitions of the vector spherical harmonics and the spherical vector waves in Reference [19]. The vector spherical harmonics used in this paper are

$$\begin{cases} \mathbf{A}_{1\sigma ml}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} \nabla \times (\mathbf{r} Y_{\sigma ml}(\hat{\mathbf{r}})) = \frac{1}{\sqrt{l(l+1)}} \nabla Y_{\sigma ml}(\hat{\mathbf{r}}) \times \mathbf{r} \\ \mathbf{A}_{2\sigma ml}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(l+1)}} r \nabla Y_{\sigma ml}(\hat{\mathbf{r}}) \\ \mathbf{A}_{3\sigma ml}(\hat{\mathbf{r}}) = \hat{\mathbf{r}} Y_{\sigma ml}(\hat{\mathbf{r}}) \end{cases}$$

The spherical harmonics are

$$Y_{\sigma ml}(\theta, \phi) = C_{lm} P_l^m(\cos \theta) \begin{cases} \cos m\phi \\ \sin m\phi \end{cases}$$

where the indices  $\sigma, m, l$  take the following values:

$$\sigma = \begin{cases} \text{e} \\ \text{o} \end{cases}, \quad m = 0, 1, 2, \dots, l, \quad l = 0, 1, \dots$$

The normalization factor  $C_{lm}$  used in this paper is

$$C_{lm} = \sqrt{\frac{\varepsilon_m}{2\pi}} \sqrt{\frac{2l+1}{2} \frac{(l-m)!}{(l+m)!}}$$

where the Neumann factor is defined as

$$\varepsilon_m = 2 - \delta_{m0}, \text{ i.e., } \begin{cases} \varepsilon_0 = 1 \\ \varepsilon_m = 2, \quad m > 0 \end{cases}$$

The vector spherical harmonics are orthonormal on the unit sphere  $\Omega$ , i.e.,

$$\iint_{\Omega} \mathbf{A}_{\tau n}(\hat{\mathbf{r}}) \cdot \mathbf{A}_{\tau' n'}(\hat{\mathbf{r}}) d\Omega = \delta_{\tau\tau'} \delta_{nn'}$$

where the multi-index  $n = \{\sigma, m, l\}$  is over  $\sigma = \text{e}, \text{o}$ ,  $m = 0, 1, 2, \dots, l$ , and  $l = 1, 2, 3, \dots$

The outgoing (or radiating) spherical vector waves  $\mathbf{u}_{\tau n}(kr)$  are

$$\begin{cases} \mathbf{u}_{1n}(kr) = h_l^{(1)}(kr) \mathbf{A}_{1n}(\hat{\mathbf{r}}) \\ \mathbf{u}_{2n}(kr) = \frac{(kr h_l^{(1)}(kr))'}{kr} \mathbf{A}_{2n}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{h_l^{(1)}(kr)}{kr} \mathbf{A}_{3n}(\hat{\mathbf{r}}) \end{cases}$$

and regular spherical vector waves  $\mathbf{v}_{\tau n}(kr)$  are

$$\begin{cases} \mathbf{v}_{1n}(kr) = j_l(kr) \mathbf{A}_{1n}(\hat{\mathbf{r}}) \\ \mathbf{v}_{2n}(kr) = \frac{(kr j_l(kr))'}{kr} \mathbf{A}_{2n}(\hat{\mathbf{r}}) + \sqrt{l(l+1)} \frac{j_l(kr)}{kr} \mathbf{A}_{3n}(\hat{\mathbf{r}}) \end{cases}$$

where  $h_l^{(1)}(z)$  and  $j_l(z)$  are the spherical Hankel and Bessel functions of order  $l$ , respectively. The spherical vector waves  $\mathbf{v}_{\tau n}(kr)$  for  $\tau = 1, 2$  satisfy

$$\begin{cases} \nabla \times \mathbf{v}_{\tau n}(kr) = k \mathbf{v}_{\bar{\tau} n}(kr) \\ \nabla \times \mathbf{u}_{\tau n}(kr) = k \mathbf{u}_{\bar{\tau} n}(kr) \end{cases}$$

The dual index  $\bar{\tau}$  is defined by  $\bar{1} = 2$  and  $\bar{2} = 1$ .

## B COMPLETENESS AND CLOSEDNESS

Let  $H$  be a Hilbert space with scalar product  $(u, v)$ . A system  $\{\phi_j\}, j \in I$ , where  $I$  is some countable index set, is closed in  $H$  if and only if for  $f \in H$

$$(f, \phi_j) = 0, \text{ for all } j \in I \quad \Rightarrow \quad f = 0$$

A system  $\{\phi_j\}$  is complete in  $H$  if and only if for every  $f \in H$  and every  $\epsilon > 0$  there are constants  $c_j(\epsilon, f)$  and an integer  $N(\epsilon, f)$  such that

$$f_\epsilon = \sum_{j=1}^{N(\epsilon, f)} c_j(\epsilon, f) \phi_j \quad \Rightarrow \quad \|f - f_\epsilon\| < \epsilon$$

The two important concepts — closedness and completeness — are equivalent in a normed linear space [11, Theorem 11.1.7, pp. 263] or [38, pp. 90–95].

## C EQUALITY OF THE EXPANSION COEFFICIENTS

From Theorem 1 in Section 3.3, we know that the system  $\{\hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L n}(\mathbf{r}), \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R n}(\mathbf{r})\}_n$  is a complete system in the space of square integrable tangential functions  $L_t^2(\Gamma)$ . The electric and the magnetic fields have expansions in terms of this system:

$$\begin{cases} \hat{\nu}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) = \sum_{n'} (\alpha_{L n'} \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L n'}(\mathbf{r}) + \alpha_{R n'} \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R n'}(\mathbf{r})) \\ i\eta_0 \hat{\nu}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) = \sum_{n'} (-\beta_{L n'} Y_- \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L n'}(\mathbf{r}) + \beta_{R n'} Y_+ \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R n'}(\mathbf{r})) \end{cases} \quad \mathbf{r} \in \Gamma$$

The convergence is in the  $L_t^2(\Gamma)$ -norm sense. In this appendix, we give a proof of  $\alpha_{L n} = \beta_{L n}$  and  $\alpha_{R n} = \beta_{R n}$ .

Start by transforming the expansions with the Bohren transformation (13) to the fields  $\mathbf{Q}_L(\mathbf{r})$  and  $\mathbf{Q}_R(\mathbf{r})$ .

$$\begin{cases} \hat{\nu}(\mathbf{r}) \times \mathbf{Q}_L(\mathbf{r}) = \frac{1}{Y_+ + Y_-} \sum_{n'} \left\{ Y_+ (\alpha_{L n'} \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L n'}(\mathbf{r}) + \alpha_{R n'} \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R n'}(\mathbf{r})) \right. \\ \quad \left. + \beta_{L n'} Y_- \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L n'}(\mathbf{r}) - \beta_{R n'} Y_+ \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R n'}(\mathbf{r}) \right\} \\ \hat{\nu}(\mathbf{r}) \times \mathbf{Q}_R(\mathbf{r}) = \frac{1}{Y_+ + Y_-} \sum_{n'} \left\{ Y_- (\alpha_{L n'} \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L n'}(\mathbf{r}) + \alpha_{R n'} \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R n'}(\mathbf{r})) \right. \\ \quad \left. - \beta_{L n'} Y_- \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{L n'}(\mathbf{r}) + \beta_{R n'} Y_+ \hat{\nu}(\mathbf{r}) \times \mathbf{v}_{R n'}(\mathbf{r}) \right\} \end{cases} \quad (\text{C1})$$

With a homogeneous biisotropic scatterer, we can apply the integral representation to the interior of the obstacle with the appropriate wavenumber. Let  $\mathbf{Q}_{L,R}(\mathbf{r}) \in F(\mathcal{D})$  satisfy the Maxwell equations in  $\mathcal{D}$ . The appropriate integral representation to be used here is [19, Sec. 3]

$$\begin{aligned} \pm \frac{1}{k_\mp} \nabla \times \left\{ \nabla \times \iint_{\Gamma} \mathbf{G}_e(k_\mp, |\mathbf{r} - \mathbf{r}'|) \cdot (\hat{\nu}(\mathbf{r}') \times \mathbf{Q}_{L,R}(\mathbf{r}')) \, dS' \right\} \\ - \nabla \times \iint_{\Gamma} \mathbf{G}_e(k_\mp, |\mathbf{r} - \mathbf{r}'|) \cdot (\hat{\nu}(\mathbf{r}') \times \mathbf{Q}_{L,R}(\mathbf{r}')) \, dS' = \begin{cases} \mathbf{Q}_{L,R}(\mathbf{r}), & \mathbf{r} \text{ inside } \Gamma \\ \mathbf{0}, & \mathbf{r} \text{ outside } \Gamma \end{cases} \end{aligned}$$

where the upper(lower) sign holds for the L(R) wave, and the Green dyadic for the electric field in free space is given in (8). With the use of (19), we rewrite as

$$\pm(k_+ + k_-) \iint_{\Gamma} \mathbf{G}_{\text{bi}}^\mp(\mathbf{r} - \mathbf{r}') \cdot (\hat{\nu}(\mathbf{r}') \times \mathbf{Q}_{L,R}(\mathbf{r}')) \, dS' = \begin{cases} \mathbf{Q}_{L,R}(\mathbf{r}), & \mathbf{r} \text{ inside } \Gamma \\ \mathbf{0}, & \mathbf{r} \text{ outside } \Gamma \end{cases}$$

Now, let the position vector  $\mathbf{r}$  be located outside the smallest circumscribed sphere of  $\Gamma$ , i.e.,  $r > r_{\max}$ . The proper expansion of the Green dyadic in spherical vector waves is now used, see (20). We obtain

$$ik_{\mp}^2 \sum_n \left\{ \mathbf{u}_{L,Rn}(\mathbf{r}) \iint_{\Gamma} \mathbf{v}_{L,Rn}(\mathbf{r}') \cdot (\hat{\mathbf{v}}(\mathbf{r}') \times \mathbf{Q}_{L,R}(\mathbf{r}')) dS' \right\} = \mathbf{0}$$

Orthogonality of the vector spherical harmonics over a spherical surface  $r > r_{\max}$  implies that the coefficients must be identically zero for all values of  $n$ . From the expression above, we conclude

$$\iint_{\Gamma} \mathbf{v}_{L,Rn}(\mathbf{r}) \cdot (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{Q}_{L,R}(\mathbf{r})) dS = 0, \quad \text{for all } n$$

Insert the expansions of the fields (C1). For the  $\mathbf{Q}_L(\mathbf{r})$  field, we obtain the following relation by the use of orthogonality in (17) in Corollary 1 (for each level of approximation, the sum in  $n'$  is finite):

$$\begin{aligned} 0 &= \iint_{\Gamma} \mathbf{v}_{Ln}(\mathbf{r}) \cdot \sum_{n'} \left\{ Y_+ (\alpha_{Ln'} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r}) + \alpha_{Rn'} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) + \beta_{Ln'} Y_- \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r}) - \beta_{Rn'} Y_+ \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r}) \right\} dS \\ &= Y_+ \iint_{\Gamma} \mathbf{v}_{Ln}(\mathbf{r}) \cdot \sum_{n'} (\alpha_{Rn'} - \beta_{Rn'}) (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) dS \end{aligned}$$

and similarly for the  $\mathbf{Q}_R(\mathbf{r})$  field

$$\begin{aligned} 0 &= \iint_{\Gamma} \mathbf{v}_{Rn}(\mathbf{r}) \cdot \sum_{n'} \left\{ Y_- (\alpha_{Ln'} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r}) + \alpha_{Rn'} \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r})) - \beta_{Ln'} Y_+ \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r}) + \beta_{Rn'} Y_- \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn'}(\mathbf{r}) \right\} dS \\ &= Y_- \iint_{\Gamma} \mathbf{v}_{Rn}(\mathbf{r}) \cdot \sum_{n'} (\alpha_{Ln'} - \beta_{Ln'}) (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln'}(\mathbf{r})) dS \end{aligned}$$

where the equality sign is interpreted in the  $L^2(\Gamma)$ -norm sense. Rewrite as

$$\iint_{\Gamma} (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln}(\mathbf{r})) \cdot \left\{ \sum_{n'} (\alpha_{Rn'} - \beta_{Rn'}) \mathbf{v}_{Rn'}(\mathbf{r}) \right\} dS = 0, \quad \text{for all } n$$

and

$$\iint_{\Gamma} (\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn}(\mathbf{r})) \cdot \left\{ \sum_{n'} (\alpha_{Ln'} - \beta_{Ln'}) \mathbf{v}_{Ln'}(\mathbf{r}) \right\} dS = 0, \quad \text{for all } n$$

Completeness of the system  $\{\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln}(\mathbf{r}), \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn}(\mathbf{r})\}_n$  in Theorem 1 gives (due to Corollary 1, adding the missing component in the two relations above does not affect the identity)

$$\sum_n (\alpha_{L,Rn} - \beta_{L,Rn}) \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{L,Rn}(\mathbf{r}) = \mathbf{0}, \quad \mathbf{r} \in \Gamma$$

since only the tangential components of  $\mathbf{v}_{L,Rn}(\mathbf{r})$  are affected.

The final step in the proof is to employ the linear independence the regular Beltrami spherical vector waves  $\{\hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Ln}(\mathbf{r}), \hat{\mathbf{v}}(\mathbf{r}) \times \mathbf{v}_{Rn}(\mathbf{r})\}_n$ , see Lemma 4. This property implies that  $\alpha_{Ln} = \beta_{Ln}$  and  $\alpha_{Rn} = \beta_{Rn}$ .

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